# A BRANCH-AND-BOUND ALGORITHM FOR ZEROONE MIXED INTEGER PROGRAMMING PROBLEMS 

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#### Abstract

This paper presents the results of experimentation on the development of an efficient branch-and-bound algorithm for the solution of zero-one linear mixed integer programming problems. An implicit enumeration is employed using bounds that are obtained from the fractional variables in the associated linear programming problem. The principal mathematical result used in obtaining these bounds is the piecewise linear convexity of the criterion function with respect to changes of a single variable in the interval $[0,1]$. A comparison with the computational experience obtained with several other algorithms on a number of problems is included.


MANY IMPORTANT practical problems of optimization in management, economics, and engineering can be posed as so-called 'zeroone mixed integer problems,' i.e., as linear programming problems in which a subset of the variables is constrained to take on only the values zero or one. When indivisibilities, economies of scale, or combinatoric constraints are present, formulation in the mixed-integer mode seems natural. Such problems arise frequently in the contexts of industrial scheduling, investment planning, and regional location, but they are by no means limited to these areas.

Unfortunately, at the present time the performance of most comprehensive algorithms on this class of problems has been disappointing. This study was undertaken in hopes of devising a more satisfactory approach. In this effort we have drawn on the computational experience of, and the concepts employed in, the Land and Doig ${ }^{[16]}$ Healy, ${ }^{[13]}$ and Driebeek ${ }^{[15]}$ algorithms. Similar approaches are contained in the work of Beale
and Small. ${ }^{[2]}$ A related enumeration scheme is contained in Graves and Whinston. ${ }^{[10]}$

Our approach is basically a branch-and-bound method of enumeration. While this is not generally the most 'glamorous' type of algorithm, our experience indicates that it works well in solving practical problems. As with any branch-and-bound algorithm, efficiency derives primarily from choosing branches and bounds in a manner that is computationally effective.

We proceed with a discussion of the algorithm, followed by a presentation of our computational results.

## 1. GENERAL THEORY

Let

$$
y=\left[\begin{array}{c}
y_{1} \\
\vdots \\
y_{n}
\end{array}\right]
$$

be an $n$-vector of zero-one variables, each of which is constrained to be zero or one. Let

$$
x=\left[\begin{array}{c}
x_{1} \\
\vdots \\
x_{m}
\end{array}\right]
$$

be an $m$-vector of continuous nonnegative variables. The standard form of the zero-one mixed integer programming problem is the following:

$$
\begin{equation*}
\operatorname{minimize} z=\phi(y, x)=c y+d x \tag{1}
\end{equation*}
$$

subject to

$$
\begin{gather*}
\mathrm{A}\left[\begin{array}{l}
y \\
x
\end{array}\right] \geqq b \quad \text { for }\left[\begin{array}{l}
y \\
x
\end{array}\right] \geqq 0  \tag{2}\\
y_{i} \leqq 1 \text { for } i=1, \cdots, n  \tag{3}\\
y_{i}=0 \text { or } 1 \text { for } i=1, \cdots, n \tag{4}
\end{gather*}
$$

There are $2^{n}$ possible $y$ vectors satisfying (4). Some of these may admit of no $x$ satisfying (2) and hence are infeasible.

Consider the solution ( $\bar{y}, \bar{x}$ ) to the problem (1), (2), (3). [We call ( $\bar{y}, \bar{x}$ ) the solution to the 'unconstrained' problem because the integer constraints (4) are missing.] If $\bar{y}$ is already a vector of zeros and ones, the problem is solved. Far more likely, it is not. In that case, we seek more information about the local properties of the unconstrained solution.

Consider the variable $y_{1}$. Let $z_{1}\left(\tilde{y}_{1}\right)$ be the optimal value of the objective function defined parametrically in terms of $\tilde{y}_{1}$ for the problem (1), (2), (3) with the additional constraint $y_{1}=\tilde{y}_{1}$ for $\tilde{y}_{1} \in[0,1]$. It can be shown that $z_{1}\left(\tilde{y}_{1}\right)$ is a piecewise linear convex function of $\tilde{y}_{1}$ (cf. DANTZIG, ${ }^{[3]}$ p. 168).

