A Brauer's theorem and related results*

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Abstract

Given a square matrix A, a Brauer's theorem [Limits for the characteristic roots of matrices IV: Applications to stochastic matrices, Duke Math. J. 19 (1952), 75-91] shows how to modify one single eigenvalue of A via a rank-one perturbation without changing any of the remaining eigenvalues. Older and new results can be considered in a framework of the above theorem. In particular, we present some applications to stabilize control systems, including the case when the system is noncontrollable. Other applications are related with the Jordan form of A, Wielandt's and Hotelling's deflations and the Google matrix. In 1955, Perfect presented an extension of such Brauer's result to change r eigenvalues of A at the same time via a rank-r perturbation without changing any of the remaining eigenvalues. The same results considered by blocks can be put into the block version framework of the above theorem.

Keywords: eigenvalues, pole assignment problem, controllability, low rank perturbation, deflation techniques, Google matrix.

1 The Brauer's theorem

The relationship among the eigenvalues of an arbitrary matrix and the updated matrix by a rank one additive perturbation was proved by A. Brauer [1] and we will refer as the Brauer's Theorem. It turns out that this result is related with older and well known results on techniques as Wielandt's and Hotelling's deflations (see [10]) and new results on the Google matrix (see [6]). Further, the eigenvalue localization problem of control theory (see [5]) can be stated as an application of the Brauer's Theorem and so, the stabilization of these control systems. In addition, the Brauer result has been first applied by Perfect [7] to construct nonnegative matrices with a prescribed spectrum.

In the first part of the paper (sections 1 and 2), we give related results that can be considered in a common framework of the Brauer's Theorem as

*Supported by Fondecyt 1085125, Chile, the Spanish grant DGI MTM2010-18228 and the Programa de Apoyo a la Investigación y Desarrollo (PAID-06-10) of the UPV.

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applications of it. A good introduction on the Brauer result and its application to the nonnegative inverse eigenvalue problem can be followed in [9] where an extended version is given. Further, this extended version, which is due to R. Rado (see [7]), is considered in the second part of the paper (sections 3 and 4) and applied to related results by blocks as the Brauer's Theorem.

Throughout the paper, we assume that the set of numbers that may be eigenvalues of a matrix are feasible in the corresponding field (i.e., closed under complex conjugation in the real field).

For completeness, we shall give a proof of the Brauer's Theorem based on the Rado's proof [7].

Theorem 1 *Let A be an $n \times n$ arbitrary matrix with eigenvalues $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Let x_k be an eigenvector of A associated with the eigenvalue λ_k, and let q be any n-dimensional vector. Then the matrix $A + x_k q^T$ has eigenvalues $\{\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + x_k^T q, \lambda_{k+1}, \ldots, \lambda_n\}$.*

Proof: There is no loss of generality in assuming that $k = 1$. Let $S = [x_1\ Y]$ be a nonsingular matrix and consider $S^{-1} = \begin{bmatrix} l_1 \\ V \end{bmatrix}$. As $Ax_1 = \lambda_1 x_1$ we have

$$S^{-1}AS = \begin{bmatrix} l_1 \\ V \end{bmatrix} A\,[x_1\ Y] = \begin{bmatrix} l_1 \\ V \end{bmatrix} [Ax_1\ AY] = \begin{bmatrix} \lambda_1 & l_1AY \\ O & VAY \end{bmatrix}$$

then $\sigma(VAY) = \{\lambda_2, \ldots, \lambda_n\}$ and

$$S^{-1}(x_1q^T)S = \begin{bmatrix} l_1 \\ V \end{bmatrix} (x_1q^T)S = \begin{bmatrix} 1 \\ O \end{bmatrix} q^TS = \begin{bmatrix} q^Tx_1 & q^TY \\ O & O \end{bmatrix}.$$

Therefore,

$$\begin{aligned} S^{-1}(A + x_1q^T)S &= S^{-1}AS + S^{-1}(x_1q^T)S \\ &= \begin{bmatrix} \lambda_1 & l_1AY \\ O & VAY \end{bmatrix} + \begin{bmatrix} q^Tx_1 & q^TY \\ O & O \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 + q^Tx_1 & l_1AY + q^TY \\ O & VAY \end{bmatrix} \end{aligned}$$

and

$$\sigma(A + x_1q^T) = \{\lambda_1 + q^Tx_1\} \cup \sigma(VAY) = \{\lambda_1 + q^Tx_1, \lambda_2, \ldots, \lambda_n\}. \qquad \blacksquare$$

The above proof gives the insight to preserve the Jordan structure when A is perturbed by a one-rank update. We see that in the following result.

