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A BRIEF PROOF OF A MAXIMAL RANK THEOREM FOR GENERIC DOUBLE POINTS IN PROJECTIVE SPACE

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Abstract. We give a simple proof of the following theorem of J. Alexander and A. Hirschowitz: Given a general set of points in projective space, the homogeneous ideal of polynomials that are singular at these points has the expected dimension in each degree of 4 and higher, except in 3 cases.

1. Introduction

Given a general collection of d points in $\mathbb{P}^n_{\mathcal{K}}$ ($\mathcal{K}$ an infinite field), consider the codimension in the space $H^0(\mathcal{O}_{\mathbb{P}^n}(m))$ of homogeneous polynomials of degree m of those that are singular at each of the d points. Since specifying that such a polynomial together with its first derivatives vanish at d points amounts to the imposition of $(n+1)d$ linear constraints on its $\binom{n+m}{m}$ coefficients, the expected codimension is $\min\left((n+1)d, \binom{n+m}{m}\right)$.

The interpolation problem may be rephrased in terms of double points. A **double point** is the scheme defined by the square of the ideal sheaf of a point. Hence a homogeneous polynomial is singular at a point precisely if it vanishes at the double point supported there. Then if X is a collection of d double points in $\mathbb{P}^n$ given by $\mathcal{I}^2$, where $\mathcal{I}$ is the ideal sheaf of a set of simple points, the following statements are equivalent:

- The vector space of homogeneous polynomials of degree m that are singular at the d points has the expected codimension.
- X has the generic Hilbert function in degree m,
$$h_{\mathbb{P}^n}(X, m) = \min\left((n+1)d, \binom{n+m}{m}\right).$$
- The natural map $H^0(\mathcal{O}_{\mathbb{P}^n}(m)) \to H^0(\mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^n}(m))$ has maximal rank (i.e. it is either surjective or injective).
- $H^1(\mathcal{I}^2 \otimes \mathcal{O}_{\mathbb{P}^n}(m)) = 0$ or $H^0(\mathcal{I}^2 \otimes \mathcal{O}_{\mathbb{P}^n}(m)) = 0$.
- X imposes independent conditions on the linear system $|\mathcal{O}_{\mathbb{P}^n}(m)|$ of m-ics in $\mathbb{P}^n$ or else X lies on no hypersurface of degree m.

The question of when a general collection of double points in projective space has the generic Hilbert function was solved completely by J. Alexander and

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A. Hirschowitz in a series of papers ([H], [A], [AH1], [AH2], [AH3]), which comprise the following:

Theorem 1. *Let $\mathcal{I}$ be the ideal sheaf of a general collection of d points in $\mathbb{P}^n$. If $m \geq 3$ then*

$$\dim H^0(\mathcal{O}_{\mathbb{P}^n}(m)) - \dim H^0(\mathcal{I}^2 \otimes \mathcal{O}_{\mathbb{P}^n}(m)) = \min\left((n+1)d, \binom{n+m}{m}\right)$$

except in the four following cases: $n = 2$, $d = 5$, $m = 4$; $n = 3$, $d = 9$, $m = 4$; $n = 4$, $d = 7$, $m = 3$; and $n = 4$, $d = 14$, $m = 4$.

In their proof, the first observation is that by semicontinuity it suffices to show that there exist collections of points in each $\mathbb{P}^n$ having the stated property. The basic technique in constructing such subsets is the *méthode d'Horace*. The main idea of this method is to specialize as many points as is convenient to a fixed hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ and apply induction on dimension and degree. More specifically, if $X \subset \mathbb{P}^n$ is a collection of double points (some of which are supported in the hyperplane), then the residual scheme $\tilde{X}$ of X with respect to $\mathbb{P}^{n-1}$ consists of the reduced points lying in $\mathbb{P}^{n-1}$ together with the remaining double points (not supported in $\mathbb{P}^{n-1}$). The restriction exact sequence

$$0 \to \mathcal{I}_{\tilde{X}}(-1) \to \mathcal{I}_X \to \mathcal{I}_{X \cap \mathbb{P}^{n-1}, \mathbb{P}^{n-1}} \to 0$$

yields the basic Castelnuovo inequality

$$h_{\mathbb{P}^n}(X, m) \geq h_{\mathbb{P}^n}(\tilde{X}, m-1) + h_{\mathbb{P}^{n-1}}(X \cap \mathbb{P}^{n-1}, m).$$

Thus, if one can specialize just enough double points of X to the hyperplane that, by suitable induction on dimension and degree, $\tilde{X}$ and $X \cap \mathbb{P}^{n-1}$ impose independent conditions on $|\mathcal{O}_{\mathbb{P}^n}(m-1)|$ and $|\mathcal{O}_{\mathbb{P}^{n-1}}(m)|$, respectively, then X imposes independent conditions on $|\mathcal{O}_{\mathbb{P}^n}(m)|$. Likewise if neither $\tilde{X}$ nor $X \cap \mathbb{P}^{n-1}$ lies on a hypersurface of degree $m-1$ or m, respectively, then X lies on no hypersurface of degree m. The problem that impedes this procedure is that since the degree of the scheme $X \cap \mathbb{P}^{n-1}$ is necessarily a multiple of n, it may be impossible to arrange that the degrees of $\tilde{X}$ and $X \cap \mathbb{P}^{n-1}$ are both less than or both greater than the dimensions of $H^0(\mathcal{O}_{\mathbb{P}^n}(m-1))$ and $H^0(\mathcal{O}_{\mathbb{P}^{n-1}}(m))$, respectively.

