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Joe Harris

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A Bound on the Geometric Genus of Projective Varieties.

JOE HARRIS

0. – Introduction.

The genus of a plane curve C is readily calculated in terms of its degree and singularities by any one of a number of elementary means. The genus of a curve in $\boldsymbol{P}^n$ is not so easily described: smooth curves of a given degree in $\boldsymbol{P}^n$ may have many different genera. One question we may reasonably hope to answer, however, is to determine the greatest possible genus of an irreducible, non-degenerate curve of degree d in $\boldsymbol{P}^n$; this problem was solved in 1889 by Castelnuovo (3), (4) (5), who went on to give a complete geometric description of those curves which achieved his bound. In this paper, we will answer the analogous question for varieties of arbitrary dimension: what is the greatest possible geometric genus of an irreducible, nondegenerate variety of dimension k and degree d in $\boldsymbol{P}^n$?

We begin in section 1 by recounting Castelnuovo's argument; a more detailed version of the argument may be found in (3) and in (4). In section 2 we use the standard adjunction sequence to relate the forms of top degree on a variety to those on its hyperplane section, and, working back to the curve case, derive a bound on the geometric genus. We do not have at this point any assurance that the bound is sharp, but we draw some conclusions about varieties which achieve the bound, called *Castelnuovo varieties*, if they exist. In particular, we see that any Castelnuovo variety of dimension l lies on a $(k+1)$-fold of minimal degree $n-k$ in $\boldsymbol{P}^n$; and section 3 is accordingly devoted to a description of these varieties, classically known as rational normal scrolls and discussed in (1) and (2).

We return in section 4 to consideration of Castelnuovo varieties. We are now able to determine their divisor classes on the minimal varieties containing them, and as a consequence to show they exist. Finally in section 5 we deduce some consequences on the geometry of Castelnuovo varieties.

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1. – Castelnuovo's bound for the genus of a space curve.

The genus of a plane curve $C \subset \boldsymbol{P}^2$ of degree d is, by any one of several elementary arguments, equal to $(d-1)(d-2)/2$ if C is smooth, strictly less if C is singular. The genus of a curve in $\boldsymbol{P}^n$, of course, is not so easily described: there are smooth curves C of a given degree d in $\boldsymbol{P}^n$ having many different genera. One question we may reasonably expect to answer, however, is to determine the greatest possible genus of an irreducible, nondegenerate curve of given degree in $\boldsymbol{P}^n$. Castelnuovo in 1889 answered this question, and in addition described in some detail the geometry of curves having maximal genus. In the following, we will first give Castelnuovo's argument for curves, and then go on to consider analogous questions for higher-dimensional varieties.

Castelnuovo, in his argument, considers not only the linear system $|\mathcal{O}_C(1)|$ on a curve $C \subset \boldsymbol{P}^n$, but the general series $|\mathcal{O}_C(l)|$. His approach is to bound from below the successive differences $h^0(C, \mathcal{O}_C(l)) - h^0(C, \mathcal{O}_C(l-1))$ and hence the dimensions $h^0(C, \mathcal{O}_C(l))$; when l is large enough to ensure $h^1(C, \mathcal{O}(l)) = 0$ he applies Riemann-Roch to obtain an upper bound on the genus of C. The mainspring of his argument is the following lemma, of which only the first assertion is necessary to deduce the bound on $g(C)$:

LEMMA. *Any set Γ of $d \geqslant kn+1$ points in $\boldsymbol{P}^n$ in general position (i.e., no $n+1$ linearly dependent) imposes at least $nk+1$ conditions on the linear system $|\mathcal{O}_{\boldsymbol{P}^n}(k)|$ of hypersurfaces of degree k in $\boldsymbol{P}^n$; and if $d > kn+1$ ($2n+2$ if $k=2$), then Γ imposes exactly $nk+1$ conditions on $|\mathcal{O}_{\boldsymbol{P}^n}(k)|$ if and only if Γ lies on a rational normal curve in $\boldsymbol{P}^n$.*

PROOF. The first statement is easy. To show that Γ imposes $kn+1$ conditions on hypersurfaces of degree k, we have to choose $kn+1$ points $\{p_i\}$ of Γ, and then for any one of these points p_i exhibit a hypersurface of degree k containing the remaining nk points but not p_i. This is immediate: given any subset $\{p_i\}$ of $nk+1$ points of Γ and one point p_i among them, we can group the remaining nk into k sets $\{q_{ij}\}_{j=1,\dots,n}$ of n apiece; the sum $\sum_{i=1}^{k} H_i$ of the hyperplanes $H_i = \overline{q_{i1}, \dots, q_{in}}$ will then be a hypersurface of degree k, by the general position hypothesis not containing p_i.

The second statement is more subtle. We first reduce to the case $k=2$, arguing that a set Γ of $d > kn+1$ points in general position which impose only $nk+1$ conditions on $|\mathcal{O}_{\boldsymbol{P}^n}(k)|$ must impose only $2n+1$ conditions on quadrics, as follows: Assume that Γ imposes only $nk+1$ condition on $|\mathcal{O}_{\boldsymbol{P}^n}(k)|$; since no subset of $nk+1$ or more points of Γ can impose fewer than $nk+1$ conditions on $|\mathcal{O}_{\boldsymbol{P}^n}(k)|$, it follows that any hypersurface of degree k in $\boldsymbol{P}^n$ containing any $nk+1$ points of Γ contains all of Γ. Now, if $Q\subset \boldsymbol{P}^n$ is any quadric containing $2k+1$ points $p_1, \ldots, p_{2k+1}$ of Γ, $q\in\Gamma$ any other point, then q must lie on Q: choosing any $k-2$ disjoint sets of points $\{p_{i1}, \ldots, p_{in}\}$ from the remaining points of Γ, the hypersurface $Q+ + H_1 + \ldots + H_{k-2}$ consisting of Q plus the hyperplanes $H_i=\overline{p_{i1}, \ldots, p_{in}}$ by hypothesis contains q, and since by general position $q\notin H_i$, it follows that $q\in Q$. We see then that Γ can impose only $2n+1$ conditions on $|\mathcal{O}_{\boldsymbol{P}^n}(2)|$.

Castelnuovo's proof of the lemma now runs as follows: labelling the points of Γ $p_1, \ldots, p_d$, let V^i be the $(n-2)$-plane in $\boldsymbol{P}^n$ spanned by the points $p_1, \ldots, \hat{p}_i, \ldots, p_n$, and let $\{V^i(\lambda)\}_\lambda$ be the pencil of hyperplanes containing V^i. Similarly, let $V\subset\boldsymbol{P}^n$ be the $(n-2)$-plane spanned by the points $p_{n+1}, \ldots, p_{2n-1}$, and $\{V(\lambda)\}$ the pencil of hyperplanes through V. For each index $\alpha=2n, 2n+1, \ldots, d$, let λ^i_α be such that

$$p_\alpha\in V^i(\lambda^i_\alpha)$$

and likewise let λ_α be such that

$$p_\alpha\in V(\lambda_\alpha);$$

by general position, of course, $p_\alpha\notin V$, V^i for $\alpha\geqslant 2n$, so λ_α and λ^i_α are uniquely determined, and the values $\{\lambda^i_\alpha\}$ are distinct for each i. Let

$$\varphi_i\colon \boldsymbol{P}^1\to\boldsymbol{P}^1$$

be the unique isomorphism such that

$$\varphi_i(\lambda_\alpha)=\lambda^i_\alpha \qquad \text{for } \alpha=2n, 2n+1, 2n+2\,,$$

and consider the hypersurfaces

$$Q_i=\bigcup_{\lambda\in\boldsymbol{P}^1} V(\lambda)\cap V^i(\varphi_i\lambda)\,.$$

Q_i is a quadric: for any line $L\subset\boldsymbol{P}^n$, φ_i defines an automorphism of L by

$$V(\lambda)\cap L\mapsto V^i(\varphi_i\lambda)\cap L\,,$$

the points of intersection of L with Q_i being just the fixed points of this automorphism. Q_i contains the subspaces V and V^i, and hence the points

$p_1, \ldots, p_i, \ldots, p_n, p_{n+1}, \ldots, p_{2n-1}$ of Γ; in addition, by the choice of φ_i, Q_i contains p_{2n}, p_{2n+1} and p_{2n+2}. Q_i thus passes through at least $2n+1$ points of Γ, and so contains Γ; this in turn implies that

$$\varphi_i(\lambda_\alpha) = \lambda_\alpha^i$$

for all $\alpha \geqslant 2n$, and not just the three values $\alpha = 2n$, $2n+1$ and $2n+2$.

Now, for each λ, the hyperplanes $\{V^i(\lambda)\}_{i=1,\ldots,n}$ meet transversely in a single point: any line contained in $\bigcap_\lambda V^i(\lambda)$ would necessarily meet every $(n-2)$-plane V^i, and hence lie in the hyperplane $W = \overline{p_1, \ldots, p_n}$; this would imply that for some $i \neq j$ and λ,

$$V^i(\varphi_i\lambda) = W = V^j(\varphi_j\lambda)\,.$$

But then the quadric

$$Q_{ij} = \bigcup_\lambda V^i(\varphi_i\lambda) \cap V^j(\varphi_j\lambda)$$

would consist of the hyperplane W plus another hyperplane W' containing $p_1, \ldots, p_i, \ldots, p_j, \ldots, p_n, p_{2n}, p_{2n+1}$ and p_{2n+2}—contrary to the hypothesis that no hyperplane in $\boldsymbol{P}^n$ contains more than n of the points of Γ. The locus

$$D = \bigcup_\lambda V^1(\lambda) \cap \ldots \cap V^n(\lambda)$$

is thus an irreducible rational curve; it has degree n, since the hyperplane W meets it transversely in exactly the n points $p_1, \ldots, p_n$ and so is a rational curve. By construction, moreover, it contains the points $p_1, \ldots, p_n$ and p_α for $\alpha \geqslant 2n$. We see, then, that we can put a rational normal curve through all but any $n-1$ points of Γ—but since Γ contains at least $2n+3$ points and a rational normal curve in $\boldsymbol{P}^n$ is determined by any $n+3$ points on it, this means that Γ lies on a rational normal curve.

Note finally that since a quadric in $\boldsymbol{P}^n$ meeting a rational normal curve in $\boldsymbol{P}^n$ in $2n+1$ points contains it, the linear system $|\mathfrak{I}_\Gamma(2)|$ of quadrics passing through Γ is exactly the linear system $|\mathfrak{I}_D(2)|$ of quadrics containing the rational normal curve D containing Γ; and that, since D is cut out by quadrics, D is exactly the base locus of the linear system of quadrics through Γ. Q.E.D. for lemma.

Now let $C \subset \boldsymbol{P}^n$ be a nondegenerate, irreducible curve of degree d, and let $\Gamma = H \cdot C$ be a generic hyperplane section of C. We have then the

LEMMA. *The points of Γ are in general position.*

PROOF. We first note that the monodromy action on Γ is the full symmetric group S_d. This follows from an induction: in case $n = 2$, we observe that projection of C from any point $p_i \in \Gamma$ expresses C as a $(d-1)$-sheeted cover of $\boldsymbol{P}^1$, and so the monodromy in the pencil of lines through p_i alone acts transitively on $\Gamma - \{p_i\}$. The full monodromy on Γ is thus twice transitive—but any subgroup of S_d which is twice transitive and contains a simple transposition contains all transpositions and so must be all of S_d. We see from this by induction that for general n, the monodromy in the linear system hyperplanes through any point p is the full symmetric group on $\Gamma - \{p\}$, and hence, since the monodromy is always transitive, the full monodromy group is S_d. Now let

$$I \subset C^n \times \boldsymbol{P}^{n*}$$

be the incidence correspondence

$$I = \{(p_1, \dots, p_n; H) : p_i \in H \ \forall i\}$$

I projects onto $\boldsymbol{P}^{3*}$ as an everywhere finite $\binom{d}{n}$-sheeted branched cover; since by the above the monodromy acts transitively on the sheets of $I \to \boldsymbol{P}^{n*}$, I is irreducible.