Corollary 1 *Consider the conditions of Theorem 1. If the following statements hold,*
(i) S is the matrix such that tal que $S^{-1}AS = J_A$, where J_A is the Jordan form of A,

(ii) λ_1 *is associated with a Jordan chain of length* 1*, that is,* $l_1AY = O_{1\times(n-1)}$*,*
(iii) if we take q *as an orthogonal vector to the remaining eigenvectors of* A*, that is,* $q^T x_i = 0$*,* $i = 2,\ldots,n$*, i.e.,* $q^T Y = O_{1\times(n-1)}$*,*
then the Jordan structures of A *and* $A + x_1q^T$ *are the same.*

Proof: Applying the statements in the proof of Theorem 1, it follows easily that

$$S^{-1}(A + x_1q^T)S = \begin{bmatrix} \lambda_1 + q^T x_1 & O \\ O & VAY \end{bmatrix} \quad \text{with}$$

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & O \\ O & VAY \end{bmatrix} = J_A.$$

Then the Jordan structure of the matrices are the same. ■

Although only the eigenvalue λ_k changes to $\mu_k = \lambda_k + x_k^T q$ and without changing any of the remaining eigenvalues, x_k is the only eigenvector of A that is preserved as eigenvector of the matrix $A + x_kq^T$. The other eigenvectors of A change as eigenvectors of $A + x_kq^T$ associated with the same eigenvalues λ_i, $i \neq k$. We can see these changes in the following result, which is well-known [8]. However we give the proof for completeness.

Proposition 1 *Let* A *be an* $n\times n$ *arbitrary matrix with eigenvalues* $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$*. Let* x_i *be an eigenvector of* A *associated with the eigenvalue* λ_i*, with* $1 \leq i \leq n$*. Let* q *be any* n*-dimensional vector and let* $\mu_k = \lambda_k + x_k^T q$*, with* $\mu_k \neq \lambda_i$*,* $i = 1, 2, \ldots, n$*. Then,* x_k *is an eigenvector of the matrix* $A + x_kq^T$ *associated with the eigenvalue* $\mu_k = \lambda_k + x_k^T q$*, and the eigenvectors of* $A + x_kq^T$ *associated with* λ_i*,* $i \neq k$*, are:*

$$w_i = x_i - \frac{q^T x_i}{\mu_k - \lambda_i}\, x_k.$$

Proof: As x_k is an eigenvector of A associated with the eigenvalue λ_k, we have $(A - \lambda_k I)x_k = 0$. Then,

$$(A + x_kq^T - (\lambda_k + x_k^T q)I)x_k = (A - \lambda_k I)x_k + (x_kq^T)x_k - (x_k^T q)x_k =$$
$$= 0 + x_k(q^T x_k) - (x_k^T q)x_k = < q, x_k > \, x_k - < x_k, q > \, x_k = 0$$

and so x_k is also an eigenvector of $A + x_kq^T$ associated with μ_k.

For the eigenvectors associated with the unchanged eigenvalues ($i \neq k$), we have

$$\begin{aligned}
(A + x_kq^T)\left(x_i - \frac{q^T x_i}{\mu_k - \lambda_i}\, x_k\right) &= Ax_i + (q^T x_i)x_k - \frac{q^T x_i}{\mu_k - \lambda_i}\, \mu_k x_k \\
&= \lambda_i x_i - \left(\frac{q^T x_i}{\mu_k - \lambda_i}\, \mu_k - q^T x_i\right) x_k \\
&= \lambda_i x_i - \lambda_i\, \frac{q^T x_i}{\mu_k - \lambda_i}\, x_k \\
&= \lambda_i \left(x_i - \frac{q^T x_i}{\mu_k - \lambda_i}\, x_k\right) \qquad ■
\end{aligned}$$

However, the changes of the left eigenvectors of A and $A + x_k q^T$ are in the opposite way as we can see in the next result for a diagonalizable matrix A.

Proposition 2 *Let A be an $n \times n$ diagonalizable matrix with eigenvalues $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Let l_i^T be a left eigenvector of A corresponding to λ_i, with $1 \leq i \leq n$. Let q be any n-dimensional vector and let $\mu_k = \lambda_k + x_k^T q$, with $\mu_k \neq \lambda_i$, $i = 1, 2, \ldots, n$. Then, the left eigenvectors of $A + x_k q^T$ corresponding to λ_i, $i \neq k$, are $r_i^T = l_i^T$, and the left eigenvector of $A + x_k q^T$ corresponding to μ_k is:*

$$r_k^T = l_k^T + \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{q^T x_i}{\mu_k - \lambda_i} \, l_i^T.$$

Proof: By l_i^T is a left eigenvector of A corresponding to λ_i, we have $l_i^T(A - \lambda_i I) = 0$, with $i \neq k$. Otherwise, $< l_i, x_k >= 0$, for all $i \neq k$, then

$$\begin{aligned} r_i^T(A + x_k q^T - \lambda_i I) &= l_i^T(A + x_k q^T - \lambda_i I) \\ &= l_i^T(A - \lambda_i I) + l_i^T(x_k q^T) = 0 + (l_i^T x_k) q^T \\ &= < l_i, x_k > q^T = 0, \end{aligned}$$

and l_i^T, $i \neq k$, is a left eigenvector of $A + x_k q^T$ corresponding to λ_i. With respect to the left eigenvector r_k^T, we have:

$$
\begin{aligned}
& r_k^T(A + x_k q^T) = \\
&= \left(l_k^T + \sum_{i=1 i \neq k}^{n} \frac{q^T x_i}{\mu_k - \lambda_i} l_i^T \right) (A + x_k q^T) \\
&= l_k^T A + l_k^T x_k q^T + \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{q^T x_i}{\mu_k - \lambda_i} l_i^T A + \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{q^T x_i}{\mu_k - \lambda_i} l_i^T x_k q^T \\
&= \lambda_k l_k^T + q^T + \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{q^T x_i}{\mu_k - \lambda_i} \lambda_i l_i^T \\
&= \lambda_k l_k^T + \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{q^T x_i}{\mu_k - \lambda_i} \lambda_i l_i^T + q^T (\underbrace{x_1 l_1^T + x_2 l_2^T + \cdots + x_n l_n^T}_{I}) \\
&= (\lambda_k + \underbrace{q^T x_k}_{\mu_k - \lambda_k}) l_k^T + \sum_{\substack{i=1 \\ i \neq k}}^{n} \left(\frac{q^T x_i}{\mu_k - \lambda_i} \lambda_i + q^T x_i \right) l_i^T
\end{aligned}
$$