The *méthode différentielle* of [AH1] gives a way around this numerical obstacle. The idea is the following: Suppose that integers u, ϵ satisfy $nu < \binom{n+m-1}{m} < n(u+\epsilon)$. Choose a collection $X \subset \mathbb{P}^n$ of double points of which exactly u have support on a hyperplane, $\mathbb{P}^{n-1}$, along with a set Ψ of ϵ points on $\mathbb{P}^{n-1}$. Then induction on dimension should allow the conclusions that $X \cap \mathbb{P}^{n-1}$ is m-independent and that $\mathbb{P}^{n-1}$ has no hypersurface of degree m containing the union of $X \cap \mathbb{P}^{n-1}$ with the ϵ double points supported on Ψ. However, the latter scheme cannot impose independent conditions on $|\mathcal{O}_{\mathbb{P}^{n-1}}(m)|$ since its degree is too large. Instead, the system of degree m hypersurfaces through $(X \cap \mathbb{P}^{n-1}) \cup \Psi$ will have a nontrivial base locus supported on Ψ, a scheme Υ containing Ψ and contained in the union of double points on Ψ. The differential lemma [AH1] reveals by a deformation argument that $h_{\mathbb{P}^n}(\tilde{X} \cup \Upsilon, m-1)$ may then be used to compute the Hilbert function of the union of X with a **general** collection of ϵ double points of $\mathbb{P}^n$.

In [AH4], the differential lemma is used to give a simpler argument proving the result of Theorem 1 in the cases with degree $m \geq 5$. The approach is to choose $\epsilon = 1$ each time, that is, to concentrate the base locus at just one point. Then the "enhanced residual scheme" consists of double points, simple points in the hyperplane, and a scheme Υ supported at a point. Since the scheme Υ depends on the choice of reduced points on $\mathbb{P}^{n-1}$, the latter collection of points plays a sinister role. Thus the induction hypotheses involve some subtlety, since it is necessary for the induction on degree to guarantee that the scheme $\tilde{X} \cup \Upsilon$ be $(m-1)$-independent.

In this paper we give a yet simpler proof of maximal rank for degrees $m \geq 4$, using a special case of the *lemme différentiel*. There are two features that make the argument proceed smoothly. First is that we use a liberal strategy for specializing points, taking whichever ϵ is dictated by the numerics. Indeed, we specialize so many extra points that each of their contributions to the base locus is all of its neighbourhood with respect to $\mathbb{P}^{n-1}$. Then since the base locus scheme does not depend on the set of reduced points $(X \cap \mathbb{P}^{n-1})_{\text{red}}$, those points are set free. This is where the second novel ingredient in the proof appears: an easy lemma (lemma 3) for adding a collection of reduced points from a hyperplane to a given scheme while preserving maximal rank.

As a result, we obtain a very tidy induction argument (lemma 7) in which the induction hypothesis is precisely the statement of interest, that double points in $\mathbb{P}^n$ impose the expected number of conditions on the linear system of hypersurfaces of degree m. By induction on dimension, we may specialize some double points to $\mathbb{P}^{n-1}$ to obtain an m-independent scheme there. According to the differential lemma, we then must verify that the union of double points, neighbourhoods of the hyperplane and points in the hyperplane is $(m-1)$-independent. First, the union of double points and neighbourhoods is $(m-1)$-independent by induction on degree. Then we use lemma 3 to show that we may add points from the hyperplane to obtain an $(m-1)$-independent scheme. This is achieved by showing that the double points do not lie on an $(m-2)$-ic, again by induction on degree. Thus, for each degree and dimension we deduce the desired statement for double points from three cases of it, each in lower dimension or lower degree.

Here is an outline of the paper. We start in section 3 with the trivial but useful lemma 3 that allows us to unite reduced points from a hyperplane with an m-independent scheme and maintain m-independence. Then in section 4 we exhibit in lemma 5 the special case of the *lemme différentiel* that is both easy to prove and convenient to apply. The main induction step (lemma 7) is given in section 5. In section 6 we give initial cases of cubics ($m = 3$) that suffice in advancing to higher degrees. Finally, in section 7, we carry out the induction step.