Let $J \subset I$ be given by

$$J = \{(p_1, \dots, p_n; H) \in I : p_1 \wedge \dots \wedge p_n = 0\} .$$

J is a closed subvariety of I, and since C is nondegenerate, $J \neq I$; thus J cannot surject onto $\boldsymbol{P}^{n*}$ and the lemma is proved. Q.E.D.

Now, by our first lemma the points of $\Gamma = H \cdot C$ impose at least $l(n-1) + 1$ conditions on the linear system of hypersurfaces of degree l in $\boldsymbol{P}^n$, for $l \leqslant M = [(d-1)/(n-1)]$; a fortiori, they impose at least $l(m+1) + 1$ conditions on the complete linear system $|\mathcal{O}_C(l)|$ on C. We see then that

$$\begin{aligned} h^0(C, \mathcal{O}_C(l-1)) &= h^0(C, \mathfrak{I}_\Gamma(l)) \\ &\leqslant h^0(C, \mathcal{O}_C(l)) - l(n-1) - 1 \end{aligned}$$

for $l \leqslant M$; of course for $l > M$ the points of Γ all impose independent conditions on $|\mathcal{O}_C(l)|$ and we have

$$h^0(C, \mathcal{O}_C(l-1)) \leqslant h^0(C, \mathcal{O}_C(l)) - d .$$

Adding up these inequalities, then, we have

$$
\begin{aligned}
&h^0(C, \mathcal{O}_C(1)) \geqslant n+1\\
&h^0(C, \mathcal{O}_C(2)) \geqslant n+1+2(n-1)+1=3n\\
&\vdots\\
&h^0(C, \mathcal{O}_C(M)) \geqslant \frac{M(M+1)}{2}(n-1)+M+1\\
&h^0(C, \mathcal{O}_C(M+k)) \geqslant \frac{M(M+1)}{2}(n-1)+M+1+kd.
\end{aligned}
$$

For large k, of course, the linear system $|\mathcal{O}_C(M+k)|$ will be nonspecial, and so we can apply Riemann-Roch:

$$
\begin{aligned}
\frac{M(M+1)}{2}(n-1)+M+1+kd \leqslant h^0(C, \mathcal{O}_C(M+k))&\\
=\deg \mathcal{O}_C(M+k)-g(C)+1&\\
=d(M+k)-g(C)+1&
\end{aligned}
$$

to obtain

$$
\begin{aligned}
g(C) &\leqslant dM-\frac{M(M+1)}{2}(n-1)-M\\
&=M\left(d-\frac{M+1}{2}(n+1)-1\right);
\end{aligned}
$$

this is our bound on the genus $g(C)$.

Now, we may note that if the genus of a curve $C \subset \boldsymbol{P}^n$ realizes this bound, then equality must hold in each of the inequalitities above. This means in particular that $h_0(C, \mathcal{O}_C(2))=3n$, so that C must lie on $\infty^{\frac{1}{2}(n+1)(n+2)-3n-1}=$ $=\infty^{\frac{1}{2}(n-1)(n-2)-1}$ quadrics; and that the points of a generic hyperplane $\Gamma=H \cdot C$ of C impose only $2n-1$ conditions on quadrics, and so lie on a rational normal curve D. Inasmuch as no quadric containing C may be reducible, it follows that the complete system $|\mathcal{J}_C(2)|$ of quadrics in $\boldsymbol{P}^n$ containing C cuts out on H the complete system $|\mathcal{J}_D(2)|$. The base locus of $|\mathcal{J}_C(2)|$ thus intersects H in D; so that the intersection of the quadrics containing $C \subset \boldsymbol{P}^n$ is a surface of degree $n-1$. We have, then, that

The genus of an irreducible, nondegenerate curve $C \subset \boldsymbol{P}^n$ of degree d is

$$
g(C) \leqslant M\left(d-\frac{M+1}{2}(n-1)-1\right);
$$

and any curve which achieves this bound lines on a surface of degree $n-1$ *in* $\boldsymbol{P}^n$ *cut out by the quadrics through* C.

As an application of Castelnuovo's bound, we may give an inequality on the geometric genus $p_g(V)$ and k-fold self-intersection $c_1^k(V)$ of the canonical bundle of a variety V of dimension k whose canonical series is birationally very ample. Let $n = p_g(V) - 1$, and $\bar{V} \subset \boldsymbol{P}^n$ the canonical variety of V; let $C = \boldsymbol{P}^{n-k+1} \cdot \bar{V}$ be a generic $(n-k+1)$-plane section of $\bar{V}$. C is then an irreducible, nondegenerate curve in $\boldsymbol{P}^{n-k+1}$, of degree $(-1)^k c_1(V)^k$; and by successive applications of the adjuntion formula we have

$$K_C = k \cdot K_V|_C$$

i.e.,

$$2g(C) - 2 = k(-1)^k c_1(V)^k$$

so the genus of C is $\big(k\cdot(-1)^k c_1(V)^k\big)/2 + 1$. Applying Castelnuovo's bound to C, then, we see that

$$\begin{aligned} n-k+1 &\leqslant \frac{2\big((k+1)\big((-1)^k c_1(V)^k - 1\big) - \big(k(-1)^k c_1(V^k)\big)/2 + 1\big)}{(k+1)(k+2)} \\ &= \frac{(-1)^k c_1(V)^k - 2}{k+1} \end{aligned}$$

so that

$$p_g(V) \leqslant \frac{(-1)^k c_1(V)^k - 2}{k+1} + k\,.$$

We will see in the fourth section that in fact this inequality is sharp, i.e., for all numbers $n \geqslant k$ there exist varieties V of dimension k, with $p_g(V) = n+1$ and

$$(-1)^k c_1(V)^k = (k+1)p_g(V) + 2\,.$$

2. – A bound on the geometric genus of projective varieties.

We ask now in general the question that has been asked and answered for curves: what is the greatest possible geometric genus of an irreducible, nondegenerate variety V of dimension k and degree d in $\boldsymbol{P}^n$?

The terms of this question require some clarification, inasmuch as we will not be restricting ourselves to smooth varieties only. Explicitly, given any variety $\tilde{V} \subset \boldsymbol{P}^n$ we can find a *resolution* of V—that is, a smooth abstract variety $\tilde{V}$ mapping holomorphically and birationally to V. We take the *geometric genus* $p_g(V)$ to be the number $h^0(\tilde{V}, \Omega^k_{\tilde{V}}) = h^k(\tilde{V}, \mathcal{O}_{\tilde{V}})$ of holomorphic forms of top degree on $\tilde{V}$; inasmuch as this is a birational invariant it does not depend on the choice of a resolution. In what follows, we will be working primarily with $\tilde{V}$, but will maintain the terminology of projective geometry: a « hyperplane section of $\tilde{V}$ » will be the pullback of hyperplane in $\boldsymbol{P}^n$ via the map $\tilde{V} \xrightarrow{\pi} V$, $\mathcal{O}_{\tilde{V}}(1) = \pi^* \mathcal{O}_{\boldsymbol{P}^n}(1)$ the corresponding sheaf. Likewise, an m-plane section « $\boldsymbol{P}^m \cap \tilde{V}$ » of V will be the intersection on $\tilde{V}$ of $n - m$ elements of the linear system $|\mathcal{O}_{\tilde{V}}(1)|$; note that since $|\mathcal{O}_{\tilde{V}}(1)|$ has no base points, by Bertini the generic m-plane section of $\tilde{V}$ will be smooth.

To answer our question about the geometric genus $p_g(V)$, we consider first a generic $(n-k+1)$-plane section $C = V \cap \boldsymbol{P}^{n-k+1}$. The curve C is nondegenerate and irreducible of degree d in $\boldsymbol{P}^{n-k+1}$; accordingly, if we set

$$M = \left[\frac{d-1}{n-k}\right]; \qquad \varepsilon = d - 1 - M(n-k)$$

then by our previous argument

$$h^0(C, \mathcal{O}(l)) - h^0(C, \mathcal{O}(l-1)) \geqslant \begin{cases} l(n-k) + 1\,, & l \leqslant M \\ d\,, & l \geqslant M \end{cases}.$$

From Riemann-Roch on C, then, we see that

$$\begin{aligned} & h^0(C, \Omega^1_C(-l+1)) - h^0(C, \Omega^1_C(-l)) \\ & \quad = g(C) - (l-1)d + 1 + h^0(C, \mathcal{O}(l-1)) - g(C) - ld + 1 + h^0(C, \mathcal{O}(l)) \\ & \quad \leqslant \begin{cases} d - l(n-k) - 1 & l \leqslant M \\ 0 & l > M\,. \end{cases} \end{aligned}$$

Since $h^0(C, \Omega^1_C(-l)) = 0$ for l sufficiently large, then, it follows that

$$\begin{aligned} & h^0(C, \Omega^1_C(-M)) = 0 \\ & h^0(C, \Omega^1_C(-M+1)) \leqslant d - M(n-k) - 1 \\ & h^0(C, \Omega^1_C(-M+2)) \leqslant 2d - (2M-1)(n-k) - 2 \\ & \vdots \\ & h^0(C, \Omega^1_C(-M+l)) \leqslant ld - \sum_{i=0}^{l-1}(M-1)(n-k) - l \\ & \qquad\qquad = \frac{l(l-1)}{2}(n-k) + l\varepsilon \end{aligned}$$

Now let S be a generic $(n-k+2)$-plane section of V containing C, and consider the standard Poincaré residue sequence tensored with $\mathcal{O}_S(-l)$:

$$0 \to \Omega_S^2(-l) \to \Omega_S^2(-l+1) \to \Omega_C^1(-l) \to 0 .$$

From the first three terms of the associated exact sequence in cohomology, we have

$$h^0(S, \Omega_S^2(-l+1)) - h^0(S, \Omega_S^2(-l)) \leqslant h^0(C, \Omega_C^1(-l))$$

and since $h^0(S, \Omega_S^2(-l)) = 0$ for $l \ll 0$, it follows that

$$\begin{aligned} h^0(S, \Omega_S^2(-M+1)) &= 0 \\ h^0(S, \Omega_S^2(-M+2)) &\leqslant h^0(C, \Omega_C^1(-M+1)) \\ &\leqslant \varepsilon \end{aligned}$$

and in general

$$\begin{aligned} h^0(S, \Omega_S^2(-M+l)) &\leqslant \sum_{i=1}^{l-1} h^0(C, \Omega_C^1(-M+1)) \\ &\leqslant \sum_{i=1}^{l-1}\left[i\varepsilon + \frac{i(i-1)}{2}(n-k)\right] \\ &= \frac{l(l-1)(l-2)}{6}(n-k) + \frac{l(l-1)}{2}\varepsilon . \end{aligned}$$

