A boundedness result for the solutions of certain third order differential equations (*).

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Summary. - *In this paper a piece-wise linear extension of the usual Liapunov type function is constructed and used to investigate the equation*

$$\dddot{x} + a\ddot{x} + g(x)\dot{x} + h(x) = p(t).$$

1. - Introduction.

The differential equations considered here are of the form

$$\dddot{x} + a\ddot{x} + g(x)\dot{x} + h(x) = p(t) \qquad \left(\dot{x} = \frac{dx}{dt}\right) \tag{1.1}$$

where a is a positive constant, g, h and p are real valued functions. All solutions are assumed to be real, and it will also be assumed that h is differentiable and that g, h' and p are continuous for all x and t.

This equation has been studied by many authors, and in particular an excellent account of may of these results regarding both stability and boundedness can be found in [6, Chap. IV].

In [1], Ezeilo constructed an interesting extension of a Liapunov function to establish conditions under which all solutions of (1.1) will be uniform-ultimately bounded. It was shown that if $a > 0$ and if

$$h(x)\,\mathrm{sgn}\,x \to +\infty \quad \text{as} \quad |x| \to \infty,$$

$$g(x) \geq b > 0 \quad \text{and} \quad h'(x) \leq c \quad \text{for} \quad |x| \geq K \quad \text{where} \quad ab > c > 0,$$

and either $|p(t)| \leq P_0$ or $|\int_0^t p(s)ds| \leq P$ for all $t \geq 0$, then the solutions of (1.1) are uniform-ultimately bounded.

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In [7], MÜLLER showed, by examining the system

$$\dot{x} = y - ax + P_1(x, y, z, t) \qquad \dot{y} = x - G(x) + P_2(x, y, z, t)$$

$$\dot{z} = -h(x) + P_3(x, y, z, t) \quad \text{where} \quad G(x) = \int_0^x g(u)du,$$

that if

$$\frac{G(x)}{x} > b > 0, \qquad \frac{aG(x)}{x} > \frac{h(x)}{x} > 0 \;\; x \neq 0, \qquad -c_0 \leq h'(x) \leq ab \; (c_0 > 0)$$

and $|P_i(x, y, z, t)| \leq P$ $(i = 1, 2, 3)$ then the solutions of (1.1) are uniform-ultimately provided

$$h(x)(G(x) - bx) > CP\,|G(x)| \quad \text{for} \quad |x| \geq K$$

where C is a sufficiently large positive constant. In both of these results there is a requirement that $h(x)$ sgn x become « sufficiently large » for large x.

For the case $g(x) \equiv b > 0$, VORÁČEK [8] has shown that this requirement on h is not necessary to obtain boundedness for the solutions of equation (1.1), although it is required there that $h(x)$ be bounded for all x. To obtain uniform ultimate boundedness for the solutions of (1.1) he again requires that $h(x)$ sgn x become « sufficiently large » for large x.

In this paper attention will be restricted to those functions $p(t)$ for which $\int_0^t p(s)ds$ is bounded for all $t \geq 0$. The LIAPUNOV function used in [2] by this author to investigate the asymptotic behavior of solutions of (1.1) is extended by the addition of a piece-wise linear function and used to show that the requirements needed on h in [1], [7] and [8] to obtain uniform ultimate boundedness for the solutions of (1.1) can be replaced by the condition

$$h(x) \operatorname{sgn} x \geq \eta > 0 \qquad |x| \geq K$$

where the choice of K is arbitrary and η is any constant satisfying $\eta > \frac{c}{2a}$ where $h'(x) \leq c$ for all x.
The following results will be established:

THEOREM. - Suppose there exist positive constants b, c, η, K and P_0 such that:

(1.2) (i) $\frac{G(x)}{x} \geq b \qquad |x| \geq K \qquad G(x) = \int_0^x g(u)du$

(1.3) (ii) $h(x) \operatorname{sgn} x \geq \eta \qquad |x| \geq K$

(1.4) (iii) $h'(x) \leq c$ for all x where $ab > c$ and $\eta > \frac{c}{2a}$

(1.5) (iv) $\left| \int_0^t p(s)ds \right| \leq P_0$ for all t,

then there exists a constant B, dependent only on equation (1.1), such that every solution $x = x(t)$ of (1.1) satisfies

$$|x(t)| + |\dot{x}(t)| + |\ddot{x}(t)| \leq B \tag{1.6}$$

for all sufficiently large t.