To map out the entire function $z_{1}\left(\tilde{y}_{1}\right)$ would require, in general, the solution of many linear programming problems. We settle instead for the more easily attainable behavior of the two linear segments to the direct right and left of $\bar{y}_{1}$. The slopes of these two segments can be read out of the linear programming tableau. (Strictly speaking, the assertion is true only if the basis is nondegenerate. If degeneracy is involved, we may be able to obtain only a lower bound on the slope of each segment, difficult to picture geometrically, but otherwise the results are not altered.)


Fig. 1. A piecewise linear convex function.
The slopes are obtained as follows. Suppose, first of all, that $y_{1}$ is in the optimal basis of the associated linear program and that $0<\bar{y}_{1}<1$. Let $\left\{w_{j}: j \epsilon N B\right\}$ be the set of nonbasic variables, let

$$
y_{1}+\sum_{j \epsilon N B} \bar{a}_{i j} w_{j}=\bar{y}_{1}
$$

be the row in the updated simplex tableau corresponding to $y_{1}$, and let

$$
z+\sum_{j \in N B} \bar{c}_{j} w_{j}=\varphi(\bar{y}, \bar{x})
$$

be the reduced-cost row (where $\bar{c}_{j} \geqq 0$ for $j_{\epsilon} N B$ ). Along the extreme rays adjacent to ( $\bar{y}, \bar{x}$ ) generated by increasing one $w_{j}$ and holding the other nonbasic variables fixed, we have:
(i) If $a_{i j}>0$, increasing $w_{j}$ by $1 / \bar{a}_{i j}$ will decrease $y_{1}$ by 1 and increase the objective function by $\bar{c}_{j} / \bar{a}_{i j}$.
(ii) If $\bar{a}_{i j}<0$, increasing $w_{j}$ by $1 /-\bar{a}_{i j}$ will increase $y_{1}$ by 1 and the objective function by $\bar{c}_{j} /-\bar{a}_{i j}$.
(iii) If $\bar{a}_{i j}=0$, increasing $w_{j}$ has no effect on $y_{1}$.

Since the feasible region is contained in the convex cone generated by
these rays (cf. Dantzig, ${ }^{[3]}$ p. 155), it follows that

$$
\begin{equation*}
m_{0}=\min _{\bar{a}_{i j}>0, j \epsilon N B} \bar{c}_{j} / \bar{a}_{i j} \quad \text { and } \quad m_{1}=\min _{-\bar{a}_{i j}>0, j \epsilon N B} \bar{c}_{j} /-\bar{a}_{i j}, \tag{5}
\end{equation*}
$$

(where $m_{0}$ is defined to be infinite if all $\bar{a}_{i j} \leqq 0$, and $m_{1}$ is defined to be infinite if all $\bar{a}_{i j} \geqq 0$ ) are lower bounds (which are exact except possibly in the case of degeneracy) on the absolute value of the slopes of $z_{1}\left(\tilde{y}_{1}\right)$ at $\bar{y}_{1}$.

If $y_{1}$ is a nonbasic variable at level $0, m_{1}=\bar{c}_{1}$; if it is a basic variable at level 0 or 1 , only $m_{1}$ (or $m_{0}$ ) need be computed from (5).

From the convexity of $z_{1}\left(\tilde{y}_{1}\right)$ it follows that $\left[q_{0}(j), q_{1}(j)\right]$ given by

$$
\begin{align*}
& q_{0}(j) \equiv z_{1}\left(\bar{y}_{1}\right)+\bar{y}_{1} \cdot m_{0} \leqq z_{1}(0)  \tag{6}\\
& q_{1}(j) \equiv z_{1}\left(\bar{y}_{1}\right)+\left(1-\bar{y}_{1}\right) \cdot m_{1} \leqq z_{1}(1)
\end{align*}
$$

are lower bounds on the objective function value if $y_{j}$ is forced to the values 0 or 1, respectively (cf. Driebeek ${ }^{[6]}$ ). Bounds for $y_{2}, \cdots, y_{n}$ can be computed analogously. These bounds are used repeatedly in the algorithm for branching decision, and are rapidly computed directly from quantities in the updated simplex tableau provided by the simplex algorithm.