2. Main Result and Notation

Let X be a subscheme of $\mathbb{P}^n = \mathbb{P}^n_{\mathcal{K}}$. We denote by $\mathcal{I}_X$ its ideal sheaf and I_X its homogeneous ideal. We write $h_{\mathbb{P}^n}(X, \cdot)$ for the **Hilbert function** of X, namely,

$$h_{\mathbb{P}^n}(X, m) = \dim H^0(\mathcal{O}_{\mathbb{P}^n}(m)) - \dim H^0(\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}^n}(m)).$$

If X is a zero-dimensional subscheme of $\mathbb{P}^n$ of degree d, we will say that X **imposes independent conditions on** m**-ics** (or is m**-independent**) if $h_{\mathbb{P}^n}(X, m) = d$. (Analogously, if $\mathcal{D}$ is a linear system we may refer to X as being $\mathcal{D}$-independent.)

When $X \subseteq \mathbb{P}^n$ and $H = \mathbb{P}^{n-1} \subset \mathbb{P}^n$ is a hyperplane we will define the **residual** of X with respect to H to be the scheme $\tilde{X}$ given by $\mathcal{I}_X : \mathcal{O}_{\mathbb{P}^n}(-H)$. The Hilbert functions of the schemes $X, \tilde{X}$, and $X \cap H$ are related by **Castelnuovo's**

inequality:

$$h_{\mathbb{P}^n}(X,m) \geq h_{\mathbb{P}^n}(\tilde{X},m-1) + h_{\mathbb{P}^{n-1}}(X \cap \mathbb{P}^{n-1},m).$$

If X is a nonsingular, reduced subscheme of a nonsingular variety V, we will denote by $X^2|_V$ its first infinitesimal neighbourhood in V, i.e. the scheme defined by the square of the ideal sheaf of X in $\mathcal{O}_V$. We will abbreviate $X^2|_V$ by X^2 when the ambient variety V is understood.

For $n,m,d \in \mathbb{N}$ we abbreviate by $AH_{n,m}(d)$ the statement:

$AH_{n,m}(d)$: There is a collection of d points $\Gamma \subset \mathbb{P}^n$ so that

$$h_{\mathbb{P}^n}(\Gamma^2,m) = \min\left((n+1)d, \binom{n+m}{m}\right).$$

We shall prove

Theorem 2. *Let $n,m,d \in \mathbb{N}$ and $m \geq 1$. Then $AH_{n,m}(d)$ holds provided that:*

- $m \geq 5$,
- $m = 4$ *and* $n \geq 5$,
- $m = 4$, $n \leq 4$ *and* $d \neq \lfloor \frac{1}{n+1}\binom{n+4}{4} \rfloor$,
- $m = 3$ *and* $(n+1)d \geq \binom{n+3}{3} + \binom{n}{2}$ *or* $(n+1)d \leq \binom{n+3}{3} - \binom{n}{2}$, *or*
- $m = 2$ *and* $d \geq n+1$.

3. A preliminary lemma

In this section we give a criterion for adding a collection of reduced points in a hyperplane to an m-independent scheme and still obtaining an m-independent scheme.

Lemma 3. *Fix $\mathbb{P}^{n-1} \subset \mathbb{P}^n$. Let $X \subset \mathbb{P}^n$ be a subscheme. Then there is a collection $\Phi \subset \mathbb{P}^{n-1}$ of u points so that*

$$h_{\mathbb{P}^n}(X \cup \Phi, m) = h_{\mathbb{P}^n}(X,m) + u$$

if and only if

$$h_{\mathbb{P}^n}(X,m) + u \leq h_{\mathbb{P}^n}(\tilde{X},m-1) + \binom{n+m-1}{m}.$$

Proof. In the homogeneous coordinate ring of $\mathbb{P}^n$, let L be a linear form defining $\mathbb{P}^{n-1}$, and take the ideals I_X, $I_{X\cup\mathbb{P}^{n-1}}$ of X, $X \cup \mathbb{P}^{n-1}$. Then

$$I_{X\cup\mathbb{P}^{n-1}} : (L) = (I_X \cap (L)) : (L) = I_X : (L),$$

so X and $X \cup \mathbb{P}^{n-1}$ have the same residual $\tilde{X}$ with respect to $\mathbb{P}^{n-1}$. The restriction exact sequence for $X \cup \mathbb{P}^{n-1}$ then gives

$$h_{\mathbb{P}^n}(X \cup \mathbb{P}^{n-1},m) = h_{\mathbb{P}^n}(\tilde{X},m-1) + \binom{n+m-1}{m},$$

from which the result follows immediately. $\square$

4. A Horace lemma

In lemma 5 we present the special case of the *lemme d'Horace différentiel* required in the main argument. (See [AH1] for the original, and see [AH5] for a generalization to points of higher multiplicities.) The lemma gives an improvement on the basic Castelnuovo inequality in computing the Hilbert function of a subscheme $X \subset \mathbb{P}^n$ that meets a hyperplane $\mathbb{P}^{n-1}$. Namely, the Hilbert function of the union of X with general double points is obtained from that of the union of $\tilde{X}$ with neighbourhoods of the hyperplane, using specialization together with analysis of curvilinear subschemes. Key to the argument is the following observation, which is proved in [C1] and is also implicit in [CTV]:

Lemma 4 ([C1], Corollary 2.4)**.** *Let Λ be a closed subscheme of a collection of double points in $\mathbb{P}^n$ and let $\mathcal{D}$ be a linear system on $\mathbb{P}^n$. Then Λ imposes independent conditions on $\mathcal{D}$ if and only if every curvilinear subscheme of Λ is $\mathcal{D}$-independent.*

Proof. Suppose that Λ is supported at a single point $p \in \mathbb{P}^n$. Suppose that every curvilinear subscheme of Λ is $\mathcal{D}$-independent. If $\deg \Lambda \leq 2$ we are done. Otherwise, by induction on degree it suffices to produce a subscheme $\Lambda' \subset \Lambda$ of degree $\deg \Lambda' = \deg \Lambda - 1$ and a section of $\mathcal{D}$ vanishing on Λ' but not Λ. Let us choose a (curvilinear) subscheme $\xi \subset \Lambda$ of degree 2. By hypothesis, we may choose a section s of $\mathcal{D}$ vanishing on p but not ξ. Then we take as Λ' the intersection of Λ with the zero locus of s.

Now if Λ is supported at points $p_1, \dots, p_d$, then we may apply the previous argument to the system $\mathcal{D}' \subset \mathcal{D}$ defined by vanishing on $\Lambda \cap \{p_1, \dots, p_{d-1}\}^2$, and induct on d. $\square$

Lemma 5. *Choose a hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$. Let $X \subset \mathbb{P}^n$, and $\tilde{X}$ its residual with respect to $\mathbb{P}^{n-1}$. Let $a = h_{\mathbb{P}^n}(\tilde{X}, m-1)$ and $b = h_{\mathbb{P}^{n-1}}(X \cap \mathbb{P}^{n-1}, m)$. Assume that $h_{\mathbb{P}^n}(X, m) = a + b$. Suppose that ϵ is a nonnegative integer so that*

$$a + n\epsilon = \binom{n+m-1}{m-1}$$

and that $q_1, \dots, q_\epsilon \in \mathbb{P}^{n-1}$ satisfy

$$h_{\mathbb{P}^n}(\tilde{X} \cup \{q_1, \dots, q_\epsilon\}^2|_{\mathbb{P}^{n-1}}, m-1) = a + n\epsilon.$$

Then there are points $p_1, \dots, p_\epsilon \in \mathbb{P}^n - \mathbb{P}^{n-1}$ such that

$$h_{\mathbb{P}^n}(X \cup \{p_1, \dots, p_\epsilon\}^2, m) = \min\left(a + b + (n+1)\epsilon, \binom{n+m}{m}\right).$$

Proof. Let $\delta = \min\left(\epsilon, \binom{n+m-1}{m} - b\right)$. We may assume that

$$h_{\mathbb{P}^{n-1}}(X \cap \mathbb{P}^{n-1} \cup \{q_1, \dots, q_\delta\}, m) = b + \delta.$$

Choose general $p_1, \dots, p_\epsilon \in \mathbb{P}^n$ together with a flat family degenerating p_i to q_i and H_i to $\mathbb{P}^{n-1}$, where H_i is a hyperplane containing p_i for $i = 1, \dots, \epsilon$. Let

$$\Lambda = \{p_1, \dots, p_\delta\}^2 \cup \{p_{\delta+1}\}^2|_{H_{\delta+1}} \cup \ldots \cup \{p_\epsilon\}^2|_{H_\epsilon}.$$

Let ξ' be any curvilinear subscheme of Λ of degree 2ϵ. Write ξ' as a disjoint union $\xi' = \zeta' \cup \gamma'$, where $\zeta' \cap p_i^2|_{H_i} \subseteq \{p_i\}$ for $i = 1, \dots, \epsilon$; in particular, $\zeta' \cap \{p_i\} = \emptyset$

if $i > \delta$. Let ζ, γ be the limits of ζ', γ' on $\{q_1, \ldots, q_\epsilon\}$. So $\gamma \subseteq \{q_1, \ldots, q_\epsilon\}^2|_{\mathbb{P}^{n-1}}$ and $\zeta \cap \mathbb{P}^{n-1} \subseteq \{q_1, \ldots, q_\delta\}$. By semicontinuity we have

$$\begin{aligned} h_{\mathbb{P}^n}(X \cup \zeta' \cup \gamma', m) &\geq h_{\mathbb{P}^n}(X \cup \zeta \cup \gamma', m) \\ &\geq h_{\mathbb{P}^n}(\tilde{X} \cup \tilde{\zeta} \cup \gamma', m-1) + h_{\mathbb{P}^{n-1}}((X \cup \zeta) \cap \mathbb{P}^{n-1}\}, m) \\ &\geq h_{\mathbb{P}^n}(\tilde{X} \cup \tilde{\zeta} \cup \gamma, m-1) + b + \deg(\zeta \cap \mathbb{P}^{n-1}) \\ &= a + b + \deg \tilde{\zeta} + \deg \zeta \cap \mathbb{P}^{n-1} + \deg \gamma. \\ &= a + b + 2\epsilon. \end{aligned}$$

Hence by lemma 4 we have

$$h(X \cup \Lambda, m) = a + b + n\epsilon + \delta = \min\left(a + b + (n+1)\epsilon, \binom{n+m}{m}\right).$$

□

5. Main induction argument

In lemma 7 we present the main induction argument for deducing $AH_{n,m}(d)$ by induction on n and m (n = dimension, m = degree, d = number of points).