If we set

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z - ay - G(x) + P(t) \\ \dot{z} &= -h(x) \end{aligned} \tag{1.7}$$

then in the terminology of PLISS [3] the following is an immediate consequence of this theorem.

COROLLARY. - If there is a positive constant ω such that $P(t + \omega) = P(t)$ for all t and if conditions (1.2) - (1.5) are satisfied, then the system (1.7) is dissipative.

2. Consider a system of differential equation

$$\frac{dx}{dt} = F(t, x) \tag{2.1}$$

where x is an n-vector and $F(t, x)$ is continuous on $[0, \infty) x R^n$.

DEFINITION [4]. - The solutions of (3.1) are uniform-ultimately bounded for bound B if they are uniform-bounded and if there exists a $B > 0$ and a $T > 0$ such that for every solution $x(t, x_0, t_0)$ of (2.1), $\|x(t, x_0, t_0)\| < B$ for all $t \geq t_0 + T$, where B is independent of the particular solution while T may depend on each solution.

The following will be used in establishing our result.

LEMMA 2.1 [4]. - Suppose that there exists a LIAPUNOV function $V(t, x)$ defined on $0 \leq t < \infty$, $\|x\| \geq H$, where H may be large, which satisfies the following conditions:

(i) $a(\|x\|) \leq V(t, x) \leq b(\|x\|)$, where $a(\tau) \in CI$ (i.e. continuous and increasing), $a(\tau) \to \infty$ as $\tau \to \infty$ and $b(\tau) \in CI$,

(ii) $\dot{V}_{(2.1)}(t, x) \leq -c(\|x\|)$, where $c(\tau)$ is positive and continuous.

Then the solutions of (2.1) are uniform-ultimately bound.

It is assumed here that $V(t, x)$ is continuous in t and satisfies a local LIPSCHITZ condition with respect to x and that $\dot{V}_{(2.1)}$ is defined as

$$\dot{V}_{(2.1)}(t, x) = \overline{\lim_{h \to 0}} \frac{1}{h} \{ V[t+h, x+hF(t, x)] - V(t, x) \}.$$

3. Equation (1.1) is equivalent to the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z - ay - G(x) + P(t) \qquad (3.1) \\ \dot{z} &= -h(x) \end{aligned}$$

where $P(t) = \int_0^t p(s)ds$.

Let β be a constant such that $b > \beta > \frac{c}{a}$ and $\beta < 2\eta$. Such a constant exists since $0 < c < ab$ and $\eta > \frac{c}{2a}$. Define the function $V_1 = V_1(x, y, z)$ as

$$2V_1 = 2a \int_0^x h(u)du + 2\beta \int_0^x G(u)du + \beta y^2 + z^2 + 2h(x)y - 2\beta xz,$$