## 2. BRANCH-AND-BOUND SEARCH

Expositions and examples of branch-and-bound-search algorithms abound in the literature (e.g., Land and Doig, ${ }^{[16]}$ Balas, ${ }^{[1]}$ and GeofFRION ${ }^{[8]}$ ); we reiterate here only that such an algorithm proceeds by sequentially partitioning the potential solution space into disjoint subsets determined by specifying the values of some of the discrete variables (called a 'partial assignment' or 'partial solution'); that a lower bound is computed for the value of the objective on all elements of each such subset (the set of completions of a partial assignment); and that the partitioning process can be represented diagramatically in the form of a tree (where nodes correspond to partial assignments).

The relative efficiency of a branch-and-bound algorithm over total enumeration derives from the ability to eliminate from consideration large subsets of potential optimal solutions. Two principal ways of eliminating such subsets are: (i) to show that no feasible completion of a given partial assignment exists; and (ii) to show that no optimal completion of a given partial assignment exists. Balas and others following him (see Fleischmann, ${ }^{[7]}$ Ivanescu and Rudeanu ${ }^{[14]}$ ) have developed a set of additive feasibility tests for these purposes that utilize the zero-one restriction on the discrete variables in an essential way. Land and Doig ${ }^{[16]}$ suggested performing subset elimination by referring to the solutions to linear programs obtained by fixing the variables in a partial assignment to their assigned values. Although developed in isolation from one another, these
two approaches are clearly not mutually exclusive; incorporation of both into the same algorithm may yield further improvements in computational efficiency (see Geoffrion ${ }^{[9]}$ ). The algorithm reported here, however, results from an investigation of the ways in which the Land and Doig approach can be enhanced by using tables of bounds derived from those described above.

Thus, if $I=\{1,2, \cdots, n\}$ is the index set of the zero-one variables, and if $I_{0}$ and $I_{1}$ are the index sets of variables in a partial assignment assigned to zero and one, respectively, let $L P\left(I_{0}, I_{1}\right)$ be the associated 'restricted' linear program (with variables indexed by $I_{0}$ and $I_{1}$ fixed at their assigned values and let $Q\left(I_{0}, I_{1}\right)$ be the associated bounds matrix defined by (6). Then if $I_{f}$ is the set of free variables ( $I_{f}=I-\left[I_{0} U I_{1}\right]$ ),

$$
\begin{equation*}
\Psi\left(I_{0}, I_{1}\right) \equiv \max _{j \epsilon I_{f}} \min \left[q_{0}(j), q_{1}(j)\right] \tag{7}
\end{equation*}
$$

is a lower bound on the value of the objective function for any completion of ( $I_{0}, I_{1}$ ). All completions of ( $I_{0}, I_{1}$ ) can thus be eliminated if (i) $L P$ ( $I_{0}, I_{1}$ ) is infeasible, or (ii) $\Psi\left(I_{0}, I_{1}\right) \geqq \bar{z}$, where $\bar{z}$ is an upper bound on the optimal solution value. If neither of these conditions obtains, but $q_{u}(j) \geqq \bar{z}$ for some $j \epsilon I_{f}, u=0$ or 1 , then all completions of ( $I_{0}, I_{1}$ ) yielding an improved solution value must satisfy $y_{j}=(1-u)$. In this case, $\left(I_{0}, I_{1}\right)$ may be enlarged to ( $I_{0}{ }^{\prime}, I_{1}{ }^{\prime}$ ), where

$$
\begin{equation*}
I_{0}{ }^{\prime}=I_{0}, I_{1}{ }^{\prime}=I_{1} \bigcup\{j\} \tag{8a}
\end{equation*}
$$