$$
\begin{aligned}
&= \mu_k l_k^T + \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{q^T x_i}{\mu_k - \lambda_i} \mu_k l_i^T \\
&= \mu_k \left(l_k^T + \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{q^T x_i}{\mu_k - \lambda_i} l_i^T \right) = \mu_k r_k^T
\end{aligned}
$$

■

2 Related results

The Brauer's Theorem [1] can be applied to prove different related results as we can see in the next subsections. Note that some results are previous to the Brauer result.

2.1 Deflation techniques

In 1944 Wielandt gave a deflation method for general matrices shifting one eigenvalue to zero (see [10]). This result is an inmediate consequence of the Brauer's Theorem.

Corollary 2 (Wielandt's deflation) *Consider the conditions of Theorem 1. Let* q *be any vector such that* $q^T x_k = -\lambda_k$*, then the matrix* $A + x_k q^T$ *has the eigenvalues* $\{\lambda_1, \ldots, \lambda_{k-1}, 0, \lambda_{k+1}, \ldots, \lambda_n\}$*.*

Proof: Apply Brauer's Theorem with a vector q such that $q^T x_k = -\lambda_k$.

Remark 1 When the general matrix A is symmetric, then A is diagonalizable and we can choose an orthogonal matrix $X = [x_1 \ x_2 \ \ldots \ x_n]$ with the eigenvectors of A. In this case the matrix $B = A + (\mu_k - \lambda_k)x_k x_k^T$ is symmetric (diagonalizable) and it can be verified that the eigenvectors of B remaining eigenvalues are the eigenvectors of A.

The above result contains an older technique due to Hotelling in 1933 for symmetric matrices that can be extended to nonsymmetric matrices.

Corollary 3 (Hotelling's deflation) *Consider the conditions of Theorem 1.*
(i) Symmetric case. Let $q = -\lambda_k x_k$*, then the symmetric matrix* $A - \lambda_k x_k x_k^T$ *has the eigenvalues* $\{\lambda_1, \ldots, \lambda_{k-1}, 0, \lambda_{k+1}, \ldots, \lambda_n\}$*, provided that* $x_k^T x_k = 1$*.*
(ii) Nonsymmetric case. Let $q = -\lambda_k l_k$*, where* l_k *is the* k*-left eigenvector of* A*, with* $l_k^T x_k = 1$*. Then the matrix* $A - \lambda_k x_k l_k^T$ *has the eigenvalues* $\{\lambda_1, \ldots, \lambda_{k-1}, 0, \lambda_{k+1}, \ldots, \lambda_n\}$*.*

Proof: Apply Brauer's Theorem with a vector $q = -\lambda_k x_k$ in the symmetric case and $q = -\lambda_k l_k$ in the nonsymmetric case.

2.2 Google matrix

In the last decade a lot of work has been done on the Google matrix for computing the PageRank vector. To check mathematical properties of existence and convergence of the Page Rank power method a stochastic matrix is updated by a rank one matrix to construct the Google matrix. Then, in [6, Theorem 5.1] the relationship between the spectrum of both matrices is given. This result can be seen as a corollay of the Brauer's Theorem 1.

Corollary 4 *Let* A *be a row stochastic matrix with eigenvalues* $\sigma(A) = \{1, \lambda_2, \ldots, \lambda_n\}$*. Denote by* e *the eigenvector associated with the eigenvalue* 1*. Then the matrix* $\alpha A + (1-\alpha)ev^T$ *has eigenvalues* $\{1, \alpha\lambda_2, \ldots, \alpha\lambda_n\}$*, where* v^T *is a probability vector and* $0 < \alpha < 1$*.*

Proof: Apply the Brauer's Theorem 1 to the matrix αA with the vector $q = (1-\alpha)v$. ∎

2.3 Pole assignment of SISO systems

Another application can be obtained from the Brauer's Theorem 1 for single-input single-output (SISO) linear time invariant control systems when the system given by the pair (A, b) is not completely controllable. Concretely, given a SISO system we use an state feedback to place the poles of the closed-loop system at specified points in the complex plane. More precisely, the pole placement problem consist of:

> *Consider the pair (A, b) and let $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Let μ_k a number. Under what conditions on (A, b) does there exist a vector f such that the spectrum of the closep-loop system $A+bf^T$, $\sigma(A+bf^T)$, is $\{\lambda_1, \ldots, \lambda_{k-1}, \mu_k, \lambda_{k+1}, \ldots, \lambda_n\}$?*

The following result answers this question.