and define $V_2 = V_2(x, y, z)$ as

$$V_2 = \begin{cases} 0 & (x, y, z) \in R_1 = \{ y \geq M \} \\ -\varepsilon y + \varepsilon M & (x, y, z) \in R_2 = \{ |y| \leq M, z \geq N \} \\ 2\varepsilon M & (x, y, z) \in R_3 = \{ y \leq -M, z \geq N \} \\ \frac{2\varepsilon M}{N} z & (x, y, z) \in R_4 = \{ y \leq -M, |z| \leq N \} \\ -2\varepsilon M & (x, y, z) \in R_5 = \{ y \leq -M, z \leq -N \} \\ \varepsilon y - \varepsilon M & (x, y, z) \in R_6 = \{ |y| \leq M, z \leq -N \} \\ 0 & (x, y, z) \in R_7 = \left\{ 0 \leq y \leq M, |z| \leq \frac{N}{M} y \right\} \\ -\varepsilon y + \varepsilon \frac{M}{N} z & (x, y, z) \in R_8 = \left\{ 0 \leq z \leq N, |y| \leq \frac{M}{N} z \right\} \\ \frac{2\varepsilon M}{N} z & (x, y, z) \in R_9 = \left\{ -M \leq y \leq 0, |z| \leq \frac{-N}{M} y \right\} \\ \varepsilon y + \frac{\varepsilon M}{N} z & (x, y, z) \in R_{10} = \left\{ -N \leq z \leq 0, |y| \leq \frac{-M}{N} z \right\} \end{cases}$$

where $\varepsilon = 2\eta - \beta > 0$ and where M and N, $2aM \leq N$, are positive constants suitably chosen to satisfy Lemmas 3.1 and 3.2.

The main tool in the proof is the function $V = V(x, y, z)$ defined by $2V = 2V_1 + 2V_2$. Before proceeding with the proof of our result, it will be convenient to make a few observations about conditions (i)–(iii) and about $V(x, y, z)$

If $\delta = a\beta - c$, then $\delta > 0$ since $b > \beta > \frac{c}{a}$ and from (1.4) we have

$$h'(x) \leq a\beta - \delta \quad \text{for all} \quad x. \tag{3.2}$$

And since $h(x) = \int_0^x h'(x)dx + h(0)$ it follows that there is $K > 0$ such that

$$\eta \leq h(x) \operatorname{sgn} x \leq a\beta x \operatorname{sgn} x \quad \text{for} \quad |x| \geq K. \tag{3.3}$$

Combining (1.2), (3.2) and (3.3) we have the following set of inequalities which will be used throughout the proof.

$$\begin{cases} \dfrac{G(x)}{x} \geq \beta + \varepsilon & |x| \geq K \\ \dfrac{\beta + \varepsilon}{2} \leq h(x) \operatorname{sgn} x \leq a\beta x \operatorname{sgn} x & |x| \geq K \\ h'(x) \leq a\beta - \delta \quad \text{for all} \quad x. \end{cases} \tag{3.4}$$

Since V_1 has continuous first partial derivatives with respect to each variable, it satisfies a local LIPSCHITZ condition with respect to the vector (x, y, z). It follows that V satisfies a local LIPSCHITZ condition since V_2 is either linear or constant in all regions of its definition and since V is clearly continuous. The partial derivatives $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$, and $\frac{\partial V}{\partial z}$ exist and are continuous for all values of x, y and z except along the planes:

$$P_1 : y = \pm M, \;\; P_2 : y = \pm N, \; y \leq M, \;\; P_3 : y = \pm \frac{M}{N} z, \; |z| < N.$$

Along these planes the upper and lower partial derivatives all exist and as a result $\dot{V}$ exist for all x, y, z and t. (See [5]).

LEMMA 3.1. - Under the hypotheses of the theorem, there exists a constant H and functions $a(\tau)$ and $b(\tau)$ in CI such that $a(\tau) \to \infty$ as $\tau \to \infty$, $a(\tau) > 0$ for $\tau > H$ and

$$a(x^2 + y^2 + z^2) \leq V(x, y, z) \leq b(x^2 + y^2 + z^2) \quad \text{for} \quad x^2 + y^2 + z^2 \geq H.$$

PROOF OF LEMMA 3.1. - The function $2V_1$ can be written in the form

$$2V_1=(\beta x-z)^2+\beta\left(y+\frac{1}{\beta}h(x)\right)^2+2\beta\int_0^x[G(u)-\beta u]du+\frac{2}{\beta}\int_0^x[a\beta-h'(u)]h(u)du-\frac{h^2(0)}{\beta}.$$