if $u=0$, or

$$
\begin{equation*}
I_{0}{ }^{\prime}=I_{0} \bigcup\{j\}, I_{1}{ }^{\prime}=I_{1} \tag{8b}
\end{equation*}
$$

if $u=1$; if $y_{j}$ is fractional valued in the solution to $L P\left(I_{0}, I_{1}\right)$, the solution to $L P\left(I_{0}{ }^{\prime}, I_{1}{ }^{\prime}\right)$ yields an improved set of bounds $Q\left(I_{0}{ }^{\prime}, I_{1}{ }^{\prime}\right)$ to which the above tests may be applied again. Augmenting partial assignments in this way can substantially reduce computation times.

Some virtually costless improvements on the bounds matrix $Q\left(I_{0}, I_{1}\right)$ follow from the observation that, since any completion of $\left(I_{0}, I_{1}\right)$ is also a completion of ( $J_{0}, J_{1}$ ) if $J_{0} \subseteq I_{0}$ and $J_{1} \subseteq I_{1}$,

$$
\begin{equation*}
\tilde{Q}\left(I_{0}, I_{1}\right)=\sup _{J_{0}} \subseteq_{I_{0}, J_{1} \subseteq I_{1}} Q\left(J_{0}, J_{1}\right) \tag{9}
\end{equation*}
$$

is another valid, and generally improved, bounds matrix. A more readily computable improvement over $Q\left(I_{0}, I_{1}\right)$, which lends itself well to the algorithm given below, is provided by the following updating procedure:

$$
\begin{align*}
& Q\left(I_{0} \cup\{j\}, I_{1}\right) \leftarrow \sup \left\{Q\left(I_{0} \cup\{j\}, I_{1}\right), Q\left(I_{0}, I_{1}\right)\right\},  \tag{10}\\
& Q\left(I_{0}, I_{1} \cup\{j\}\right) \leftarrow \sup \left\{Q\left(I_{0}, I_{1} \cup\{j\}\right), Q\left(I_{0}, I_{1}\right)\right\} .
\end{align*}
$$

The results reported below were obtained using bounds matrices updated in this way.

To obtain large lower bounds and avoid redundant assignments of variables connected by constraints, partitioning of the completions of a partial assignment $\left(I_{0}, I_{1}\right)$ that cannot be enlarged or eliminated is accomplished by choosing that fractional valued variable $y_{j}$ in the solution to $\operatorname{LP}\left(I_{0}, I_{1}\right)$ that has maximum zero or one bound, and considering the two sets of completions of ( $I_{0}, I_{1}$ ) determined by assigning $y_{j}$ to 0 or $y_{j}$ to 1 .

## 3. STATEMENT AND DISCUSSION OF THE ALGORITHM

Let $\bar{z}$ be the current upper bound on the optimal solution value, and $S$ be the set of partial assignments that remain to be considered at iteration $\nu$. $S_{0}$ is defined to consist of the vacuous assignment with both $I_{0}$ and $I_{1}$ empty (which is the 'specified' node for iteration 1). Then the $\nu$ th iteration of the algorithm proceeds as follows:
A. Node Evaluation

Solve $L P\left(I_{0}, I_{1}\right)$ for the currently specified assignment ( $I_{0}, I_{1}$ ) and, if feasible, compute $Q\left(I_{0}, I_{1}\right)$ and $\psi\left(I_{0}, I_{1}\right)$, using (6), (10), and then (7).
(1) If $L P\left(I_{0}, I_{1}\right)$ is infeasible or $\psi\left(I_{0}, I_{1}\right) \geqq \bar{z}$, form $S_{v}$ by removing $\left(I_{0}, I_{1}\right)$ from $S_{v-1}$ and go to step C.
(2) If $y$ satisfies (4), record the feasible solution as the best yet found, set $\bar{z}$ to the current objective function value, form $S_{v}$ by removing ( $I_{0}, I_{1}$ ) from $S_{v \sim 1}$, and go to step C.
(3) If $q_{0}(j) \geqq \bar{z}$ or $q_{1}(j) \geqq \bar{z}$ for any free variable $y_{j}, j \epsilon I_{f}$, enlarge ( $I_{0}, I_{1}$ ) according to (8), and, if any such $y_{j}$ is fractional valued in the solution to $L P\left(I_{0}, I_{1}\right)$, repeat step A. Otherwise, continue with step B.
B. Node Formation (or branching)