Proposition 3 *Consider the pair (A, b), let $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and let x_k be an eigenvector of A^T associated with λ_k. If $b^T x_k \neq 0$, then there exists a vector f such that $\sigma(A + bf^T) = \{\lambda_1, \ldots, \lambda_{k-1}, \mu_k, \lambda_{k+1}, \ldots, \lambda_n\}$.*

Proof: As $\sigma(A^T) = \sigma(A)$ and by the Brauer's Theorem 1 applied to A^T, the matrix $A^T + x_k q^T$ has eigenvalues $\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + q^T x_k, \lambda_{k+1}, \ldots, \lambda_n$, where q is any n-dimensional vector. It is clear that $\sigma(A + qx_k^T) = \{\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + q^T x_k, \lambda_{k+1}, \ldots, \lambda_n\}$.

Consider $q = b$ and $f = x_k$. If $b^T x_k \neq 0$, we have:

$$\lambda_k + q^T x_k = \lambda_k + b^T x_k = \mu_k \quad \Longrightarrow \quad b^T x_k = \mu_k - \lambda_k$$

then $\sigma(A + bf^T) = \{\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + q^T x_k, \lambda_{k+1}, \ldots, \lambda_n\}$. ■

Remark 2 (a) Note that the assumption of $b^T x_k \neq 0$ is needed only to assure the change of the eigenvalue λ_k. Otherwise no eigenvalue changes.

(b) By this result we can say that the pole assignment problem has a solution if x_k is not orthogonal to the vector b (that is, $b^T x_k \neq 0$) (see [2]). When this condition holds for all eigenvectors of A^T, then it is said that the pair (A, b) is completely controllable, in this case the solution is unique [3].

(c) If $\mu_k \neq \lambda_i$, for $i = 1, 2, \ldots, n$, $i \neq k$, then the eigenvectors of A^T change as Proposition 1 shows, that is, the eigenvector of A^T associated with λ_k is the same. Further, the eigenvectors of A^T corresponding to λ_i, $i \neq k$, such that $b^T x_i = 0$ remain unchanged.

(d) If $\lambda_i \neq \lambda_j$ for each $i \neq j$, and $b^T x_i \neq 0$, then it is also obtained that $b^T w_i \neq 0$, where w_i is defined in Proposition 1.

Example 1 Consider the pair (A, b)

$$A = \begin{bmatrix} -2 & -3 & -2 & 0 \\ 2 & 3 & 2 & 0 \\ 3 & 3 & 3 & 0 \\ 0 & 1 & -2 & 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

This pair (A, b) is not completely controllable since the rank of the controllability matrix

$$\mathcal{C}(A, b) = [b \ Ab \ A^2b \ A^3b] = \begin{bmatrix} 0 & -2 & -8 & -26 \\ 0 & 2 & 8 & -26 \\ 1 & 3 & 9 & 27 \\ 1 & 0 & -4 & -18 \end{bmatrix}$$

is 3. Note that $\sigma(A) = \sigma(A^T) = \{0, 1, 2, 3\}$ and the eigenvectors of A^T are:

$$\begin{aligned} x^T_{\lambda=0} &= (\alpha_1, -\alpha_1, 0, 0) \quad \forall \alpha_1 \neq 0 \implies b^T x_{\lambda=0} = 0 \\ x^T_{\lambda=1} &= (\alpha_2, 0, \alpha_2, 0) \quad \forall \alpha_2 \neq 0 \implies b^T x_{\lambda=1} = \alpha_2 \\ x^T_{\lambda=2} &= (\alpha_3, 2\alpha_3, 0, \alpha_3) \quad \forall \alpha_3 \neq 0 \implies b^T x_{\lambda=2} = \alpha_3 \\ x^T_{\lambda=3} &= (\alpha_4, \alpha_4, \alpha_4, 0) \quad \forall \alpha_4 \neq 0 \implies b^T x_{\lambda=3} = \alpha_4 \end{aligned}$$

Although the system is not completely controllable, we can change all the eigenvalues of A, but $\lambda = 0$. For instance, if we change $\lambda = 3$ by $\mu = 0.7$ we consider the eigenvector of A^T associated with $\lambda = 3$ and obtain

$$b^T x_{\lambda=3} = \alpha_4 = 0.7 - 3 = -2.3 \quad \implies \quad \alpha_4 = -2.3$$

Then, $f^T = (-2.3, -2.3, -2.3, 0)$ and

$$A + bf^T = \begin{bmatrix} -2 & -3 & -2 & 0 \\ 2 & 3 & 2 & 0 \\ 0.7 & 0.7 & 0.7 & 0 \\ -2.3 & -1.3 & -4.3 & 2 \end{bmatrix} \text{ with } \sigma(A + bf^T) = \{0, 0.7, 1, 2\}.$$

Consider a SISO discrete-time (or continuous-time) invariant linear system given by the pair (A^T, b). Let $\sigma(A^T) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. The system is asymptotically stable if all eigenvalues λ_i of A^T satisfy $|\lambda_i| < 1$ (or $\mathrm{Re}(\lambda_i) < 0$), see for instance [3, 5]. Then applying properly Proposition 3 to an unstable pair (A, b) we can obtain the closed-loop system $A + bf^T$ with the feedback vector f equals to the eigenvector associated with the eigenvalue λ_k such that $|\lambda_k| \geq 1$ (or $\mathrm{Re}(\lambda_k) \geq 0$).

The following algorithm gives the stabilization of the SISO system (A^T, b) by Proposition 3 and the Power Method [8] assuming that A^T has a dominant eigenvalue. The advantage of the proposed method is that we do not need, in general, the system be completely controllable.