Since $\int_0^x[a\beta-h'(u)]h(u)du$ and $\int_0^x[G(u)-\beta u]du$ are continuous on $[-K, K]$, there exists $D_1>0$ such that

$$(3.5)\qquad \frac{2}{\beta}\int_0^x[a\beta-h'(u)]h(u)du\geq -D_1 \quad\text{and}\quad \beta\int_0^x[G(u)-\beta u]du\geq -D_1$$

on that interval. From (3.4) we have

$$(3.6)\qquad 2\beta\int_0^x[G(u)-\beta u]du\geq\beta\varepsilon x^2-D_1-\varepsilon K^2$$

and

$$\frac{2}{\beta}\int_0^x[a\beta-h'(u)]h(u)du\geq -D_1 \quad\text{for all}\quad x.$$

Let $D=2D_1+\varepsilon K^2+\dfrac{h^2(0)}{\beta}$, then

$$(3.7)\qquad 2V_1\geq(\beta x-z)^2+\beta\left[y+\frac{1}{\beta}h(x)\right]^2+\beta\varepsilon x^2-D \quad\text{for all}\quad x.$$

It follows from (1.4) and the MEAN Value Theorem that there is a constant D_2 such that

$$(3.8)\qquad |h(x)|\leq C|x|+D_2 \quad\text{for all } x.$$

Using (3.7) and (3.8) it is clear that

$$(3.9)\qquad V_1(x, y, z)\to\infty \quad\text{as}\quad x^2+y^2+z^2\to\infty.$$

Since V_1 is continuous and independent of t, it follows from (3.9) that there exist functions $a_1(\tau)$, $b_1(\tau)\in CI$ such that

$$a_1(x^2+y^2+z^2)\leq V_1(x, y, z)\leq b_1(x^2+y^2+z^2) \quad\text{and}\quad a_1(\tau)\to\infty \quad\text{as}\quad \tau\to\infty.$$

It follows from an easy examination of the individual cases in the definition of V_2 that

$$-2\varepsilon M, \leq 2V_2 \leq 2\varepsilon M \quad \text{for all} \quad (x, y, z) \in R^3,$$

and therefore if we set $a(\tau) = a_1(\tau) - \varepsilon M$, and $b(\tau) = b_1(\tau) + \varepsilon M$, then

$$a(x^2 + y^2 + z^2) \leq V(x, y, z) \leq b(x^2 + y^2 + z^2),$$

$a(\tau) \to \infty$ as $\tau \to \infty$ and that there is an $H_1 > 0$ such that $a(\tau) > 0$ for $\tau \geq H_1$.

LEMMA 3.2. - Under hypotheses of Theorem 1, there is an H_2 such that

$$\dot{V}_{(3.1)}(t, x, y, z) < -1 \quad \text{for} \quad x^2 + y^2 + z^2 \geq H_2 \quad \text{and} \quad t \geq 0.$$

PROOF OF LEMMA 3.2. - If (x, y, z) is any solution of (3.1), then

$$\dot{V}_1 = -[a\beta - h'(x)]y^2 + \beta P(t)y - [G(x) - \beta x - P(t)]h(x).$$

The value of $\dot{V}_2$ is easily calculated for all values of x, y and z in the interior of each of the regions of definition and the exact value on the boundary is not important since $\dot{V}_2$ will be estimated from both of the possible expressions at each boundary point. For example if $0 < z < N$ and $-M < y < 0$, then

$$\dot{V}_2 = \begin{cases} -\varepsilon \dot{y} + \varepsilon \dfrac{M}{N}\dot{z} & \text{if} \quad Mz > -Ny \\ 2\varepsilon \dfrac{M}{N}\dot{z} & \text{if} \quad Mz < -Ny \end{cases}$$

and for $Mz = -Ny$, from the definition of $\dot{V}_2$, $\dot{V}_2$ is either $-\varepsilon \dot{y} + \varepsilon \dfrac{M}{N}\dot{z}$ or $2\varepsilon \dfrac{M}{N}\dot{z}$.