Given n, m, d, we specify in definition 1 the number $u = u_{n,m}(d)$ of points to specialize to a hyperplane and the number $\epsilon = \epsilon_{n,m}(d)$ used in applying lemma 5. Then $AH_{n,m}(d)$ reduces to $AH_{n-1,m}(u)$ together with the assertion that in $\mathbb{P}^n$ the general union of $d - u - \epsilon$ double points, ϵ neighbourhoods of $\mathbb{P}^{n-1}$ and u points of $\mathbb{P}^{n-1}$ impose the expected number of conditions in degree $m-1$. For the latter, $AH_{n,m-1}(d-u)$ implies that the union of double points and neighbourhoods is as expected in degree $(m-1)$. Then lemma 3 is applied to $AH_{n,m-2}(d-u-\epsilon)$ to add the u simple points. Hence $AH_{n,m}(d)$ is obtained from three instances of AH, one in lower dimension and two in lower degree.

Definition 1. Let $n, m, d \in \mathbb{N}$. Let $u = u_{n,m}(d), \epsilon = \epsilon_{n,m}(d)$ be the integers given as follows: Suppose that there is a $u \leq d$ so that either

$$nu \leq \binom{n+m-1}{m} \text{ and } (n+1)(d-u) + u \leq \binom{n+m-1}{m-1}$$

or

$$nu \geq \binom{n+m-1}{m} \text{ and } (n+1)(d-u) + u \geq \binom{n+m-1}{m-1}.$$

Then fix the minimal such u and set $\epsilon = 0$. Otherwise, perform division with remainder to write

$$nu + \epsilon = (n+1)d - \binom{n+m-1}{m-1}, \qquad 0 \leq \epsilon \leq n-1.$$

Lemma 6 (Numerics). *Let $n \geq 2$, $m \geq 4$, $d \geq 0$. Then*

$$u_{n,m}(d) + n\epsilon_{n,m}(d) \leq \binom{n+m-2}{m-1}.$$

Proof. Set $\epsilon = \epsilon_{n,m}(d), u = u_{n,m}(d)$.

For $n = 2$, $u + n\epsilon \leq 2 + \frac{m}{2} \leq m$.

Now suppose $n \geq 3$.

The cases $m = 4, n = 3, 4$ may be checked by hand, observing that $\epsilon = 0$ throughout.

In general, we have $\epsilon \leq n-1$ and

$$u \leq \left\lceil \frac{1}{n}\binom{n+m-1}{m} \right\rceil < \frac{1}{n}\binom{n+m-1}{m} + 1,$$

so that

$$u + n\epsilon < \frac{1}{n}\binom{n+m-1}{m} + 1 + n(n-1).$$

Let

$$P(n,m) = \binom{n+m-2}{m-1} - \frac{1}{n}\binom{n+m-1}{m} - 1 - n(n-1).$$

Then

$$P(n,4) = \frac{1}{8}(n-5)(n^2-n+2),$$

so $P(n,4) \geq 0$ for $n \geq 5$.

We have $P(3,5) > 0, P(4,5) > 0$,

$$P(n,m) - P(n,m-1) = \binom{n+m-3}{m-1}\frac{nm-n-m+2}{nm} > 0,$$

and hence $P(n,m) > 0$ for $m \geq 5$ as well. □

Lemma 7 (Main Induction Argument)**.** *Let n, m, d be given, with $m \geq 4$. Let $u = u_{n,m}(d)$, $\epsilon = \epsilon_{n,m}(d)$ be the integers given by Definition 1.*

Suppose that $AH_{n-1,m}(u)$, $AH_{n,m-1}(d-u)$ and $AH_{n,m-2}(d-u-\epsilon)$ all hold. Then so does $AH_{n,m}(d)$.

Proof. Let us focus on the hypothesis that $\epsilon > 0$ or $(n+1)d \geq \binom{n+m}{m}$, since the remaining (easier) case follows the same path.

According to $AH_{n,m-1}(d-u)$ we may choose a set Ψ of ϵ points of $\mathbb{P}^n$ along with a collection $\Sigma \subset \mathbb{P}^n$ of $d-u-\epsilon$ points so that

$$h_{\mathbb{P}^n}(\Sigma^2 \cup \Psi^2, m-1) = \min\left((n+1)(d-u), \binom{n+m-1}{m-1}\right).$$

Fix a hyperplane $\mathbb{P}^{n-1}$ containing Ψ.