Algorithm Input: (A^T, b).

Step 1. Set $A_0 = A_1 = A$, $i = 1$ and f_0 the zero vector.

Step 2. Apply the power method to A_i, and obtain the dominant eigenvalue λ_i and the corresponding eigenvector x_i.

Step 3. If $|\lambda_i| < 1$, then the pair (A_i, b) is asymptotically stable, where $A_i = A_{i-1} + f_{i-1}b^T$. **END.**

Otherwise,

Step 4. If $< x_i, b >= 0$, then the pair (A_i, b) can not be stabilized (Proposition 3) **END.**

Otherwise,

Step 5. Choose an scalar α_i such that the new eigenvalue $\mu_i = \lambda_i + (\alpha_i x_i^T)b$ satisfies $|\mu_i| < 1$. Let $f_i = f_{i-1} + \alpha_i x_i$.

Step 6. Let $A_{i+1} = A_i + \alpha_i x_i b^T$. Note that $\sigma(A_{i+1}) = \{\lambda_1, \ldots, \lambda_{i-1}, \mu_i, \lambda_{i+1}, \ldots, \lambda_n\}$ with $|\mu_i| < 1$. Let $i = i + 1$, GOTO **Step 2.**

3 The Rado's theorem

Perfect [7] in 1955 presented the following result, due to R. Rado, which shows how to modify, in only one step, r eigenvalues of an arbitrary matrix A wihtout changing any of the remaining $(n - r)$ eigenvalues. The Rado's Theorem is an extension of the Brauer's Theorem and it has been applied to generate sufficient conditions for the existence and construction of nonnegative matrices with prescribed spectrum. As in the previous case, the immediate consequences of this result are the block deflation methods and the pole assignment problem when the MIMO linear control system is not completely controllable.

Theorem 2 *[9, Brauer Extended, Theorem 5] Let A be an $n \times n$ arbitrary matrix with eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Let $X = [x_1\, x_2 \ldots x_r]$ be an $n \times r$ matrix such that* $\operatorname{rank}(X) = r$ *and $Ax_i = \lambda_i x_i$, $i = 1, 2, \ldots, r$, $r \leq n$. Let C be an $r \times n$ arbitrary matrix. Then the matrix $A + XC$ has eigenvalues $\{\mu_1, \mu_2, \ldots, \mu_r, \lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_n\}$, where $\mu_1, \mu_2, \ldots, \mu_r$ are eigenvalues of the matrix $\Omega + CX$ with* $\Omega = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r)$.

Proof: It is very similar to the proof of Theorem 1 just working by blocks. ■

Theorem 2 shows how to change r eigenvalues of A in only one step. In general, the eigenvector x_i associated with λ_i of A, $i = 1, 2, \ldots, r$, is not the eigenvector associated with the new eigenvalue μ_i of $A + XC$. If the matrix $\Omega + CX$ is diagonalizable the way in which x_i changes is given below.

Proposition 4 *Let A be an $n \times n$ arbitrary matrix with eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Let $X = [x_1\, x_2\, \dots\, x_r]$ be an $n \times r$ matrix where its column vectors satisfy $Ax_i = \lambda_i x_i$, $i = 1, 2, \dots, r$, $r \leq n$. Let C be an $r \times n$ arbitrary matrix and let $\Omega = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_r)$.*

If $\mu_1, \mu_2, \dots, \mu_r$ are eigenvalues of the diagonalizable matrix $\Omega + CX$ and T is the transition matrix to its Jordan form, then the column vectors of the matrix product XT are the eigenvectors of $A + XC$ associated with $\mu_1, \mu_2, \dots, \mu_r$.

Proof: From the transition matrix T we have

$$(A + XC)X = X(\Omega + CX) = XT\,\mathrm{diag}(\mu_1, \mu_2, \dots, \mu_r)T^{-1}$$

then

$$(A + XC)XT = XT\,\mathrm{diag}(\mu_1, \mu_2, \dots, \mu_r)$$

and the result follows. ■

Remark 3 If we take the arbitrary matrix C such that

$$CX = \mathrm{diag}(\mu_1 - \lambda_1, \mu_2 - \lambda_2, \dots, \mu_r - \lambda_r)$$

then $\Omega + CX = \mathrm{diag}(\mu_1, \mu_2, \dots, \mu_r)$, and the matrix T, of Proposition 4, is equal to the identity matrix. Therefore, the eigenvector x_i associated with λ_i of A, $i = 1, 2, \dots, r$, is the eigenvector associated with the new eigenvalue μ_i of $A + XC$.

In this case, the eigenvectors associated with the eigenvalues $\lambda_{r+1}, \dots, \lambda_n$ change in the following way.

Proposition 5 *Assume the conditions of Theorem 2 and Remark 3. Let x_i be the eigenvector of A associated with the eigenvalue λ_i, $r + 1 \leq i \leq n$. Then, the eigenvectors of $A + XC$ associated with λ_i are given by:*

$$w_i = x_i - \sum_{j=1}^{r} \frac{c_j x_i}{\mu_j - \lambda_i}\, x_j \qquad r + 1 \leq i \leq n$$

where c_j is the j-th row of the matrix C.