In order to establish the lemma it will be shown that $\dot{V} < -1$ on each of the ten regions of R^3 used in the definition of V_2.

For $(x, y, z) \in R_1$, $2V = 2V_1$, and

$$\dot{V} = \dot{V}_1 = -[a\beta - h'(x)]y^2 + \beta P(t)y - [G(x) - \beta x - P(t)]h(x).$$

It follows from (3.4) that $[G(x) - \beta x] \operatorname{sgn} x \geq \varepsilon |x|$ for $|x| \geq K$ which along with (1.5) and (3.4) imply the existence of a constant $K_1 \geq K$ such that

$$\text{(3.10)} \quad [G(x) - \beta x - P(t)]h(x) \geq [G(x) - \beta x - P_0 \operatorname{sgn} x]h(x) > 0 \quad \text{for} \quad |x| \geq K_1.$$

Since $[G(x) - \beta x - P(t)]h(x)$ is continuous for all x and $|P(t)| \leq P_0$ for all t, (3.10) implies the existence of a positive constant D_4 such that $[G(x) - \beta x - P(t)]h(x) \geq - D_4$ for all x and t.

So on R_1

$$\dot V \leq - \delta y^2 + \beta P_0 |y| + D_4,$$

and clearly an M_1 can be chosen such that

$$- \delta y^2 + \beta P_0 |y| + D_4 < - 1 \quad \text{for} \quad y \geq M_1 .$$

Now if we assume that $M \geq M_1$, then

$$\dot V < - 1 \quad \text{for} \quad (x, y, z) \in R_1 . \tag{3.12}$$

For $(x, y, z) \in R_3 \cup R_5$, $\dot V = \dot V_1$, and the same arguments as used above can be used to find an $M_2 > 0$, such that if $M \geq M_2$, then

$$\dot V < - 1 \quad \text{for} \quad (x, y, z) \in R_3 \cup R_5 . \tag{3.13}$$

On R_4, $\dot V = \dot V_1 - \frac{2\varepsilon M}{N} h(x) = -[a\beta - h'(x)]y^2 + \beta P(t) y - \left[G(x) - \beta x - P(t) + \frac{2\varepsilon M}{N}\right]h(x)$.

Proceeding just as in the preceeding case, recalling that $2aM \leq N$, there is $K_2 \geq K$ such that

$$\left[G(x) - \beta x - P(t) + \frac{2 \varepsilon M}{N}\right] h(x) \geq \{ G(x) - \beta x - [P_0 + 2a\varepsilon] \operatorname{sgn} x \} h(x) > 0$$

for $|x| \geq K_2$, and the continuity of this function again implies the existence of a positive constant D_5 such that

$$\left[G(x) - \beta x - P(t) + \frac{2\varepsilon M}{N}\right] h(x) \geq - D_5 \quad \text{for all } x \text{ and } t.$$

As in the preceeding case, a positive constant M_2 can now be selected such that

$$- \delta y^2 + \beta P_0 |y| + D_5 < - 1 \quad \text{for} \quad y \leq M_2 ,$$

so that if $M \geq M_2$, then

$$\dot V < - 1 \quad \text{for} \quad (x, y, z) \in R_4 . \tag{3.15}$$

Let $M = \max(M_1, M_2, M_3)$, then it follows from (3.12), (3.13) and (3.15) that

$$\dot V < - 1 \quad \text{for} \quad (x, y, z) \in R_1 \cup R_3 \cup R_4 \cup R_5 \tag{3.16}$$