The procedure in general is just a repetition of this first step. If T is a generic $(n-k+3)$-plane section of $\tilde{V}$ containing S, then we have similarly

$$h^0(T, \Omega_T^3(-l+1)) - h^0(T, \Omega_T^3(-l)) \leqslant h^0(S, \Omega_S^2(-l))$$

hence

$$h^0(T, \Omega_T^3(-M+2)) = 0$$

and in general

$$\begin{aligned} h^0(T, \Omega_T^3(-M+l)) &\leqslant \sum_{i=2}^{l-1} h^0(S, \Omega_S^2(-M+1)) \\ &\leqslant \sum_{i=1}^{l-1}\left[\frac{i(i-1)}{2}\varepsilon + \frac{i(i-1)(i-2)}{6}(n-k)\right] \\ &= \frac{i(i-1)(i-2)(i-3)}{24}(n-k) + \frac{i(i-1)(i-2)}{6}\varepsilon . \end{aligned}$$

Continuing in this fashion, we find that

$$h^0(\tilde{V}, \Omega^k_{\tilde{V}}(-MM+l)) \leqslant \binom{l}{k+1}(n-k) + \binom{l}{k}\varepsilon$$

and in particular,

$$\begin{aligned} p_g(V) &= h^0(\tilde{V}, \Omega^k_{\tilde{V}}) \\ &\leqslant \binom{M}{k+1}(n-k) + \binom{M}{k}\varepsilon \end{aligned}$$

where of course the binomial coefficient $\binom{a}{b} = 0$ when $b > a$. This, then, is our bound on the geometric genus of a variety; we note as an immediate consequence that

The geometric genus of a variety of dimension k and degree d in $\boldsymbol{P}^n$ is zero whenever

$$d \leqslant k(n-k) + 1 .$$

We will call varieties $V^k_d \subset \boldsymbol{P}^n$ of degree $d \geqslant k(n-k) + 2$ which achieve our bound on p_g *Castelnuovo varieties.*

Our principal object for the remainder of this discussion will be to show that Castelnuovo varieties exist, and to describe their properties. To start with, we consider by analogy with the curve case the linear system of quadrics containing such a variety; again, the central point is to show that

The points of a generic $(n-k)$-plane section of a Castelnuovo variety $V^k \subset \boldsymbol{P}^n$ lie on a rational normal curve.

To see this, we observe that if V is Castelnuovo, then the generic hyperplane section D of V must satisfy

$$h^0(D, \Omega^{k-1}_D(-M+l)) = \binom{l}{k}(n-k) + \binom{l}{k-1}\varepsilon$$

for all $l \leqslant M-1$; hence the double hyperplane section D' of V satisfies

$$h^0(D', \Omega^{2-k}_{D'}(-M+l)) = \binom{l}{k-1}(n-k) + \binom{l}{k-2}\varepsilon$$

for all $l \leqslant M-2$ and so on; finally, we conclude that the generic $(n-k+1)$-plane section C of V satisfies

$$h^0(C, \Omega^1_C(-M+l)) = \frac{l(l-1)}{2}(n-k) + l\varepsilon$$

for all $l \leqslant M - k + 1$. Assume for the moment that $d - 1 \not\equiv 0\,(n-k)$. Since $M \geqslant k$, this gives us in particular

$$h^0(C, \Omega_C^1(-M)) = 0$$

and

$$h^0(C, \Omega_C^1(-M+1)) = d - M(n-k) - 1\,.$$

Applying Riemann-Roch, we have

$$h^0(C, \mathcal{O}(M)) - h^0(C, \mathcal{O}(M-1)) = M(n-k) + 1$$

i.e., the points of a hyperplane section Γ of C impose only $M(n-k)+1$ conditions on the linear system $|\mathcal{O}_C(M)|$. Since $d > M(n-k)+1$, this tells that the points of Γ lie on a rational normal curve.

In fact, this last argument tells us something more: C—and hence every generic plane section of V—must be projectively normal, since otherwise the points of a divisor $\Gamma \in |\mathcal{O}_C(1)|$ would impose strictly more than $M(n-k)+1$ conditions on $|\mathcal{O}_C(M)|$. Now, since V is projectively normal, we see from the sequence

$$0 \to \mathfrak{I}_{V,\boldsymbol{P}^n}(1) \to \mathcal{O}_{\boldsymbol{P}^n}(1) \to \mathcal{O}_V(1) \to 0$$

that

$$H^1(\boldsymbol{P}^n, \mathfrak{I}_{V,\boldsymbol{P}^n}(1)) = 0\,.$$

Let $D = V \cdot H$ again be a hyperplane section of V and consider the sequence

$$0 \to \mathfrak{I}_{V,\boldsymbol{P}^n}(1) \to \mathfrak{I}_{V,\boldsymbol{P}^n}(2) \to \mathfrak{I}_{D,\boldsymbol{P}^{n-1}}(2) \to 0$$

obtained by restriction to H. Since $h^1(\boldsymbol{P}^n, \mathfrak{I}_{V,\boldsymbol{P}^n}(1)) = 0$, $h^0(\boldsymbol{P}^n, \mathfrak{I}_{V,\boldsymbol{P}^n}(2))$ must surject onto $h^0(\boldsymbol{P}^{n-1}, \mathfrak{I}_{D,\boldsymbol{P}^{n-1}}(2))$; that is, the linear system of quadrics containing V cuts out on H the complete system of quadrics containing D. Reiterating, we see that the restriction of $|\mathfrak{I}_V(2)|$ to a generic $\boldsymbol{P}^{n-k}$ is the complete system $|\mathfrak{I}_\Gamma(2)|$ of quadrics containing the points of $\Gamma = V \cdot \boldsymbol{P}^{n-k}$; the base locus of this system—and hence the φ intersection of the base locus of $|\mathfrak{I}_V(2)|$ with $\boldsymbol{P}^{n-k+1}$—is thus a rational normal curve. It follows that base locus of $|\mathfrak{I}_V(2)|$ is a $(k+1)$-dimensional variety of (minimal) degree $n-k$ in $\boldsymbol{P}^n$; we have, then, that

A Castelnuovo variety $V \subset \boldsymbol{P}^n$ of dimension k lies on a variety S of dimension $k+1$ and degree $n-k$ in $\boldsymbol{P}^n$, cut out by the quadrics containing V.

In case $d-1 \equiv 0 \ (n-k)$, the same argument works with indices shifted: we have in this case $M \geqslant k+1$, hence

$$h^0(C, \Omega_C^1(-M+1)) = d-M(n-k)-1 = 0$$
$$h^0(C, \Omega_C^1(-M+2)) = 2d-(2M-1)(n-k)-2$$
$$= d-(M-1)(n-k)-1$$

so

$$h^0(C, \mathcal{O}_C(M-1))-h^0(C, \mathcal{O}_C(M-1)) = (M-1)(n-k)-1$$

and the argument proceeds as before.

Now, it is clear from the above that in order to study Castelnuovo varieties further we must have a description of « minimal varieties » X of dimension $k+1$ and degree $n-k$ in $\boldsymbol{P}^n$. We will give such a description in the following section, and then return to our discussion of Castelnuovo varieties in section IV.

3. – Minimal varieties.

We will begin by constructing a class of $(k+1)$-dimensional varieties of degree $n-k$ in $\boldsymbol{P}^n$, as follows. Choose $k+1$ complementary linear subspaces $\{W_i \cong \boldsymbol{P}^{a_i} \subset \boldsymbol{P}^n\}_{i=1,\ldots,k+1}$. Note that since the inverse images $\boldsymbol{C}^{a+1} \subset \boldsymbol{C}^{n+1}$ of the subspaces $W_i \subset \boldsymbol{P}^n$ give a direct sum decomposition of $\boldsymbol{C}^{n+1}$, we must have

$$\sum_{i=1}^{k+1}(a_i+1) = n+1$$

i.e.,

$$\sum a_i = n-k\,.$$

Let $E_i \subset W_i$ be a rational normal curve, and choose isomorphisms

$$\varphi_i\colon \boldsymbol{P}^1 \to E_i\,, \qquad i=1,\ldots,k+1$$

and consider the variety

$$S_{a_1,\ldots,a_{k+1}} = \bigcup_{p\in\boldsymbol{P}^1} \overline{\varphi_1(p), \ldots, \varphi_{k+1}(p)}$$

swept out by the k-planes spanned by corresponding points of the curves E_i. To see that the degree of $S = S_{a_1,\ldots,a_{k+1}}$ is $\sum a_i = n-k$, we use an induction on k: assume the result for $k' \leqslant k$ and any n, and let $H \subset \boldsymbol{P}^n$ be a generic hyperplane containing the subspaces $W_1, \ldots, W_k$. H then intersects S transversely in the variety

$$S_{a_1,\ldots,a_{k+1}} = \bigcup_{p\in\boldsymbol{P}^1} \overline{\varphi_1(p), \varphi_2(p), \ldots, \varphi_k(p)}$$

and the a_{k+1} k-planes

$$\{\overline{\varphi_1(p), \ldots, \varphi_{k+1}(p)}\}_{\varphi_{k+1}(p)\in H}\,.$$

By induction, $S_{a_1,\ldots,a_k}$ has degree $\sum_{i=1}^{k} a_i$, and so $H\cdot S$—and hence S—has degree $\sum_{i=1}^{k+1} a_i = n-k$.

Note that the construction of the variety $S_{a_1,\ldots,a_{k+1}}$ makes sense (of a sort) when some of the integers a_i are zero: if $a_i = 0$, we simply take the « curve » E_i to be the point W_i, the map $\varphi_i\colon \boldsymbol{P}^1 \to E_i$ the only one. It's not hard to see that, in case all the a_i's are positive, the variety $S_{a_1,\ldots,a_{k+1}}$ will be smooth, while if $a_i = 0$ then $S_{a_1,\ldots,a_{k+1}}$ will be the cone, with vertex W_i, over the variety $S_{a_1,\ldots,\hat{a}_i,\ldots,a_{k+1}} \subset \boldsymbol{P}^{n-1}$.

An alternative description of the varieties $S_{a_1,\ldots,a_{k+1}}$ may be given as follows: denote by $H \to \boldsymbol{P}^1$ the hyperplane (i.e, point) bundle on $\boldsymbol{P}^1$, and for any collection of $k+1$ non-negative integers a_i with $\sum a_i = n-k$, consider the vector bundle

$$E = H^{-a_1} \oplus \ldots \oplus H^{-a_{k+1}} \to \boldsymbol{P}^1$$

and the associated $\boldsymbol{P}^k$-bundle $\boldsymbol{P}(E)$. A global section $\sigma \in \Gamma(E^*)$ of the dual bundle $E^* \simeq H^{a_1} \oplus \ldots \oplus H^{a_{k+1}}$ gives a divisor $(\sigma = 0)$ on $\boldsymbol{P}(E)$; the divisors $\{(\sigma)\colon \sigma \in \Gamma(E^*)\}$ form a linear system of dimension

$$h^0(\boldsymbol{P}^1, \mathcal{O}(E^*)) = \sum_i h^0(\boldsymbol{P}^2, \mathcal{O}(a_i)) = \sum (a_i + 1) = n+1$$

and we let $S_{a_1,\ldots,a_{k+1}}$ be the image of $\boldsymbol{P}(E)$ under the map to $\boldsymbol{P}^n$ given by this linear system. Note that the degree of $S_{a_1,\ldots,a_{k+1}}$—that is, the $(k+1)$-fold self-intersection of the divisors (σ)—is just the number of points in $\boldsymbol{P}^1$ where $k+1$ generic section of E^* fail to be linearly independent; this is of course just the first chern class

$$c_1(E^*) = \sum c_1(H^{a_i}) = \sum a_i = n-k$$

of E^*.