Proof: For x_i, $r+1 \leq i \leq n$, we have

$$
\begin{aligned}
(A+XC)(x_i - \sum_{j=1}^{r} \frac{c_j x_i}{\mu_j - \lambda_i}\, x_j) = & \\
&= Ax_i + XCx_i - \sum_{j=1}^{r} (A+XC) \frac{c_j x_i}{\mu_j - \lambda_i}\, x_j \\
&= \lambda_i x_i + \sum_{j=1}^{r} (c_j x_i) x_j - \sum_{j=1}^{r} \frac{c_j x_i}{\mu_j - \lambda_i} \mu_j x_j \\
&= \lambda_i x_i - \sum_{j=1}^{r} \left(-(c_j x_i) + \frac{c_j x_i}{\mu_j - \lambda_i} \mu_j \right) x_j \\
&= \lambda_i \left(x_i - \sum_{j=1}^{r} \frac{c_j x_i}{\mu_j - \lambda_i} \right) x_j
\end{aligned}
$$

■

4 Applications of the Rado's Theorem

In this section we give the applications of the Rado's Theorem to deflation techniques and to the pole assignment problem for MIMO systems.

4.1 Block deflation techniques

Now using the Rado's Theorem 2 we can obtain a block version of the deflation results working with particular matrices C.

Corollary 5 (Wielandt's deflation) *Consider the conditions of Theorem 2. Let C be a matrix such that $\Omega + CX$ has all the eigenvalues zero. Then the matrix $B = A + XC$ has eigenvalues $\{0, 0, \ldots, 0, \lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_n\}$.*

Proof: It is a direct application of Rado's Theorem.

Remark 4 When the general matrix A is symmetric, then A is diagonalizable and we can choose an orthogonal matrix $X = [x_1 \ldots x_r \, x_{r+1} \ldots x_n] = [X_r \, X_{n-r}]$ with the eigenvectors of A. Consider $\Theta = \text{diag}(\mu_1 - \lambda_1, \mu_2 - \lambda_2, \ldots, \mu_r - \lambda_r)$, then the matrix $B = A + X_r \Theta X_r^T$ is symmetric (diagonalizable) and it can be verified that its eigenvectors associated with the eigenvalues $\lambda_{r+1}, \ldots, \lambda_n$ are the eigenvectors of A.

Corollary 6 (Hotelling's deflation) *Consider the conditions of Theorem 2.* *(i) Symmetric case. Let* $C = -\Omega X^T$*, then the symmetric matrix* $A + XC$ *has the eigenvalues* $\{0, 0, \ldots, 0,\ \lambda_{r+1},\ \lambda_{r+2},\ \ldots,\ \lambda_n\}$*, provided that* $X^T X = I_r$*.*
(ii) Nonsymmetric case. Let $X = [x_1, x_2, \ldots, x_r]$ *and* $L = [l_1, l_2, \ldots, l_r]$ *be* $n \times r$ *matrices such that* $\text{rank}(X) = \mathit{rank}(L) = r$*,* $Ax_i = \lambda_i x_i$*,* $l_i^T A = \lambda_i l_i^T$ *and* $L^T X = I$*. Let* $C = -\Omega L^T$*, then the matrix* $B = A + XC$ *has eigenvalues* $\{0, 0, \ldots, 0, \lambda_{r+1}, \lambda_{r+2}, \ldots,\ \lambda_n\}$*.*

Proof: Apply Rado's Theorem with $C = -\Omega X^T$ for the symmetric case and with $C = -\Omega L^T$ for the nonsymmetric case.

Remark 5 It is easy to check that the matrices A and $A + XC$ have the same eigenvectors and the same Jordan structure associated with the eigenvalues $\lambda_{r+1},\ \lambda_{r+2},\ \ldots,\ \lambda_n$.

4.2 Pole assignment of MIMO systems

An immediate application of the Rado's Theorem 2 to the Control Theory in multi-input, multi-output (MIMO) systems defined by the pair (A, B) is the following problem, where we assume that the new eigenvalues μ_i are different from the eigenvalues to be changed λ_j, $1 \le i, j \le r$. .

> *Consider the pair* (A, B) *with* A $n \times n$ *and* B $n \times m$ *matrices and the set of numbers* $\{\mu_1, \mu_2, \ldots,\ \mu_r\}$*, and let* $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$*. What are the conditions on* (A, B) *so that the spectrum of the closed loop matrix* $A{+}BF^T$*,* $\sigma(A{+}BF^T)$*, coincides with the set* $\{\mu_1, \mu_2, \ldots,$ $\mu_r,\ \lambda_{r+1},\ \lambda_{r+2},\ \ldots,\ \lambda_n\}$*, for some matrix* F*?*

The following result answers this question.

Proposition 6 *Consider the pair* (A, B)*, with* A $n \times n$ *and* B $n \times m$ *matrices. Let* $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$*. Let* $X = [x_1\, x_2\, \ldots\, x_r]$ *be an* $n \times r$ *matrix such that* $\text{rank}(X) = r$ *and* $A^T x_i = \lambda_i x_i$*,* $i = 1, 2, \ldots, r$*,* $r \le n$*. If there is a column* b_{j_i} *of the matrix* B *such that* $b_{j_i}^T x_i \ne 0$*, for all* $i = 1, 2, \ldots, r$*, then there exists a matrix* F *such that* $\sigma(A + BF^T) = \{\mu_1, \mu_2, \ldots,\ \mu_r,\ \lambda_{r+1},\ \lambda_{r+2},\ \ldots,\ \lambda_n\}$*.*

Proof: As $\sigma(A^T) = \sigma(A)$ and by the Rado's Theorem 2 applied to A^T, we have that $\sigma(A^T + XC) = \{\mu_1, \mu_2, \ldots, \mu_r, \lambda_{r+1},\ \lambda_{r+2}, \ldots,\ \lambda_n\}$, where $\{\mu_1, \mu_2,$ $\ldots,\ \mu_r\}$ are the eigenvalues of $\Omega + CX$, with $\Omega = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r)$. Then, $\sigma(A + C^T X^T) = \{\mu_1, \mu_2, \ldots, \mu_r,\ \lambda_{r+1},\ \lambda_{r+2},\ \ldots,\ \lambda_n\}$.