For $(x, y, z) \in R_2$, $\dot{V} = \dot{V}_1 - \frac{\varepsilon}{2} z + \frac{\varepsilon}{2} ay + \frac{\varepsilon}{2} G(x) - \frac{\varepsilon}{2} P(t)$, and since $|y| \leq M$ there is a positive constant D_6 such that

$$-[a\beta - h'(x)]y^2 + \beta P_0 |y| + \frac{\varepsilon}{2} a |y| + \frac{\varepsilon}{2} P_0 \leq D_6 \quad \text{on} \quad R_2,$$

and thus

$$\dot{V} \leq -[G(x) - \beta x + P(t)]h(x) + D_6 - \frac{\varepsilon}{2} z \quad \text{on} \quad R_2. \tag{3.17}$$

If $x \leq -K_1$, then (3.4) and (3.10) imply that

$$-[G(x) - \beta x - P_0 \operatorname{sgn} x]h(x) < 0 \quad \text{and} \quad \frac{\varepsilon}{2} G(x) < 0, \tag{3.18}$$

while for positive x we can write the first two terms of (3.17) as

$$-[G(x) - \beta x - P_0 \operatorname{sgn} x]\left[h(x) - \frac{\varepsilon}{2}\right] + \frac{\beta\varepsilon}{2} x + \frac{P_0 \varepsilon}{2} \tag{3.19}$$

and again, if $x \geq K_1$, since $h(x) \geq \frac{\beta + \varepsilon}{2}$ each term of this expression is negetive. So combining (3.18) and (3.19) we have

$$-[G(x) - \beta x - P_0 \operatorname{sgn} x]h(x) + \frac{\varepsilon}{2} G(x) < 0 \quad \text{if} \quad |x| \geq K_1,$$

and since this expression is continuous in x we have shown the existence of a positive D_7 with the property

$$-[G(x) - \beta x - P_0 \operatorname{sgn} x]h(x) + \frac{\varepsilon}{2} G(x) \leq D_7 \quad \text{for all } x.$$

Thus on R_2 $\dot{V} \leq D_6 + D_7 - \frac{\varepsilon}{2} z$, and we can clearly choose N_1 such that

$$\dot{V} < -1 \quad \text{for} \quad z \geq N_1; \quad \text{and so if} \quad N \geq N_1$$

then

$$\dot{V} < -1 \quad \text{for} \quad (x, y, z) \in R_2. \tag{3.20}$$

If $(x, y, z) \in R_6$, then $\dot{V} = \dot{V} + \frac{\varepsilon}{2} z - \frac{\varepsilon a}{2} y - \frac{\varepsilon}{2} G(x) + \frac{\varepsilon}{2} P(t)$, and since $z \leq 0$, we find $N_2 > 0$, using the same method as used in the preceding case, for which $N \geq N_2$ implies that

(3.21) $$\dot V < -1 \quad \text{for} \quad (x, y, z) \in R_6 .$$

Combining (3.20) and (3.21) if $N \equiv \max(N_1, N_2)$ then

(3.22) $$\dot V < -1 \quad \text{for} \quad (x, y, z) \in R_2 \cup R_6 .$$

For $(x, y, z) \in R_8$, $\dot V = \dot V_1 - \frac{\varepsilon}{2} z + \frac{\varepsilon a}{2} y + \frac{\varepsilon}{2} G(x) - \frac{\varepsilon}{2} P(t) - \frac{\varepsilon M}{N} h(x)$, and since $|y| \leq M$ and $|z| \leq N$ and $\dot V$ is continuous in y and z on R_8, there is a positive constant D_8 such that on R_8

$$-[a\beta - h'(x)]y^2 + \beta P(t)\,|y| + \frac{\varepsilon a}{2} y - \frac{\varepsilon}{2} P(t) - \frac{\varepsilon}{2} z \leq D_8$$

and thus on R_8

(3.23) $$\dot V \leq D_8 - [G(x) - \beta x - P_0 \operatorname{sgn} x] h(x) + \frac{\varepsilon}{2} G(x) - \frac{\varepsilon M}{N} h(x).$$