To see that these descriptions of $S_{a_1,\ldots,a_{k+1}}$ are the same, note that the linear system $|(\sigma)|$ on $\boldsymbol{P}(E)$ cuts out on the image E_i in $\boldsymbol{P}(E)$ of each factor H^{-a_1} the complete linear system $|\mathcal{O}_{\boldsymbol{P}^1}(a_i)|$, and on the fibers of $\boldsymbol{P}(E)$ the complete linear system $|\mathcal{O}_{\boldsymbol{P}^k}(1)|$: The map given by $|(\sigma)|$ thus sends each curve E_i to a rational normal curve of degree a_i in $\boldsymbol{P}^n$, and the fibers of $\boldsymbol{P}(E)$ to k-planes in $\boldsymbol{P}^n$ joining corresponding points of the curves E_i. The varieties $S_{a_1,\ldots,a_{k+1}}$ are called *rational normal scrolls.*

We claim now that, with a couple of exceptions which we shall describe later, *all nondegenerate, irreducible $(k+1)$-dimensional varieties of degree $n-k$ in $\boldsymbol{P}^n$ are rational normal scrolls.* The argument consists of two steps. We will first show that any irreducible nondegenerate $(k+1)$-dimensional variety $V \subset \boldsymbol{P}^n$ of degree $n-k$—with two classes of exceptions—is swept out by k-planes; then that any such variety swept out by k-planes is in fact a rational normal scroll.

For the first part, we start with a surface $S \subset \boldsymbol{P}^n$ of degree $n-1$. We claim that *unless S is a smooth quadric in $\boldsymbol{P}^3$ of the Veronese surface in $\boldsymbol{P}^5$, through a generic point of S there passes exactly one line of S.* To see this, first note that any line l meeting S three times (not all at the same point) lies in S: if it instead met S in isolated points, the image of S under projection from l would either be a nondegenerate irreducible surface of degree $\leqslant n-4$ in $\boldsymbol{P}^{n-2}$—an impossibility—or a curve, in which case S is a cone over some point of $S \cap l$, and hence $l \subset S$. In particular, if S is singular at p, S must be a cone with vertex p; in this case our assertion is clearly true.

Now let p be a generic point of S, and consider the linear system $|\mathfrak{I}_p^2(1)|$ of hyperplane sections of S tangent to S at p. These are generically nondegenerate curves of degree $n-1$ in $\boldsymbol{P}^{n-1}$, and being singular they must be reducible; indeed, inasmuch as they are all connected, they must generically consist of two rational normal curves meeting at the point p. (We see in particular that, except when $n=3$, there can be at most one line on S through a smooth point of S). By cases, then,

i) if $n=3$ or 4, then clearly one or both of the components of a generic tangent hyperplane section of S will be lines;

ii) If $n \geqslant 6$, then $T_p(S)$ must intersect S in a curve: otherwise, projection of S from $T_p(S)$ would yield an irreducible, nondegenerate surface of degree $\leqslant n-5$ in $\boldsymbol{P}^{n-3}$. But S contains the line joining p to any point $q \neq p \in T_p(S) \cap S$; thus $T_p(S) \cap S$ is a line.

iii) If $n=5$, the two components of a generic tangent hyperplane section of S are both conics, and S meets $T_p(S)$ only at p, we let Z_p denote the family of conics on S containing p, $\boldsymbol{P}^{2*}$ the space of hyperplanes in $\boldsymbol{P}^5$

tangent to S at P, and consider the incidence correspondence

$$I \subset Z_p \times \boldsymbol{P}^{2*}$$

given by

$$I = \{(C, H) \colon C \subset H\} .$$

I maps $2 - 1$ onto $\boldsymbol{P}^{2*}$, and so has either 1 or 2 irreducible components, each of dimension 2; on the other hand, the fibers of the projection $\pi_1 \colon I \to Z_p$ are all $\boldsymbol{P}^1$'s, so Z_p has one or two 1-dimensional components depending on whether I is irreducible or not. In fact, I must be irreducible: if I had two components $\{C_\lambda\}$ and $\{C'_\lambda\}$ then every tangent hyperplane section of S would contain one of each; so

$$C_\lambda \cdot C'_\lambda = 1 .$$

But each family $\{C_\lambda\}$ and $\{C'_\lambda\}$ sweeps out S: so, given any curve C_λ in the first family, we can choose a point $q \neq p \in C_\lambda$ and find a curve C'_λ through q. C_λ and C'_λ will then have a component in common, and so must be equal: otherwise both would consist of a pair of lines, contrary to the hypothesis that S contains no lines through p. It follows that S contains irreducible one-dimensional family $Z_p = \{C_\lambda\}$ of conics through a generic point $p \in S$; and hence altogether S contains an irreducible 2-dimensional family of conics $\{C_\lambda\}$. But now the conics $\{C_\lambda\}$ are all homologous, and hence linearly equivalent since S is rational; and so we see that S has a linear system of dimension $\geqslant 2$ and self-intersection $C_\lambda^2 = 1$. S must therefore be $\boldsymbol{P}^2$, the curves C_λ the images of the lines in $\boldsymbol{P}^2$, and S the Veronese surface.

Now we conclude from this that, with two exceptions, *a $(k+1)$-fold $X \subset \boldsymbol{P}^n$ of degree $n-k$ contains a family of k-planes*: let $p \in X$ be a generic point of X, and let $L_p(X)$ be the variety swept out by the lines on X passing through p. Let $\Lambda \in \boldsymbol{P}^n$ be a generic $(n-k+1)$-plane through the point p. Inasmuch as p is generically chosen on X, the surface $S = \Lambda \cap X$ is a generic $(k-1)$-fold hyperplane section of X, and hence a nondegenerate irreducible surface of degree $n-k$ in $\boldsymbol{P}^{n-k-1}$. If we assume for the moment that it is neither a smooth quadric in $\boldsymbol{P}^3$ or the Veronese surface in $\boldsymbol{P}^5$, then since p is generic on S it follows from our description of minimal surfaces that there is exactly one line on S through p, i.e., that Λ intersects $L_p(X)$ in a single line. $L_p(X)$ is thus a k-plane, and hence X is swept out by k-planes.

Leaving aside for the moment our two exceptional cases—varieties $X \subset \boldsymbol{P}^n$ whose generic $(n-k+1)$-plane section is a quadric in $\boldsymbol{P}^3$ or a Veronese surface—we now show that *any variety $X \subset \boldsymbol{P}^n$ of degree $n-k$ swept out by ∞^1 k-planes is a rational normal scroll.* We prove this by induction on k:

it is clearly true for $k = 0$; given X as above, choose $[n/(k+1)]$ of the k-planes of X, and consider the intersection D of S with a hyperplane H containing those k-planes. D must contain exactly one component D_0 intersecting each k-plane of X in a $(k-1)$-plane; all other components of D are disjoint from a generic k-plane of X and so are k-planes themselves. We can thus write

$$C = D_0 + W_1 + \dots + W_m, \qquad m \geqslant \left[\frac{n}{k+1}\right],$$

$W_1, \dots, W_m$ all k-planes. Clearly the degree of D_0 is $n-k-m$. We claim that the span of D_0 is an $(n-m-1)$-plane in $\boldsymbol{P}^n$: having degree $n-k-m$ and dimension k, it cannot span more than an $(n-m-1)$-plane; if, on the other hand, D_0 were contained in a $\boldsymbol{P}^{n-m-2}$, we could simply choose $m+1$ points $p_1, \dots, p_{m+1}$ of X lying on distinct k-planes $W_1', \dots, W_{m+2}'$ of X and lying off D_0; the hyperplane $\overline{\boldsymbol{P}^{n-m-2}, p_1, \dots, p_{m+1}} \subset \boldsymbol{P}^n$ would then contain the variety $D_0 + W_1' + \dots + W_{m+1}' \subset X$ of degree $n-k+1$ and so contain X. Thus D_0 spans a $\boldsymbol{P}^{n-m-1}$; and being swept out by $(k-1)$-planes, it is by induction hypothesis of the form $S_{a_1,\dots,a_k} \subset \boldsymbol{P}^{n-m-1}$. Let a_1 be the smallest of the a_i's, and E_1 the corresponding rational normal curve in D_0; we have

$$\begin{aligned}\deg E_1 = a_1 &\leqslant \frac{n-k-m}{k}\\ &\leqslant \frac{(n-k)(k+1)-n}{k(k+1)}\\ &= \frac{n}{k+1} - 1\,.\end{aligned}$$

Now each k-plane of X meets E_1 in a point; we can accordingly choose a_1 k-planes $W_1' \dots W_{a_1}'$ of X and find a hyperplane $H' \subset \boldsymbol{P}^n$ containing $W_1' \dots W_{a_1}'$ but not containing E_1. As before, the divisor $D' = H \cdot X$ will contain exactly one component D_0' meeting each k-plane of S in a $(k-1)$-plane; and again D' will be of the form $S_{b_1,\dots,b_k}$, swept out by the $(k-1)$-planes spanned by corresponding points of k rational normal curves $F_1, \dots, F_k$. Note that in fact D_0' is the only component of D' other than the planes W_i'. Any other component of D' would necessarily be a k-plane, and so meet E_1; H' would then meet E_1 in $a_1 + 1$ points and so contain E_1, contrary to hypothesis. We thus have

$$D' = D_i' + W_1 + \dots + W_{a_1};$$

in particular, D_0' is of degree $n-k-a$, and spans a $\boldsymbol{P}^{n-a_1-1}$. Finally, E_1 is disjoint from D'_0—again, if D'_0 met E_1 H would contain $a_1 + 1$ points of E_1—so each k-plane of S meets D'_0 and E_1 in a hyperplane and point disjoint from one another; thus S may be described as the variety swept out by the k-planes spanned by corresponding points of the rational normal curves $E_1, F_1, \ldots, F_k$.

Lastly, we consider the two exceptional cases: varieties $X \subset \boldsymbol{P}^n$ whose generic $(n-k+1)$-plane sections are smooth quadrics in $\boldsymbol{P}^3$—in case $k = n-2$—or Veronese surfaces in $\boldsymbol{P}^5$, in case $k = n-4$. In the former case X is of course a quadric hypersurface of rank $\geqslant 4$; for the latter, we claim that

Any $(n-3)$-fold $X \subset \boldsymbol{P}^n$ whose generic 5-plane section is a Veronese surface, is a cone over a Veronese surface.

To see this, it is sufficient to show that X is singular: it will then follow that X is a cone over a variety $X' \subset \boldsymbol{P}^{n-1}$ whose generic 5-plane section is a Veronese surface, and hence by an induction a cone over the Veronese surface. Now, suppose that X were smooth, $S = X \cdot \boldsymbol{P}^5$ a generic 5-plane section. By the Lefschetz theorem on hyperplane sections, then, the map

$$H_2(S, \boldsymbol{Z}) \to H_2(X, \boldsymbol{Z})$$

induced by the inclusion $S \subset X$ would be a surjection. But $S \cong P^2$, and $H_2(S, \boldsymbol{Z})$ is thus generated by the class of a line in $\boldsymbol{P}^2$—that is, a conic curve in S. It would follow then that every curve on X was homologous to an integral multiple of a conic curve on X, and in particular that X contained no lines. This is impossible: a generic singular 5-plane section of X will consist of a minimal surface other than the Veronese, and so must contain lines.

Summing up, then, we have seen that

An irreducible nondegenerate variety X of dimension $k+1$ and degree $n-k$ in $\boldsymbol{P}^n$ is either

i) *a rational normal scroll;*

ii) *a quadric of rank $\geqslant 4$; or*

iii) *a cone over a Veronese surface.*

3′. – A note on degeneration of minimal varieties.