If $\epsilon > 0$, then by Definition 1 we have

$$(n+1)(d-u-\epsilon) + u + n\epsilon = \binom{n+m-1}{m-1}$$

and $u \geq n$, so

$$(n+1)(d-u) \leq \binom{n+m-1}{m-1}.$$

Hence $\Sigma^2 \cup \Psi^2$ is $(m-1)$-independent, and therefore

$$h_{\mathbb{P}^n}(\Sigma^2 \cup \Psi^2|_{\mathbb{P}^{n-1}}, m-1) = (n+1)(d-u-\epsilon) + n\epsilon.$$

Next, by lemma 6 we have

$$(n+1)(d-u-\epsilon) \geq \binom{n+m-2}{m-2},$$

so $AH_{n,m-2}(d-u-\epsilon)$ ensures that Σ^2 lies on no $(m-2)$-ic. We deduce by lemma 3 that there is a collection $\Phi \subset \mathbb{P}^{n-1}$ of u points so that

$$h_{\mathbb{P}^n}(\Sigma^2 \cup \Phi \cup \Psi^2|_{\mathbb{P}^{n-1}}, m-1) = \binom{n+m-1}{m-1}.$$

Then by $AH_{n-1,m}(u)$ (together with a suitable choice of Φ) we have

$$h_{\mathbb{P}^{n-1}}(\Phi^2 \cup \Psi, m) = \min\left(nu + \epsilon, \binom{n+m-1}{m}\right).$$

Applying lemma 5 (or Castelnuovo's inequality) to the scheme $X = \Sigma^2 \cup \Phi^2$, we see that there is a set $\Psi_1 \subset \mathbb{P}^n$ of ϵ points so that

$$h_{\mathbb{P}^n}(\Sigma^2 \cup \Phi^2 \cup {\Psi_1}^2, m) = \min\left((n+1)d, \binom{n+m}{m}\right).$$

□

6. Degrees two and three

Here we collect the results from degrees two and three required in higher degrees. The main result, given in lemma 9, is that a collection of d double points imposes the expected number of conditions on cubics if its degree is not "too close" to $\dim H^0(\mathcal{O}_{\mathbb{P}^n}(3))$. With this allowance (which the higher degree cases happily provide) the proof proceeds using only the basic Castelnuovo inequality.

If Γ is a reduced subscheme of $\mathbb{P}^n$ we write $\operatorname{Sec}\Gamma$ for the union of the lines joining pairs of points of Γ.

Lemma 8. *Let $n \geq 0$. Fix a flag of projective spaces $\mathbb{P}^n \subset \mathbb{P}^{n+1}$. Let s, b be nonnegative integers so that $s + b \leq n+1$. Suppose that $\Gamma \subset \mathbb{P}^{n+1} - \mathbb{P}^n$ and $\Sigma \subset \mathbb{P}^n$ are collections of s and b points, respectively, so that $\Gamma \cup \Sigma$ is in linearly general position. Then*

$$h_{\mathbb{P}^n}(\operatorname{Sec}\Gamma \cap \mathbb{P}^n \cup \Sigma^2, 2) = \binom{s}{2} + (n+1)b - \binom{b}{2}.$$

In particular, $AH_{n,2}(d)$ holds when $d \geq n+1$.

Proof. For $n = 0$ the result holds trivially, so we may assume that $n \geq 1$.

The case $b = 0$ follows from an elementary argument that appears in, e.g. [C1].

Suppose, then, that $b \geq 1$. Let $M = \operatorname{span}\Sigma$. It is easy (by Bézout considerations, e.g.) to see that any quadric that vanishes on Σ^2 must also vanish on M^2. Hence

$$h_{\mathbb{P}^n}(\Sigma^2, 2) = h_{\mathbb{P}^n}(M^2, 2) = \binom{n+2}{2} - \binom{n+2-b}{2} = (n+1)b - \binom{b}{2}.$$

Set $S = \operatorname{Sec}\Gamma \cap \mathbb{P}^{n-1}$. Choose a hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ containing S and so that $M \not\subset \mathbb{P}^{n-1}$. Then by induction on n we have

$$\begin{aligned} h_{\mathbb{P}^n}(S \cup \Sigma^2, 2) &= h_{\mathbb{P}^n}(S \cup M^2, 2) \\ &\geq h_{\mathbb{P}^n}(M^2, 1) + h_{\mathbb{P}^{n-1}}(S \cup (M \cap \mathbb{P}^{n-1})^2, 2) \\ &= n + 1 + \binom{s}{2} + n(b-1) - \binom{b-1}{2} \\ &= \binom{s}{2} + (n+1)b - \binom{b}{2}, \end{aligned}$$

and therefore equality holds throughout. □

Lemma 9. *Let s, d be nonnegative integers. Assume that $s \leq \frac{n+2}{2}$ and either*

$$\binom{s}{2} + (n+1)d \leq \binom{n+3}{3} - \binom{n}{2}$$

or

$$\binom{s}{2} + (n+1)d \geq \binom{n+3}{3} + \binom{n}{2}.$$

Then there are collections of points $\Gamma \subset \mathbb{P}^{n+1}$ and $\Psi \subset \mathbb{P}^n$ of degrees s and d, respectively, so that

$$h_{\mathbb{P}^n}(\operatorname{Sec}\Gamma \cap \mathbb{P}^n \cup \Psi^2, 3) = \min\left(\binom{s}{2} + (n+1)d, \binom{n+3}{3}\right).$$

In particular (taking $s = 0$), $AH_{n,3}(d)$ is satisfied in each of the cases listed in Theorem 2.