Since $2M \leq N$, $\frac{\varepsilon M}{N} \leq \frac{\varepsilon}{2a}$ and thus if $x \leq -K$ it follows from (3.4) that

$$\frac{\varepsilon}{2} G(x) - \frac{\varepsilon M}{N} h(x) < 0 \quad \text{and} \quad -[G(x) - \beta x - P_0 \operatorname{sgn} x] h(x) \leq -[\varepsilon x + P_0] h(x).$$

From this last expression and (3.4) it is clear that a positive constant L_1 can be selected such that

(3.24) $$D_8 - [G(x) - \beta x - P_0 \operatorname{sgn} x] h(x) + \frac{\varepsilon}{2} G(x) - \frac{\varepsilon M}{N} h(x) < -1 \quad \text{for} \quad x < -L_1$$

For positive values of x we can rewrite (3.23) as

$$\dot V \leq D_8 - [G(x) - \beta x - P_0]\left[h(x) - \frac{\varepsilon}{2}\right] + \frac{\beta\varepsilon}{2} x + \frac{P_0 \varepsilon}{2} - \frac{\varepsilon M}{N} h(x),$$

and if $x \geq K_1$ then (3.4) and (3.10) imply that $h(x) \geq \frac{\beta + \varepsilon}{2}$ and thus that

$$-[G(x) - \beta x - P_0]\left[h(x) - \frac{\varepsilon}{2}\right] + \frac{\varepsilon M}{N} h(x) < 0.$$

So $\dot V \leq D_8 - \frac{P_0 \varepsilon}{2} - \frac{\beta\varepsilon}{2} x$ on R_8 if $x \geq K_1$ and clearly an $L_2 \geq K_1$ can be selected such that

$$D_8 - \frac{P_0 \varepsilon}{2} - \frac{\beta\varepsilon}{2} x < -1 \quad \text{if} \quad x \geq L_2 \tag{3.25}$$

Now if $L_3 = \max(L_1, L_2)$ and if $L \geq L_3$, then combining (3.24) and (3.25) we have

$$\dot{V} < -1 \quad \text{for} \quad (x, y, z) \in R_8 \tag{3.26}$$

The details for the cases R_7, R_9 and R_{10} are very similar to R_8 and will not be repeated. Let $L_4 > 0$ be such that

$$\dot{V} < -1 \quad \text{for} \quad (x, y, z) \in R_7 \cup R_9 \cup R_{10} \quad \text{and} \quad |x| \geq L_4.$$

Let $L = \max(L_3, L_4)$ then

$$\dot{V} < -1 \quad \text{if} \quad (x, y, z) \varepsilon \bigcup_{i=7}^{10} R_i \quad \text{and} \quad |x| \geq L. \tag{3.27}$$

Now set $H_2 = 3 \max(L^2, M^2, N^2, 1)$, then if $x^2 + y^2 + z^2 \geq H_2$ one of the following must hold:

(i) $|y| \geq M$

(ii) $|y| < M$ and $|z| \geq N$

(iii) $|y| < M$, $|z| < N$ and $|x| \geq L$.

It is clear from (3.16), (3.22) and (3.26) that $\dot{V} < -1$ in each case which completes the proof of Lemma 3.2.

If $H = \max(H_1, H_2)$ then combining Lemmas 2.1, 3.1 and 3.2 it follows that there is a costant B_1 such that if $(x(t), y(t), z(t))$ is any solution of (3.1), then

$$x^2(t) + y^2(t) + z^2(t) \leq B_1 \tag{3.28}$$

for all sufficiently large t. Now if $x(t)$ is a solution of (1.1), then $\dot{x}(t) = y$ and $x(t) = z - ay - G(x) + P(t)$, where (x, y, z) is a solution of (3.1).

Now since $|P(t)| \leq P_0$ for all t and G is continuous it follows from (3.28) that there is a constant B dependent only on equation (1.1) such that (1.6) is satisfied.

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