Let $C^T = [b_{j_1}\ b_{j_2}\ \ldots\ b_{j_r}]$, where $b_{j_i}^T x_i \ne 0$ for $i = 1, 2, \ldots, r$. Then

$$A + C^T X^T = A + [b_{j_1}\ b_{j_2}\ \ldots\ b_{j_r}]X^T = A + B[e_{j_1}\ e_{j_2}\ \ldots\ e_{j_r}]X^T$$

where the matrix $[e_{j_1}\ e_{j_2}\ \ldots\ e_{j_r}]$ is formed by the corresponding unit vectors. Setting $F^T = [e_{j_1}\ e_{j_2}\ \ldots\ e_{j_r}]X^T$, we have

$$\sigma(A + C^T X^T) = \sigma(A + BF^T) = \{\mu_1, \mu_2, \ldots, \mu_r, \lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_n\}$$

where $\{\mu_1, \mu_2, \ldots, \mu_r\}$ are the eigenvalues of $\Omega + [e_{j_1}\ e_{j_2}\ \ldots\ e_{j_r}]^T B^T X$, with $\Omega = \mathrm{diag}(\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_r)$. ■

Remark 6 (a) Note that the assumption of the existence of a column b_{j_i} of the matrix B such that $b_{j_i}^T x_i \neq 0$, for $i = 1, 2, \ldots, r$, is needed only to assure the change of the eigenvalue λ_i. Otherwise no eigenvalue changes.

(b) In the MIMO systems the solution of the pole assignment is not unique as we can see in the next example. Further, note that Proposition 6 indicates that we can locate poles even in the case of uncontrollable systems.

Proposition 6 is illustrated with the following example.

Example 2 Consider the pair (A, B) where

$$A = \begin{bmatrix} -2 & -3 & -2 & 0 \\ 2 & 3 & 2 & 0 \\ 3 & 3 & 3 & 0 \\ 0 & 1 & -2 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Note that this pair is not completely controlable since the rank of the matrix

$$\mathcal{C}(A, B) = [B\ AB\ A^2B\ A^3B] = \begin{bmatrix} 0 & 0 & -2 & -2 & -8 & -8 & -26 & -26 \\ 0 & 0 & 2 & 2 & 8 & 8 & -26 & -26 \\ 1 & 1 & 3 & 3 & 9 & 9 & 27 & 27 \\ 1 & 1 & 0 & 0 & -4 & -4 & -18 & -18 \end{bmatrix}$$

is 3. The spectral computation gives $\sigma(A) = \sigma(A^T) = \{0, 1, 2, 3\}$ and the eigenvectors of A^T are:

$$\begin{aligned} x_{\lambda=0}^T &= (\alpha_1, -\alpha_1, 0, 0) \quad \forall \alpha_1 \neq 0 \implies B^T x_{\lambda=0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ x_{\lambda=1}^T &= (\alpha_2, 0, \alpha_2, 0) \quad \forall \alpha_2 \neq 0 \implies B^T x_{\lambda=1} = \begin{bmatrix} \alpha_2 \\ \alpha_2 \end{bmatrix} \\ x_{\lambda=2}^T &= (\alpha_3, 2\alpha_3, 0, \alpha_3) \quad \forall \alpha_3 \neq 0 \implies B^T x_{\lambda=2} = \begin{bmatrix} \alpha_3 \\ \alpha_3 \end{bmatrix} \\ x_{\lambda=3}^T &= (\alpha_4, \alpha_4, \alpha_4, 0) \quad \forall \alpha_4 \neq 0 \implies B^T x_{\lambda=3} = \begin{bmatrix} \alpha_4 \\ \alpha_4 \end{bmatrix} \end{aligned}$$

Since the above products are different from zero for the eigenvalues $\lambda = 1, \lambda = 2$ and $\lambda = 3$, we have considered three cases according with the number of eigenvalues we want to change and the number of columns of the matrix B.

Case 1. Suppose we want to change the eigenvalues $\lambda = 2$ and $\lambda = 3$, by $\mu = 0.5$ and $\mu = 0.7$, respectively. Then, $r = m$.