Proof. We first observe that if $\Psi \subset \mathbb{P}^n$ is a collection of points, then any cubic that vanishes on Ψ^2 must vanish on $\operatorname{Sec}\Psi$ also. Indeed, any such cubic vanishes 4 times on each line between a pair of points of Ψ, so by Bézout's theorem it must vanish identically on each such line. Thus we have

$$h_{\mathbb{P}^n}(\Psi^2, 3) = h_{\mathbb{P}^n}(\Psi^2 \cup \operatorname{Sec}\Psi, 3).$$

Now assume that the result holds in dimension $n-1$ (observing the case $n = 1$ to start). Let s, d be given, and define

$$f(b) = \binom{s}{2} + nb - \binom{b}{2} + d$$

and

$$g(b) = \binom{b}{2} + n(d-b).$$

We shall focus on the hypothesis

$$\binom{s}{2} + (n+1)d \leq \binom{n+3}{3} - \binom{n}{2};$$

the proof in the other situation is its mirror image.

We show that there is an integer $b \leq d$ with $1 \leq b \leq \frac{n+1}{2}$ so that

$$f(b) \leq \binom{n+2}{2} \text{ and } g(b) \leq \binom{n+2}{3} - \binom{n-1}{2}.$$

Clearly it suffices to verify this when d has the extreme value

$$d = \left\lfloor \frac{1}{n+1}\left(\binom{n+3}{3} - \binom{n}{2} - \binom{s}{2}\right)\right\rfloor.$$

Then (after a little calculation) we see that $f(1) < \binom{n+2}{2}$, $d > \frac{n+1}{2}$,

$$g\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right) \leq \binom{n+2}{3} - \binom{n-1}{2},$$

and hence there is a b, $1 \leq b \leq \frac{n+1}{2} < d$, that is minimal to satisfy the property $g(b) \leq \binom{n+2}{3} - \binom{n-1}{2}$. If $b = 1$ we are done. Otherwise, minimality reveals that

$$g(b) + n - b + 1 = g(b-1) \geq \binom{n+2}{3} - \binom{n-1}{2} + 1,$$

and hence

$$f(b) \leq \binom{n+3}{3} - \binom{n}{2} - g(b) \leq \binom{n+2}{2} - b + 1 \leq \binom{n+2}{2}.$$

(For the case $\binom{s}{2} + (n+1)d \geq \binom{n+3}{3} + \binom{n}{2}$, one may compute: $f(0) < \binom{n+2}{2}$, $f(\lfloor \frac{n+1}{2} \rfloor) \geq \binom{n+2}{2}$, choose b minimal so that $f(b) \geq \binom{n+2}{2}$, and then proceed just as below.)

By the induction hypothesis we may choose a general set of $d-b$ points $\Phi \subset \mathbb{P}^{n-1}$ and a set $\Sigma \subset \mathbb{P}^n$ of b points so that

$$h_{\mathbb{P}^{n-1}}(\operatorname{Sec}\Sigma \cap \mathbb{P}^{n-1} \cup \Phi^2, 3) = \binom{b}{2} + n(d-b).$$

Further, by lemma 8 together with lemma 3 we may arrange that there is a set of s points $\Gamma \subset \mathbb{P}^{n+1}$ so that

$$h_{\mathbb{P}^n}(\operatorname{Sec}\Gamma \cap \mathbb{P}^n \cup \Sigma^2 \cup \Phi, 2) = \binom{s}{2} + (n+1)b - \binom{b}{2} + d - b.$$

Applying Castelnuovo's inequality to $X = \operatorname{Sec}\Gamma \cap \mathbb{P}^n \cup \Sigma^2 \cup \operatorname{Sec}\Sigma \cup \Phi^2$, we conclude that

$$h_{\mathbb{P}^n}(\operatorname{Sec}\Gamma \cap \mathbb{P}^n \cup \Sigma^2 \cup \Phi^2, 3) = \binom{s}{2} + (n+1)d.$$

$\square$

7. Conclusion of the proof of Theorem 2

We carry out the induction procedure of lemma 7 to complete the proof of Theorem 2, from the initial cases of degrees 2 and 3 (in Section 6) and those of dimension 1 (standard one-variable interpolation).

Let $n \geq 2$, $m \geq 4$, and d be given as in Theorem 2. (Here n is the dimension, m the degree, and d the number of points.) We use lemma 7 to verify $AH_{n,m}(d)$.

We may assume without loss of generality that d satisfies

$$(n+1)(d-1) < \binom{n+m}{m} < (n+1)(d+1)$$

except when $m = 4$ and $n \leq 4$, where we assume that $d = \lfloor \frac{1}{n}\binom{n+4}{4} \rfloor \pm 1$. (That is, $AH_{n,m}(d)$ holds if there is a set of d double points of $\mathbb{P}^n$ that either is contained in an m-independent scheme or contains a scheme that lies on no m-ics.)