We have seen that, with two exceptions, the isomorphism classes of irreducible nondegenerate varieties X of dimension $k+1$ and degree $n-k$ in $\boldsymbol{P}^n$ are described by sequences of integers $0 \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_{k+1}$ with $\sum a_i = n-k$. We ask now a question about the relations among these varieties: when is the variety $S_{b_1,\dots,b_{k+1}}$ a degeneration of the variety $S_{a_1,\dots,a_{k+1}}$; or, equivalently, if we denote by $C^{k+1}_{n-k}(\boldsymbol{P}^n)$ the Chow variety of degree $n-k$ $(k+1)$-folds in $\boldsymbol{P}^n$ and by $C_{a_1,\dots,a_{k+1}}$ the subvariety of those isomorphic to $S_{a_1,\dots,a_{k+1}}$, when does $C_{b_1,\dots,b_{k+1}}$ lie in the closure of $C_{a_1,\dots,a_{k+1}}$? The answer is that

$S_{b_1,\dots,b_{k+1}}$ is a degeneration of $S_{a_1,\dots,a_{k+1}}$ if and only if, when we arrange the indices so that $a_1 \leqslant a_2 \leqslant \dots \leqslant a_{k+1}$ and $b_1 \leqslant \dots \leqslant b_{k+1}$,

$$\sum_{i=1}^{\alpha} b_i \leqslant \sum_{i=1}^{\alpha} a_i$$

for all α.

The first step in proving this is to consider the automorphism group of the variety $\boldsymbol{P}(E) = \boldsymbol{P}(H^{-a_1} \oplus \dots \oplus H^{-a_{k+1}})$. Now, inasmuch as projective space $\boldsymbol{P}^k$ does not contain two disjoint divisors, it cannot map holomorphically to $\boldsymbol{P}^1$; thus the only divisors isomorphic to $\boldsymbol{P}^k$ in the variety $\boldsymbol{P}(E)$ are the fibers of the map $\boldsymbol{P}(E) \to \boldsymbol{P}^1$, and correspondingly any automorphism of $\boldsymbol{P}(E)$ preserves these fibers. The automorphism group of $\boldsymbol{P}(E)$ is thus an extension, by $\mathrm{Aut}(\boldsymbol{P}^1) = PGL(2)$, of the subgroup G of automorphisms of $\boldsymbol{P}(E)$ fixing each fiber; and G may be described as follows: On each fiber $\boldsymbol{P}(E)_t$ of $\boldsymbol{P}(E)$, let $Y_1, \dots, Y_{k+1}$ be homogeneous coordinates corresponding to the decomposition $E^* = H^{a_1} \oplus \dots \oplus H^{a_{k+1}}$. If $\varphi\colon \boldsymbol{P}(E) \to \boldsymbol{P}(E)$ is any automorphism carrying $\boldsymbol{P}(E)_t$ to itself for all t, the automorphism of $\boldsymbol{P}(E)_t$ induced by φ is given by a matrix $(\sigma_{ij}(T))$ (defined up to scalars), where the entry $\sigma_{ij}(t)$ is a section of the bundle $\mathrm{Hom}(H^{a_i}, H^{a_j}) = H^{a_j - a_i} \to \boldsymbol{P}^1$; conversely, the generic collection of sections $\{\sigma_{ij} \in H^0(\boldsymbol{P}^1, \mathcal{O}(a_j - a_i))\}$ defines in this way an automorphism of $\boldsymbol{P}(E)$ preserving individual fibers.

The above description of $\mathrm{Aut}(\boldsymbol{P}(E))$ enables us to compute the dimension of $C_{a_1,\dots,a_{k+1}}$: since the line bundle $\mathcal{O}(1)$ on $S_{a_1,\dots,a_{k+1}} \cong \boldsymbol{P}(E)$—being the only line bundle on $\boldsymbol{P}(E)$ intersecting each fiber in a hyperplane and having $(k+1)$-fold self-intersection $n-k$—is preserved by any automorphism of $\boldsymbol{P}(E)$, every automorphism of $S \subset \boldsymbol{P}^n$ is projective, and the dimension

$$\dim C_{a_1,\dots,a_{k+1}} = \dim PGL(n+1) - \dim\big(\mathrm{Aut}(\boldsymbol{P}(E))\big)\,.$$

But we have seen that

$$\begin{aligned}
&\dim\big(\mathrm{Aut}(\boldsymbol{P}(E))\big) = 3 + \dim G\\
&= 3 + \sum h^0\big(\boldsymbol{P}^1, \mathcal{O}(a_j - a_i)\big) - 1\\
&= 2 + \frac{(k+1)(k+2)}{2} + \sum_{j \geqslant i} (a_j - a_i) + \#\{(i,j)\colon i<j,\ a_i = a_j\}\\
&= 2 + \frac{(k+1)(k+2)}{2} - (k+2)(n-k) + 2\sum i a_i + \#\{(i,j)\colon i<j,\ a_i = a_j\}
\end{aligned}$$

and hence

$$\dim C_{a_1,\ldots,a_{k+1}} = (n+1)^2 - 3 + \frac{(k+2)(2n+k-1)}{2} - \\ -2\sum i a_i + \#\{(i,j)\colon i<j,\ a_i = a_j\}.$$

The main point of our description of Aut $(\boldsymbol{P}(E))$, however, is the following observation: if for each $\alpha = 1, \ldots, k+1$ we let S_α denote the image of the sub-bundle $H^{-a_1} \oplus \ldots \oplus H^{-a_\alpha}$ in $S = \boldsymbol{P}(H^{-a_1} \oplus \ldots \oplus H^{-a_{k+1}})$ (that is, the intersection of $S \subset \boldsymbol{P}^n$ with the subspace spanned by the curves $E_1, \ldots, E_\alpha$) then we see from the above argument that *the automorphisms of* S *act transitively on* $S_\alpha - S_{\alpha-1}$—or, to put it differently, if $p \in S_\alpha - S_{\alpha-1}$, *then* S *may be realized as the join of rational normal curves* E_i *of degree* a_i, *with* $p \in E_\alpha$.

We are now prepared to degenerate minimal varieties. As a first step, we will show that

$$S_{a_1,\ldots,a_{k+1}} \textit{ degenerates to } S_{a_1,\ldots,a_{i-1}-1,a_i+1,\ldots,a_{k+1}}.$$

To do this, we consider in general the projection of the variety $S_{a_1,\ldots,a_{k+1}} \subset \boldsymbol{P}^n$ from a point p lying on the rational normal curve $E_i \subset S$ into a hyperplane. Under this map, all the curves E_j, $j \neq 1$, are mapped into rational normal curves $\bar{E}_j$ of degree a_i; E_i, on the other hand, is mapped to a rational normal curve $\bar{E}_i$ of degree $a_i - 1$. The k-planes spanned by corresponding points of the curves E_i are, with the exception of the one passing through p, mapped to k-planes joining the curves $\{\bar{E}_i\}$. The k-plane through p is collapsed to a $(k-1)$-plane, with the exceptional divisor of the projection taking its place; we recognize the image as the variety $S_{a_1,\ldots,a_i-1,\ldots,a_{k+1}}$. Finally, since any point $p \in S_i - S_{i-1}$ may be considered a point of E_i we see that *the image of* $S_{a_1,\ldots,a_{k+1}}$ *under projection from a point* $p \in S_i - S_{i-1}$ *is the variety* $S_{a_1,\ldots,a_i-1,\ldots,a_{k+1}}$.

This gives us explicitly our primary degeneration: let

$$S = S_{a_1,\ldots,a_i+1,a_{i+1},\ldots,a_{k+1}} \subset \boldsymbol{P}^n,$$

let $\gamma(t)$ be an arc in S_i with $\gamma(0) \in S_{i-1}$, $\gamma(t) \notin S_{i-1}$ for $t \neq 0$, and let $\boldsymbol{P}^n \subset \boldsymbol{P}^{n+1}$ be a hyperplane disjoint from the arc γ. If S_t is the image of S under projection from $\gamma(t)$ to $\boldsymbol{P}^n$, then, the varieties $\{S_t\}$ form a family with $S_t \cong S_{a_1,\ldots,a_{k+1}}$ for $t \neq 0$, and $S_0 \cong S_{a_1,\ldots,a_{i-1}-1,a_i+1,\ldots,a_{k+1}}$.

We claim now that by a sequence of these primary degenerations, we can degenerate a variety $S_{a_1,\ldots,a_{k+1}}$ into a variety $S_{b_1,\ldots,b_{k+1}}$ whenever $\sum_{i=1}^{\alpha} a_i \geqslant$

$\geqslant \sum_{i=1}^{\alpha} b_i$ for all α. This is fairly clear: a priori we have $a_1 \geqslant b_1$; for $a_1 > b_1$, degenerate $S_{a_1,\ldots,a_{k+1}}$ to $S_{a_1-1,a_2+1,\ldots,a_{k+1}}$ and re-order the indices if necessary. This preserves the inequalities and reduces $a_1 - b_1$, so after a few steps we have $a_1 = b_1$; continuing in this way we arrive at $S_{b_1,\ldots,b_{k+1}}$.

To prove the converse, we first have to establish one point: that *no plane V of dimension less than $\sum_{i=1}^{\alpha} a_i + \alpha - 1$ in $\boldsymbol{P}^n$ meets every k-plane of a variety $S = S_{a_1,\ldots,a_{k+1}}$ in an $(\alpha-1)$-plane*. This may be seen by induction: projecting from a point of $V \cap S$ gives a $\left(\sum_{i=1}^{\alpha} a_i + \alpha - 2\right)$-plane in $\boldsymbol{P}^{n-1}$ meeting every k-plane of a variety $S_{a_1,\ldots,a_j-1,\ldots,a_{k+1}}$ in an $(\alpha-1)$-plane. It follows also—in case $a_{\alpha+1} > a_\alpha$—that *the linear span of S_α is the unique $\left(\sum a_i + \alpha - 1\right)$-plane in $\boldsymbol{P}^n$ meeting every k-plane of S in an $(\alpha-1)$-plane*: if any such $\left(\sum_{i=1}^{\alpha} a_i + \alpha - 1\right)$-plane contained a point p of $S_{\alpha+1} - S_\alpha$, projection from p onto a hyperplane would yield a $\left(\sum_{i=1}^{\alpha} a_i + \alpha - 2\right)$-plane meeting every k-plane of a variety $S_{a_1,\ldots,a_\alpha,a_{\alpha+1}-1,\ldots,a_{k+1}}$ in an $(\alpha-1)$-plane. Now, if we have a family of varieties S_t with $S_t \cong S_{a_1,\ldots,a_{k+1}}$ for $t \neq 0$, $S_0 \cong S_{b_1,\ldots,b_{k+1}}$, $a_{\alpha+1} > a_\alpha$, then the limiting position of the $\left(\sum_{i=1}^{\alpha} a_i + \alpha - 1\right)$-planes $V_t = \overline{S_{t_\alpha}}$ is a $\left(\sum_{i=1}^{\alpha} a_i + \alpha - 1\right)$-plane meeting every k-plane of the variety S_0 in an $(\alpha-1)$-plane; this shows that $\sum_{i=1}^{\alpha} b_i \leqslant \sum_{i=1}^{\alpha} a_i$ if $a_{\alpha+1} > a_\alpha$. Finally, it follows that $\sum_{i=1}^{\alpha} b_i \leqslant \sum_{i=1}^{\alpha} a_i$ for all α: if α_0 were the first number of which this inequality failed to hold, we would have $a_{\alpha+1} = a_\alpha$ and $b_\alpha > a_\alpha$, so $\sum_{i=1}^{\alpha_0+1} b_i > \sum_{i=1}^{\alpha_0+1} a_i$, hence $a_{\alpha+2} = a_{\alpha+1}$, $b_{\alpha+1} > a_{\alpha+2}$ and $\sum_{i=j}^{\alpha_0+2} b_i > \sum_{i=1}^{\alpha_0+2} a_i$, and so on until we arrive at $n - k = \sum_{i=1}^{k+1} b_i > \sum_{i=1}^{k+1} a_i = n - k$.