For the current (possibly enlarged) assignment ( $I_{0}, I_{1}$ ), find $j^{*} \epsilon I_{f}$ such that $y_{j *}$ is the fractional-valued variable in the solution to $\operatorname{LP}\left(I_{0}, I_{1}\right)$ with the largest zero or one bound, and form $S_{v}$ by replacing ( $I_{0}, I_{1}$ ) in $S_{v-1}$ with the two assignments ( $I_{0} U\left\{j^{*}\right\}, I_{1}$ ) and ( $I_{0}, I_{1} U\left\{j^{*}\right\}$ ). If $q_{u}\left(j^{*}\right)<q_{1-u}\left(j^{*}\right)$ ( $u=0$ or 1 ), a lower bound on the completions with $y_{j}=u$ is $\psi\left(I_{0}, I_{1}\right)$, and, with $y_{j}=1-u$, is $q_{1-u}\left(j^{*}\right)$.

## C. Node Selection

Choose the next partial assignment to be evaluated from $S_{v}$ according to one of the following rules:

Option (1). If ( $I_{0}, I_{1}$ ) was eliminated or yields an integer solution, use option (2). Otherwise specify the assignment determined in B where $y_{j} *$ is assigned to the value with the smaller bound.

Option (2). Specify the node in $S_{v}$ with the least lower bound. If $S_{v}$ is empty or none with lower bound less than $\bar{z}$ exist, the search is complete.

In option (1), the current assignment is partially completed at successive iterations until it becomes infeasible, is bounded off by the current upper bound $\bar{z}$, or produces an improved feasible solution. A disadvantage of this approach is that many more nodes may be evaluated than are necessary to prove convergence. One advantage is that storage require-
ments are minimal. Option (2) is the 'flooding' strategy of always evaluating the partial assignment with lowest lower bound. The advantage of this procedure is that only those assignments that are necessary to prove convergence are evaluated. A disadvantage is that large storage capacity may be required. Use of random-access disk-removed computer memory is a limiting factor for us, so this approach was usually used.

The efficacy of the tests in A depends on obtaining a good upper bound $\bar{z}$ early in the search. One of the most successful approaches tried was to run the algorithm with option (1) and $\bar{z}=+\infty$ in the very beginning. This usually resulted in a near-optimal feasible integer solution; then option (2) could be applied.

## 4. EXPERIENCE WITH THE ALGORITHM

Table I gives a resumé of computational experience with our algorithm ( $D K W$ ) using updated bounds matrices and the 'flooding' strategy search. The table also gives comparisons with the experience of others where possible. Option 2 was used for all the runs of our algorithm reported here.

Solution time in the table refers to the time required on an IBM 7094 to obtain the optimum mixed integer solution, including the solution to the unconstrained problem. In the solution of the Kendrick problem with the Healy algorithm, all but four of the variables could have been bounded off after 24 minutes using an upper bound obtained from other solutions of the problem. Though such a bound is not available in the Healy algorithm, it is used here for comparative purposes. It is assumed that the enumeration of the remaining 16 lattice points would have required 10 minutes. The same type of adjustment is made for the number of $L P$ 's and iterations. In solving the same problem with Driebeek's algorithm, the run was halted after enumerating 100 of the 128 'unbounded' lattice points in 44 minutes. Enumeration of the remaining 28 lattice points is assumed to have required 12 minutes. The same type of adjustment is made for the number of $L P$ 's and iterations.

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by Davis (see Davis ${ }^{[5]}$ ). We are indebted to Jun Onaka for able programming assistance in solving the problems referred to in Table I.

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