Since $b_1^T x_{\lambda=2} \neq 0$ and $b_1^T x_{\lambda=3} \neq 0$ then

$$C^T = [b_1\, b_1] = B \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and the matrix

$$\Omega + CX = \Omega + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} B^T X = \begin{bmatrix} 2+\alpha_3 & \alpha_4 \\ \alpha_3 & 3+\alpha_4 \end{bmatrix}$$

has the eigenvalues $\mu_1 = 0.5$ y $\mu_2 = 0.7$ when $\alpha_3 = 1.95$ and $\alpha_4 = -5.75$, so the feedback matrix F is

$$F^T = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} X^T = \begin{bmatrix} -3.8 & -1.85 & -5.75 & 1.95 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, the close-loop matrix

$$A + BF^T = \begin{bmatrix} -2 & -3 & -2 & 0 \\ 2 & 3 & 2 & 0 \\ -0.8 & 1.15 & -2.75 & 1.95 \\ -3.8 & -0.85 & -7.75 & 3.95 \end{bmatrix}$$

has the spectrum $\sigma(A + BF^T) = \{0, 0.5, 0.7, 1\}$.

Note that working with the two column vectors of the matrix B, we will obtain the feedback matrix

$$F^T = \begin{bmatrix} 1.95 & 3.9 & 0 & 1.95 \\ -5.75 & -5.75 & -5.75 & 0 \end{bmatrix}.$$

Case 2. Now, we want to change only the eigenvalue $\lambda = 3$ by $\mu = 0.7$, in this case $r < m$.

Since $b_1^T x_{\lambda=3} \neq 0$ then

$$C^T = [b_1] = B \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the matrix $\Omega + CX = \Omega + [1\,0]\, B^T X = 3 + \alpha_4$ has the eigenvalue $\mu = 0.7$ if $\alpha_4 = -2.3$, so the feedback matrix F is

$$F^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} X^T = \begin{bmatrix} -2.3 & -2.3 & -2.3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, the close-loop matrix is

$$A + BF^T = \begin{bmatrix} -2 & -3 & -2 & 0 \\ 2 & 3 & 2 & 0 \\ 0.7 & 0.7 & 0.7 & 0 \\ -2.3 & -1.3 & -4.3 & 2 \end{bmatrix}$$

whose spectrum is $\sigma(A + BF^T) = \{0, 0.7, 1, 2\}$.

Case 3. Finally, we want to change the three eigenvalues $\lambda = 1$, $\lambda = 2$ and $\lambda = 3$, by $\mu_1 = 0.2$, $\mu_2 = 0.5$ and $\mu_3 = 0.7$, respectively. In this case $r > m$.

Since $b_1^T x_{\lambda=1} \neq 0$, $b_1^T x_{\lambda=2} \neq 0$ and $b_1^T x_{\lambda=3} \neq 0$ then

$$C^T = [b_1\, b_1\, b_1] = B \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and the matrix

$$\Omega + CX = \Omega + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} B^T X = \begin{bmatrix} 1+\alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 & 2+\alpha_3 & \alpha_4 \\ \alpha_2 & \alpha_3 & 3+\alpha_4 \end{bmatrix}$$

has eigenvalues $\mu_1 = 0.2$, $\mu_2 = 0.5$ and $\mu_3 = 0.7$ when $\alpha_2 = -0.06$, $\alpha_3 = 3.51$ and $\alpha_4 = -8.05$, so the feedback matrix F is

$$F^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} X^T = \begin{bmatrix} -4.6 & -1.03 & -8.11 & 3.51 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, the close-loop matrix

$$A + BF^T = \begin{bmatrix} -2 & -3 & -2 & 0 \\ 2 & 3 & 2 & 0 \\ -1.6 & 1.97 & -5.11 & 3.51 \\ -4.6 & -0.03 & -10.11 & 5.51 \end{bmatrix}$$

with spectrum $\sigma(A + BF^T) = \{0,\, 0.2,\, 0.5,\, 0.7\}$.

Remark 7 As before a MIMO discrete-time (or continuous-time) invariant linear system, given by the pair (A^T, B) is asymptotically stable if all eigenvalues λ_i of A^T satisfy $|\lambda_i| < 1$ (or Re $(\lambda_i) < 0$), see for instance [3, 5]. Then applying properly Proposition 6 to an unstable pair (A^T, B) we can obtain the closed-loop system $A + BF^T$ with the feedback matrix F computed as in the proof of the above proposition.

References

[1] A. Brauer. *Limits for the characteristic roots of matrices IV: Applications to stochastic matrices.* Duke Math. J., vol. 19 (1952), 75-91.

[2] P. E. Crouch. *Introduction to Mathematical Systems Theory.* Lecture notes for a joint course at the Universities of Warwick and Bremen. Control Theory Centre Publication. Forschungsschwerpunkt Dynamische Systeme, 1983.

[3] D. F. Delchamps. *State-Space and Input-Output Linear Systems*, Springer-Verlag, New York, 1988.

[4] M. L. J. Hautus. *Controllability and observability condition of linear autonomous systems.* Ned. Akad. Wetenschappen, Proc. Ser. A, vol. 72, (1969), 443-448.

[5] T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, 1980.

[6] A. N. Langville, C. D. Meyer. *Deeper Inside PageRank*. Internet Mathematics, vol. 1 (2004), 1-33.

[7] H. Perfect. *Methods of constructing certain stochastic matrices. II.* Duke Math. J., vol. 22 (1955), 305-311.

[8] Y. Saad. *Numerical Methods for large eigenvalue problems*, SIAM, 2011.

[9] R. L. Soto, O. Rojo. *Applications of a Brauer theorem in the nonnegative inverse eigenvalue problem*. Linear Algebra and its Applications, vol. 416 (2006), 844-856.

[10] J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Oxford Science, Oxford, England, 1965.