A note: the degeneration of the surface $S = S_{a,b}$ into the surface $S_{a-1,b+1}$ may be performed geometrically as follows. Let E_i be the rational normal curves of degree a and b on S, $\varphi_i\colon \boldsymbol{P}^1 \to E_i$ the isomorphisms used in the construction of S. If we take a generic hyperplane H containing only $a-1$ lines of S, the residual intersection of H with S will be a rational normal curve E_2' of degree $b+1$, meeting E_1 once at a point p. Every line on S will meet E_2' once, and so S may be realized as the union of lines joining corresponding points of E_1 and E_2' (or, more precisely, the closure of the union of lines joining points $q \neq p \in E_1$ with corresponding points of E_2'.) Now, let $\{E_1(\lambda)\}_{\lambda \in \Delta}$ be a family of rational normal curves lying in the span $W_1 \cong \boldsymbol{P}^a$ and passing through p degenerating to the sum of a rational normal curve $E_1' = E_1(0)$ of degree $a-1$ and a line l; let

$$\varphi_\lambda\colon \boldsymbol{P}^1 \to E_1(\lambda)$$

be a family of maps with $\varphi_\lambda(\varphi_1^{-1}(p)) = p$. (Explicitly, if with suitable coordinates t on $\boldsymbol{P}^1$ and on $W_1 \cong \boldsymbol{P}^a$, we have

$$\varphi_1(t) = [1, t, t^2, \ldots, t^a], \qquad p = \varphi_1(\infty) = [0, \ldots, 0, 1]$$

we may take

$$\varphi_\lambda(t) = [1, t, t^2, \ldots, t^{a-1}, \lambda t^a]$$

and $E_1(\lambda)$ the image $\varphi_\lambda(\boldsymbol{P}^1)$; $E_1(0)$ will be the image $\varphi_0(\boldsymbol{P}^1)$ plus the line $\{[0, \ldots, 0, \mu_1, \mu_2]\}$.) Let

$$S(\lambda) = \bigcup_{t \in \boldsymbol{P}^1} \overline{\varphi_\lambda(t), \varphi_2(t)};$$

for $\lambda \neq 0$, then, $S(\lambda)$ will be isomorphic to $S_{a,b}$ while $S(0)$ will be the surface $S_{a-1,b+1}$ obtained by joining the curves E_1' and E_2', the line split off as $E_1(\lambda)$ degenerates to E_1' appearing as a fiber of the ruled surface $S(0)$.

All degenerations of varieties may be seen from this: to degenerate $S = S_{a_1,\ldots,a_{k+1}}$ into $S_{a_1,\ldots,a_{i-1}-1,a_i+1,\ldots,a_{k+1}}$, we realize S as the join of the surface S_{a_{i-1},a_i} with the variety $S_{a_1,\ldots,a_{i-2},a_{i+1},\ldots,a_{k+1}}$ and, holding the latter fixed, degenerate the former into $S_{a_{i-1}-1,a_i+1}$.

Note finally that since every family $\{E_t \to \boldsymbol{P}^1\}_{t\in\Delta}$ of vector bundles of rank $k+1$ and chern class $-n+k$ on $\boldsymbol{P}^1$ generates a family of varieties S_t of minimal degree; and conversely any family of such varieties S_t lifts to a family of vector bundles, the result applies as well to degeneration of vector bundles of fixed chern class on $\boldsymbol{P}^1$.

4. – Castelnuovo varieties.

We now wish to consider each of the types of minimal varieties constructed in the last section, and show that they do indeed contain Castelnuovo varieties.

We start with a rational normal scroll $S = S_{a_1,\ldots,a_{k+1}}$.

The group of divisors on S is freely generated by the hyperplane section H and a k-plane $W \subset S$; we first compute

LEMMA. *The canonical class K_S is*

$$K_S \sim -(k+1)H + (n-k-2)W.$$

PROOF. We start with the case $k = 1$, that is, with a ruled surface $S = S_{a_1,a_2}$. The intersection pairing in S is given by

$$H^2 = n - k = n - 1\,, \quad H\cdot W = 1\,, \quad W\cdot W = 0;$$

if we write

$$K_S = \alpha H + \beta W$$

then we can solve the equations

$$0 = \pi(W) = \frac{W\cdot W + K\cdot W}{2} + 1 = \frac{\alpha}{2} + 1$$

and

$$0 = \pi(H) = \frac{H\cdot H + K\cdot H}{2} = \frac{(n+1) + \alpha(n-1) + \beta}{2} + 1$$

to find that $\alpha = -2$, $\beta = n - 3$.

Now, if S is any minimal variety of the form $S_{a_1,\ldots,a_{k+1}}$, the hyperplane section $S' = S\cdot H$ of S is likewise; moreover the divisors H and W on S restrict to H and W respectively on S'. By adjunction,

$$K_{S'} = (K_S + H)|_{S'}$$

but by an induction hypothesis

$$\begin{aligned} K_{S'} &= -kH + \big((n-1) - (k-1) - 2\big)W \\ &= -kH(n-k-2)W \end{aligned}$$

and it follows that $K_S = -(k+1)H + (n-k-2)W$.

We claim next that

A Castelnuovo variety $V^k \subset S$ of degree d must have class either $(M+1)H - (n-k-1-\varepsilon)W$ or $M\cdot H + W$ (in case $\varepsilon = 0$), *where* $M = [(d-1)/(n-k)]$ *and* $\varepsilon = d - 1 - M(n-k)$.

PROOF. We may a priori write

$$V \sim (M + 1 + a)H + \big(d - (M+1+a)(n-k)\big)W$$

for some a. Let C be a generic $(n-k+1)$-plane section of V, T the corresponding $(n-k+1)$-plane section of S. We have seen that if V is

Castelnuovo,

$$h^0(C, \Omega_C^1(-M+1)) = d - M(n-k) - 1 .$$

But now we have

$$K_T = -2H + (n-k-2)W$$

so

$$K_C = \big((M-1+a)H + (d-(M+a)(n-k)-2)W\big)|_C$$

and

$$\begin{aligned} K_C - (M-1)H &= \big(aH + (d-(M+a)(n-k)-2)W\big)|_C \\ &= \big(aH + (e-1-a(n-k))W\big)|_C \end{aligned}$$

where $e = d - 1 - M(n-k)$. Moreover, since $q(T) = 0$, the Poincaré residue map

$$H^0(T, \Omega_T^2(C)) \to H^0(C, \Omega_C^1)$$

is onto, and a fortiori the map

$$H^0(T, \Omega_T^2(C)(-M+1)) \to H^0(C, \Omega_C^1(-M+1))$$

is. Thus we must have

$$h^0\Big(T, \mathcal{O}_T\big(aH + (e-1-a(n-k))W\big)\Big) \geq e .$$

Let us restrict ourselves for the moment to the case $e \neq 0$, that is, $d-1 \not\equiv \not\equiv 0(n-k)$. We see immediately that a must be non-negative: otherwise the divisor $D = aH + (e-1-a(n-k))W$ would have negative intersection number with the fiber W of T, and so could not be effective. On the other hand, if a were strictly positive, D would have negative intersection with the smaller rational normal curve $E_1 = H - a_2 W$ on $T \cong S_{a_1,a_2}$; indeed, we would have then

$$\begin{aligned} |D| &= aE_1 + |(e-1-a(n-k-a_2))W| \\ &= aE_1 + |(e_1 - 1 - a\cdot a_1)W| \end{aligned}$$

and correspondingly $h^0(D) < e$ unless $a_1 = 0$ in which case $h^0(D) = e$. Finally, the case $a > 0$, $a_1 = 0$ can be eliminated out of hand: in this case, $C \sim (M+1+a)H + \big(d-(M+1+a)(n-k)\big)W$ would have negative intersection number with E_1, and so contain E_1 as a component. The argument in case $d \equiv 1(n-k)$ is the same: here we must have

$$\begin{aligned} n-k &= d-(M-1)(n-k)-1 \\ &= h^0(C, \Omega_C^1(-M+2)) \\ &\leqslant h^0\big(T, \mathcal{O}_T((a+1)H-(a(n-k)-1)W\big) \end{aligned}$$

which is satisfied if and only if $a = 0$ or -1.

Let us now check that an irreducible divisor

$$V \sim (M+1)H + \big(d-(M+1)(n-k)\big)W$$

on S, whose singularities impose no adjoint conditions, does indeed achieve our bound on p_g. We note first that since S is rational, $H^0(S, \Omega_S^{k+1}) = H^1(S, \Omega_S^{k+1}) = 0$ and so the Poincaré residue map

$$H^0(S, \tilde{V}_S^{k+1}(V)) \to H^0(V, \tilde{V}_k^V)$$

is an isomorphism. Thus

$$\begin{aligned} p_g(V) &= h^0(S, \Omega_S^{k+1}(V)) \\ &= h^0\Big(S, \mathcal{O}\big((M-k)H + (d-M(n-k)-2)W\big)\Big)\,. \end{aligned}$$

Now, to compute this number, we consider first sections of the line bundle $(M-k)H$ on the variety $S = \boldsymbol{P}(E) = \boldsymbol{P}(H^{-a_1} \oplus \ldots \oplus H^{-a_{k+1}})$. Such a section σ gives, on each fiber $\boldsymbol{P}(E)_t$ of the bundle $\boldsymbol{P}(E) \to \boldsymbol{P}^1$, a polynomial of degree $M-k$. If as before we take $Y_1, \ldots, Y_{k+1}$ to be homogeneous coordinates on the fiber $\boldsymbol{P}(E)_t$ corresponding to the decomposition $\boldsymbol{P}(E)_t^* = \boldsymbol{P}(E_t^*) = \boldsymbol{P}(H_t^{a_1} \oplus H_t^{a_2} \oplus \ldots \oplus H_t^{a_{k+1}})$, and write

$$\sigma|_{\boldsymbol{P}(E)_t} = \sum_{i_1+\ldots+i_{k+1}=M-k} \sigma_{i_1,\ldots,i_{k+1}}(t)\, Y_1^{i_1} \ldots Y_{k+1}^{i_{k+1}}$$

then, the coefficients $\sigma_{i_1,\ldots,i_{k+1}}(t)$ are sections of the bundle

$$(H^{a_1})^{\otimes i_1} \otimes (H^{a_2})^{\otimes a_2} \otimes \ldots \otimes (H^{a_{k+1}})^{\otimes i_{k+1}} = H^{\sum_\alpha a_\alpha i_\alpha}$$

conversely, any collection of sections

$$\Big\{\sigma_{i_1,\ldots,i_{k+1}} \in H^0\Big(\boldsymbol{P}^1, \mathcal{O}\big(\sum_\alpha a_\alpha i_\alpha\big)\Big)\Big\}_{\Sigma i_\alpha = m-k}$$

gives a section of $\mathcal{O}_S((M-k)H)$. In other words, we have an isomorphism

$$\begin{aligned} H^0(\boldsymbol{P}(E), \mathcal{O}(M-k)H) &\cong H^0(\boldsymbol{P}^1, \mathcal{O}\,\mathrm{Sym}_{(M-k)}(E^*)) \\ &\cong H^0\Big(\boldsymbol{P}^1, \mathcal{O}\Big(\sum_{\substack{J\subset\{1,\dots,k+1\} \\ \#J=M-k}} H^{a_{j_1}}\otimes\dots\otimes H^{a_{j_{M-k}}}\Big)\Big) \\ &\cong \sum_{\substack{J\subset\{1,\dots,k+i\} \\ \#J=M-k}} H^0(\boldsymbol{P}^1, \mathcal{O}(a_{j_1}+\dots+a_{j_{M-k}}))\,. \end{aligned}$$

Similarly, a section of $(M-k)H-eW$ is given by a section of $(M-k)H$ vanishing identically on chosen fibers $\boldsymbol{P}_{t_1}(E), \dots, \boldsymbol{P}(E)_{t_e}$, in turn given by a collection of coefficient functions $\sigma_{i_1,\dots,i_{k+1}}\in H^0\big(\boldsymbol{P}^2, \mathcal{O}\big(\sum_\alpha i_\alpha a_\alpha\big)\big)$, where all σ_I vanish at $t_1, \dots, t_e$—i.e., by a collection of sections

$$\sigma_{i_1,\dots,i_{k+1}}\in H^0\big(\boldsymbol{P}_1, \mathcal{O}\big(\textstyle\sum i_\alpha a_\alpha - e\big)\big)\,.$$

We have, then,

$$\begin{aligned} p_g(V) &= h^0\big(S, \mathcal{O}(M-k)H + (d-M(n-k)-1)\,W\big) \\ &= \sum_{i_1+\dots+i_{k+1}=a-k-1} h^0\big(\boldsymbol{P}^1, \mathcal{O}\big(\textstyle\sum i_\alpha a_\alpha + d - M(n-k)-2\big)\big) \\ &= \binom{M}{k}(d-M(n-k)-1) + \sum_{i_1+\dots+i_{k+1}=m-k}\ \sum_\alpha i_\alpha a_\alpha \\ &= \binom{M}{k}(d-M(n-k)-1) + \binom{M}{k+1}\cdot\sum a_\alpha \\ &= \binom{M}{k}(d-M(n-k)-1) + \binom{M}{k+1}(n-k) \end{aligned}$$

so V does indeed achieve the bound.

It remains to check that we can actually find, in the divisor class $|aH-bW|$ on a suitably chosen $S=S_{a_1,\dots,a_{k+1}}$ $(a\geqslant k+1, b\leqslant n-k-1)$, irreducible divisors whose singularities impose no adjoint conditions. For this purpose we will take $S=S_{a_1,\dots,a_{k+1}}$ the « generic » minimal variety, i.e.,

$$a_1=\dots=a_{k-m+1}=\left[\frac{n-k}{k+1}\right] m = n-k-(k+1)\left(\frac{n-k}{k+1}\right)$$
$$a_{k-m+2}=\dots=a_{k+1}=\left[\frac{n-k}{k+1}\right]+1\,.$$

Again, if we let $Y_1\dots Y_{k+1}$ be homogeneous co-ordinates on each fiber of $S\cong\boldsymbol{P}(E)$ corresponding to the decomposition $E^*=H^{a_1}\otimes\dots\otimes H^{a_{k+1}}$, then a section σ of $|aH - bW|$ is given by a polynomial

$$\sigma = \sum_{i_1+\ldots+i_{k+1}=a} \sigma_{i_1,\ldots,i_{k+1}}(t)\, Y_1^{i_1} \ldots Y_{k+1}^{i_{k+1}}$$

where the coefficient functions

$$\sigma_{i_1,\ldots,i_{k+1}} \in H^0\Big(\boldsymbol{P}^1, \mathcal{O}\Big(\sum_\alpha i_\alpha a_\alpha - b\Big)\Big)\,.$$

Now, for any multi-index $I = i_1, \ldots, i_{k+1}$ such that

$$(*)\qquad \begin{aligned} b &\leqslant \sum_\alpha i_\alpha a \\ &= a\cdot\left[\frac{n-k}{k+1}\right] + \sum_{\alpha \geqslant k-m+2} i_\alpha \end{aligned}$$

the value of the corresponding coefficient function σ_I may be prescribed at any given value t; thus the system $|aH - bW|$ cuts out on each k-plane $\boldsymbol{P}(E)_t$ of $S \cong \boldsymbol{P}(E)$ the subsystem of $|\mathcal{O}_{\boldsymbol{P}^k}(a)|$ generated by monomials Y^I satisfying $(*)$ above. In particular, in case

$$a\left[\frac{n-k}{k+1}\right] \geqslant b$$

—as will occur most often, inasmuch as $a \geqslant k+1$ and $b \leqslant n-k-1$—we see that the system $|aH - bW|$ has no base points on S. By Bertini, then, the generic element of $|aH - bW|$ will be smooth and hence irreducible: writing a reducible element V of $aH - bW$ as a sum $V = V' + V''$ we see that V' will intersect V'' in codimension 1.

In case $a[(n-k)/(k+1)] < b$, the linear system will have base locus (though as we shall see, if $a[(n-k)/(k+1)] \geqslant b - m$, the generic element of $|aH - bW|$ is still smooth): since the coefficient of any monomial having fewer than $b - a[(n-k)/(k+1)]$ factors among $Y_{k-m+2}, \ldots, Y_{k+1}$ must be zero, the generic divisor $V \in |aH - bW|$ will have multiplicity $b - a\cdot$ $\cdot[(n-k)/(k+1)]$ along the locus $Y_{k-m} = \ldots = Y_{k+1} = 0$ in S, that is, the subvariety S_{k-m+1} spanned by the curves $E_1, \ldots, E_{k-m+1}$. We consider in this case the blow-up $\tilde{S}$ of S along the subvariety S_{k-m+1} (note that the case $m = 1$ concerns us only if either $[(n-k)/(k+1)] = 0$—that is, $n - k = 1$ and we are dealing with a hypersurface—or $a = k+1$—in which case our bound is zero). In the fiber $F_p \cong \boldsymbol{P}^{m-1}$ of the exceptional divisor F of the blow-up $\tilde{S} \to S$ over a point $p \in S'$, the proper transform of a divisor

$V \in |aH - bW|$ given by

$$\sigma = \sum_{i_1 + \ldots + i_{k+1} = v} \sigma_{i_1, \ldots, i_{k+1}}(t) \cdot Y_1^{i_1} \ldots Y_{k+1}^{i_{k+1}}$$

is given, in terms of homogeneous co-ordinates $\tilde{Y}_1, \ldots, \tilde{Y}_m$, by

$$\tilde{V} = \sum_{i_{k-m+2} + \ldots + i_{k+1} = b - a[(n-k)/(k+1)]} \sigma_{i_{k-m+2}, \ldots, i_{k+1}}(t) \cdot \tilde{Y}_{k-m+2}^{i_{k-m+2}} \ldots \tilde{Y}_{k+1}^{i_{k+1}} .$$

The proper transform $|a \cdot \pi^* H - b\pi^* W - (b - a[(n H k)/(k+1)])F|$ of the linear system $|aH - bW|$ thus cuts out on each fiber $F_p \cong \pi^{-1}(p) \cong \boldsymbol{P}^{m-1}$ of the exceptional divisor the complete system $\mathcal{O}_{F_p}(b - [(n-k)/(k+1)])$; and so has no base points on $\tilde{S}$. It follows that the proper transform $\tilde{V}$ of the generic element V of $|aH - bW|$ is smooth (and hence, as before, irreducible), and we may use the Poincaré residue formula on $\tilde{S}$ to compute $p_g(V) = p_g(\tilde{V})$. We have

$$K_{\tilde{S}} = \pi^* K_S + (m H 1)F$$

and

$$\tilde{V} = \pi^* V - \left(b - a\left[\frac{n-k}{k+1}\right]\right)F$$

so

$$K_{\tilde{S}} + \tilde{V} = \pi^*(K_S + V) + \left(m - b + a\left[\frac{n-k}{k+1}\right]\right)F .$$

But $b \leqslant n - k - 1$, and since $a \geqslant k+1$,

$$a\left[\frac{n-k}{k+1}\right] \geqslant n - k - m ,$$

so we have

$$m - b + a\left[\frac{n-k}{k+1}\right] \geqslant 0 .$$

Thus

$$p_g(\tilde{V}) = h_0(\tilde{S}, \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + \tilde{V})) = h^0(S, \mathcal{O}_S(K_S + V))$$

which, by our calculation, achieves the bound on p_g.

Finally, we wish to check the two exceptional cases: quadric hypersurfaces and cones over the Veronese surface. As for quadrics, a quadric of rank 4 or less (that is, a cone over a quadric in $\boldsymbol{P}^3$) is of the form $S_{1,1,0,\ldots,0}$, $S_{2,0,0,\ldots,0}$ and so covered by our first computation.

Consider now a quadric Q of rank $r \geqslant 5$ in $\boldsymbol{P}^n$. The desingularization $\tilde{Q}$ of Q—obtained by taking the proper transform of Q in the blow-up $\tilde{\boldsymbol{P}}^n$ of $\boldsymbol{P}^n$ along the vertex $\boldsymbol{P}^{n-r}$ of Q—is a $\boldsymbol{P}^{n-r-1}$-bundle over a smooth quadric Q' of dimension $r-2$: in fact, since $r-2=3$, by Lefschetz the second cohomology $H^2(Q')$ is generated by the class H of a hyperplane section, and

$$\tilde{Q} = \boldsymbol{P}(E)$$

where

$$E = [H]^{-1} \oplus \boldsymbol{C} \oplus \ldots \oplus \boldsymbol{C} \to Q' ,$$

the map $\tilde{Q} \to Q \subset \boldsymbol{P}^n$ being given by the dual of the tautological bundle on $\boldsymbol{P}(E)$, i.e., by the linear system

$$\{(\sigma)\}_{\sigma \in H^0(Q', \mathcal{O}(E^*))} .$$

The group of divisors on $\tilde{Q}$ has two generators: the inverse image H' of a hyperplane section of Q' (this is the proper transform of the intersection of Q with a hyperplane containing the vertex of Q) and the hyperplane class. The difference $H - H'$ is just the exceptional divisor F of the blow-up $\tilde{Q} \to Q$, that is, the image in $\boldsymbol{P}(E)$ of the sub-bundle $C \oplus \ldots \oplus \boldsymbol{C} \subset E$; in fact it is the basis H and F we will use.

To find the canonical bundle $K_{\tilde{Q}}$, let $\boldsymbol{P}^n$ be the blow-up of $\boldsymbol{P}^n$ along the vertex $V \cong \boldsymbol{P}^{n-r}$ of Q. Then

$$\begin{aligned} K: &= K_{\boldsymbol{P}^n} + \tilde{Q}|_{\tilde{Q}} \\ &= H\,(n+1)H + (r-1)F + 2H - 2F \\ &= -(n-1)H + (r-3)F . \end{aligned}$$

Now, let $\tilde{V}$ be a smooth divisor on $\tilde{Q}$; if the degree of the image V of $\tilde{V}$ in $\boldsymbol{P}^n$ is d, we may write

$$\tilde{V} \sim \frac{d}{2} H - bF$$

so

$$K_{\tilde{Q}} + \tilde{V} \sim \left(\frac{d}{2} - n + 1\right) H + (-b + r - 3) F$$

i.e., as long as $b \leqslant r-3$, the canonical series on $\tilde{V}$ is simply cut out by hypersurfaces of degree $d/2 - n + 1$ in $\boldsymbol{P}^n$. There are $\binom{d/2+1}{n}$ such hypersurfaces; and since any such hypersurface contains V if and only if it contains Q, $\binom{d/2-1}{n}$ of them pass through V. The geometric genus of V is thus

$$\binom{d/2+1}{n}-\binom{d/2-1}{n}=\binom{M+2}{n}-\binom{M}{n}\left(M=\frac{d-1}{n-k}=\frac{d}{2}-1\right)$$

$$=\frac{(M+2)(M+1)\dots(M-n+3)-M(M-1)\cdot\dots\cdot(M-n+1)}{n!}$$

$$=\frac{M(M-1)\cdot\dots\cdot(M-n+3)}{n!}\big((M+2)(M+1)-(M-n+2)(M-n+1)\big)$$

$$=\frac{M(M-1)\cdot\dots\cdot(M-n+3)}{n!}\big(2(M-n+2)n+n(n-1)\big)$$

$$=2\binom{M}{n-1}+\binom{M}{n-2}$$

which equals our bound. Thus we see that *the generic complete intersection of Q with a hypersurface passing $r-3$ or fewer times through the vertex of Q is a Castelnuovo variety.*

Lastly, we consider divisors on a cone $S\subset \boldsymbol{P}^n$ over the Veronese surface. Again, the desingularization $\tilde{S}$ of such a cone S is obtained by blowing up the vertex of the cone once; again, $\tilde{S}$ is a $\boldsymbol{P}^{n-5}$-bundle over $\boldsymbol{P}^2$. Letting $H_{\boldsymbol{P}^n}$ be the hyperplane class on $\boldsymbol{P}^2$, we see that

$$\tilde{S}=\boldsymbol{P}(E)$$

where

$$E=H_{\boldsymbol{P}^n}^{-2}\oplus\boldsymbol{C}\oplus\dots\oplus\boldsymbol{C}\to\boldsymbol{P}^2;$$

again, the map $\tilde{S}\xrightarrow{\pi}S\subset\boldsymbol{P}^n$ is given by the dual of the tautological bundle on $\boldsymbol{P}(E)$. The divisors on $\tilde{S}$ are generated by two classes: the hyperplane section of S, and the inverse image L of a line in $\boldsymbol{P}^2$, that is, one half the proper transform of a hyperplane section of S containing the vertex. (Note that the proper transforms in $\tilde{S}$ of complete intersections with S have index 2 in the group of divisors on $\tilde{S}$, being the subgroup generated by H and $2L$.)

To compute the canonical class K_S of $\tilde{S}$, we may argue as follows: the divisor L on $S\to\boldsymbol{P}^n$ is a variety $S_{2,0,0,\dots,0}$, spanning an $(n-3)$-plane; the class of L restricted to $L\cong S_{2,0,\dots,0}$ is the class W of an $(n-5)$-plane in $S_{2,0,\dots,0}$. Now we have seen that

$$(K_{\tilde{S}}+L)|_L=K_L=-(n-4)H$$

so

$$K_{\tilde{S}} = -(n-4)H - L\,.$$

Now suppose $\tilde{V}$ is an irreducible divisor whose image $V = \pi(\tilde{V})$ in $\boldsymbol{P}^n$ has degree d. Inasmuch as H has degree 4 and L degree 2 in $\boldsymbol{P}^n$, we can then write

$$\tilde{V} \sim aH + \frac{d-4a}{2} L\,.$$

Note that both coefficients must be non-negative; otherwise $\tilde{V}$ would contain the exceptional divisor of the blow-up $\pi\colon \tilde{S} \to S$ as a component. Following the same argument as in the main case, we see that

$$\begin{aligned}
p_g(\tilde{V}) &= h^0\left(\tilde{S}, \mathcal{O}_{\tilde{S}}\left((a-n+4)H + \left(\frac{d}{2} - 2a - 1\right)L\right)\right) \\
&= h^0\left(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(\mathrm{Sym}^{a-n+4}E^*)\left(\frac{d}{2} - 2a - 1\right)\right) \\
&= \sum_{i_1+\ldots+i_{n-4}=a-n+4} h^0\left(\boldsymbol{P}^2, \mathcal{O}\left(2i_1 + \frac{d}{2} - 2a - 1\right)\right) \\
&= \sum_{i_1+\ldots+i_{n-4}=a-n+4} \frac{(2i_1 + d/2 - 2a)(2i_1 + d2 - 2/a + 1)}{2} \\
&= 4 \sum_{i_1+\ldots+i_{n-4}=a-n+4} \frac{i_1(i_1-1)}{2} + (3 + d - 4a) \sum_{i_1+\ldots+i_{n-4}=a-n+4} i_1 \\
&\qquad + \frac{1}{2} \sum_{i_1+\ldots+i_{n-4}=a-n+4} \left(\frac{d}{2} - 2a\right)\left(\frac{d}{2} - 2a + 1\right) \\
&= 4\binom{a-1}{n-2} + (3 + d - 4a)\binom{a-1}{n-3} + \frac{1}{8}(d-4a)(d-4a+2)\binom{a-1}{n-4}.
\end{aligned}$$

Now, in case d is divisible by 4 and we take $a = d/4$—that is, take V to be the complete intersection of S with a hypersurface—this achieves one bound; for $a < d/4$ it falls short. On the other hand, for $d \equiv 2 \bmod 4$, this number achieves the bound only when $a = (d+2)/4$—which is not the class of an irreducible divisor on $\tilde{S}$. Thus, a divisor V on S is a Castelnuovo variety if and only if it is a complete intersection of S with a hypersurface not passing through the vertex of S, whose singularities impose no adjoint conditions. Since the linear system all on $\tilde{S}$ has no base points, such divisors clearly exist.

To sum up, then,

The greatest possible geometric genus $p_g(V)$ of a nondegenerate irreducible *variety $V \subset \boldsymbol{P}^n$ of dimension k and degree d is*

$$\binom{M}{k+1}(n-k)+\binom{M}{k}\varepsilon$$

where $M=(d-1)/(n-k)$, $\varepsilon=d-1-M(n-k)$. A variety which achieves this bound is either

i) *residual to $n-k-1-\varepsilon$ k-planes in the complete intersection of a rational normal scroll S with a hypersurface of degree $M+1$, for any d, n and k;*

i′) *homologous on S to a k-plane plus a complete intersection of S with a hypersurface of degree m;*

ii) *the complete intersection of a quadric $Q \subset \boldsymbol{P}^n$ of rank $r \geqslant 5$ with a hypersurface passing $r-3$ times or fewer through the vertex of Q, in case $k=n-2$; or*

iii) *the complete intersection of a cone S over the Veronese surface in $\boldsymbol{P}^5$ with a hypersurface not containing the vertex of the case S, in case $k=n-4$.*

Finally, we can now see that our inequality on $p_g(V)$ and $C_1^k(V)$ for a variety V of dimension k whose canonical bundle is birationally very ample is sharp; for any n and k we may take a generic divisor

$$V \sim (k+2)H-(n-k-2)W$$

on a minimal surface $S_{a_1,\ldots,a_{k+1}} \subset \boldsymbol{P}^n$.

5. – Some properties of Castelnuovo varieties.

To conclude, we wish to list a few properties of Castelnuovo varieties, for the most part clear from our description of Castelnuovo varieties as divisors on minimal varieties.

The first observation is that

The generic hyperplane section of a Castelnuovo variety V is again a Castelnuovo variety.

This is easy: if V sits on a rational normal scroll S then we have seen that V has class

$$V \sim (M+1)H+\big(d-(M+1)(n-k)\big)W$$

and that the singularities of V impose no adjoint conditions; the generic hyperplane section $V'=V\cdot H$ then sits on the minimal variety $S'=S\cdot H$, again has class $V' \sim (M+1)H + (d-(M+1)(n-k))W$ on S', and its singularities impose no adjoint conditions. Likewise, if V is the complete intersection of a quadric Q of rank r with a hypersurface passing no more than $r-3$ times through the vertex of Q, then so is $V' = V \cdot H$; and if V is the complete intersection of a cone X over a Veronese surface with a hypersurface not containing the vertex of X, then V' is as well.

In particular, we see that the generic $(n-k+1)$-plane section of a Castelnuovo variety V is a Castelnuovo curve C; as we saw in section 1, C must be arithmetically normal and so

A Castelnuovo variety V is arithmetically normal.

We note that we may expect a Castelnuovo variety $V \subset S = S_{a_1,\dots,a_{k+1}}$ to contain a number of subvarieties which are also Castelnuovo: in addition to the hyperplane sections of V, the intersection of V with any of the subvarieties $S_{a_{i_1},\dots,a_{i_m}} \subset S$ are Castelnuovo if they are not too singular, and of course the intersection of V with the k-planes of S are hypersurfaces in $\boldsymbol{P}^k$, trivially Castelnuovo varieties.

Next, we observe that since the linear system $|\mathcal{O}_S(V)|$ of a Castelnuovo variety V on the minimal variety containing it (or, more properly, on the blow-up of S along the base locus of $|\mathcal{O}_S(V)|$, as described in the previous section) has no base locus and positive $(k+1)$-fold self-intersection,

$$H^i(S, \mathcal{O}(-V)) = 0 \qquad \text{for } i = 0, \dots, k .$$

Accordingly, from the sequence

$$0 \to \mathcal{O}_S(-V) \to \mathcal{O}_S \to \mathcal{O}_V \to 0$$

we see that

$$H^i(V, \mathcal{O}_V) = H^i(S, \mathcal{O}_S) = 0 \qquad \text{for } i = 1, \dots, k-1 .$$

On the other hand, since we know that the sequence

$$0 \to \Omega^k_V(-l) \to \Omega^k_V(-l+1) \to \Omega^{k-1}_{V'}(-l) \to 0$$

($V' = V \cdot H$ the generic hyperplane section of V) is exact on global sections for $l \geqslant 1$, it follows by an induction that

$$H^i(V, \mathcal{O}^k_V(-l)) = H^{k-1}(V, \mathcal{O}^k_V(-l)) = 0$$

for all l, $i = 1, \dots, k-1$.

Finally, in case V is smooth, we have the following characterization:

A smooth variety $V^k \subset \mathbf{P}^n$ is Castelnuovo if and only if its generic hyperplane section $V' = V \cdot H$ is.

PROOF. Suppose V' is Castelnuovo. We go back to the sequences

$$0 \to \Omega_V^k(-l) \to \Omega_V^k(-l+1) \to \Omega_{V'}^{k-1}(-l) \to 0;$$

we note first that the coboundary map

$$H_1(V', \Omega_{V'}^{k-1}(-l)) \to H^2(V\text{m}\, \Omega_V^k(-l))$$

is injective: in case $k > 2$, this is true simply because $H^1(V', \Omega_{V'}^{k-1}(-l)) = 0$; if $k = 2$, the map is dual to the restriction map

$$H^0(V, \mathcal{O}_V(l)) \to H^0(V', \mathcal{O}_{V'}(l))$$

which is surjective since V' is arithmetically normal. We see that $H^1(V, \Omega_V^k(-l))$ surjects onto $H^1(V, \Omega_V^k(-l+1))$ for all l; since V is smooth, however, $H^1(V, \Omega_V^k(-l)) = 0$ for $l \ll 0$ and so $H^1(V, \tilde{V}_V^k(l)) = 0$ $\forall l$. It follows then that

$$H^0(V, \Omega_V^k(-l+1)) - H^0(V, \Omega_V^k(-l)) = H^0(V', \Omega_{V'}^{k-1}(-l)) = \\ = \binom{M-l}{k-1}\varepsilon + \binom{M-l}{k}(n-k)$$

and hence

$$h^0(V, \Omega_V^k) = \binom{M}{k+1}(n-k) + \binom{M}{k}\varepsilon$$

so V is Castelnuovo.

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Mathematical Department
Brown University
Providence, R.I. 02912