Let $u = u_{n,m}(d), \epsilon = \epsilon_{n,m}(d)$ be the specialisation numbers given by Definition 1. Then we must verify that $AH_{n-1,m}(u)$, $AH_{n,m-1}(d-u)$, and $AH_{n,m-2}(d-u-\epsilon)$ are all valid.

For $m \geq 6$ and $n \geq 6$ these are automatic. But for lower dimension and lower degree some extra checking of the numbers $d-u$ and $d-u-\epsilon$ is required to assure that $AH_{n,m-1}(d-u)$ and $AH_{n,m-2}(d-u-\epsilon)$ are known. The only case that cannot be deduced from lemma 7 is that of $n = 5$, $m = 4$, and $d = 21$. Indeed, we have $u = 14$, $\epsilon = 0$, and the premise $AH_{4,4}(14)$ (we shall soon see) is not achieved. We must therefore deal with $AH_{5,4}(21)$ *ad hoc*.

We divide the proof by degree $m \geq 6$, $m = 5$, and $m = 4$, followed by the special case in $\mathbb{P}^5$.

Degrees 6 and higher. Assume that the degree m is at least 6.

Then $AH_{n-1,m}(u)$ holds by induction on n, as does $AH_{n,m-1}(d-u)$ by induction on m. Likewise $AH_{n,m-2}(d-u-\epsilon)$ is known with the possible exception of $m=6$ and $n \leq 4$. For these we observe that $d-u-\epsilon > \lfloor \frac{1}{n}\binom{n+4}{4} \rfloor$:

n	2	2	3	4
d	9	10	21	42
u	3	4	9	21
ϵ	0	0	1	0

Thus, $AH_{n,m}(d)$ holds subject to the lower degree cases of Theorem 2, according to lemma 7.

Degree 5. Consider the case $m=5$. We have $AH_{n-1,5}(u)$ by induction on n, and we have $AH_{n,4}(d-u)$, except perhaps when $n \leq 4$. We check these:

n	2	3	4	4
d	7	14	25	26
u	3	7	14	14
ϵ	0	0	0	0

Next, a calculation reveals that

$$(n+1)(d-u-\epsilon) \geq \binom{n+3}{3} + \binom{n}{2},$$

so that $AH_{n,3}(d-u-\epsilon)$ holds throughout. Hence, by lemma 7, we will obtain $AH_{n,5}(d)$ upon verification of the degree 4 cases of Theorem 2.

Degree 4. For $m=4$ we must verify the following:

(a) $AH_{n,2}(d-u-\epsilon)$, for which it suffices (by lemma 8) that $d-u-\epsilon \geq n+1$,

(b) $AH_{n,3}(d-u)$, which (by lemma 9) holds provided that

$$(n+1)(d-u) \leq \binom{n+3}{3} - \binom{n}{2}.$$

(c) $AH_{n-1,4}(u)$ which we obtain by induction on n unless $n \leq 5$ and $u = \lfloor \frac{1}{n+1}\binom{n+4}{4} \rfloor$.

By lemma 7, we are left with performing some computations together with special checking in low dimensions.

Computations. By inspection, the fraction $\frac{1}{n+1}\binom{n+4}{4} = \frac{(n+4)(n+3)(n+2)}{24}$ can be written in the form $\frac{\text{integer}}{4}$, and it is an integer if n is even. Thus, for $n \geq 5$ our numerical hypotheses yield the inequalities

$$\frac{(n+3)(n+2)}{8} - \frac{3}{4} \leq d-u \leq \frac{(n+3)(n+2)}{8} + \frac{3}{4},$$

and $\epsilon = 0$ if n is odd.

For $n \geq 11$, a little calculation yields

$$d-u-2n \geq \frac{n(n-11)}{8} \geq 0$$

and

$$(n+1)(d-u) \le (n+1)\frac{(n+3)(n+2)+6}{8} \le \binom{n+3}{3} - \binom{n}{2},$$

so we have verified conditions (a) and (b) for $n \ge 11$.

We list the remaining cases (except for dimension 5 which is handled separately):

n	2	2	3	3	4	4	6	7	7	8	9	9	10
d	4	6	8	10	13	15	30	41	42	55	71	72	91
u	1	3	4	6	8	10	21	30	30	41	55	55	71
ϵ	0	0	0	0	0	0	0	0	0	2	0	0	5

Evidently, we have $d - u - \epsilon \ge n + 1$, and $(n+1)(d-u) \le \binom{n+3}{3} - \binom{n}{2}$, so that $AH_{n,2}(d-u-\epsilon)$ and $AH_{n,3}(d-u)$ both hold by lemma 8 and lemma 9. Hence we have verified the cases $n \le 4$ and $n \ge 6$ (subject to the case $n = 5$).

Exceptional cases. We remark that if $(n,d) = (2,5), (3,9)$, or $(4,14)$, a general collection of d double points lies on a unique quartic in $\mathbb{P}^n$, although we have $(n+1)d \ge \binom{n+4}{4}$. Since $d = \binom{n+2}{2} - 1$ in each case, there is a quadratic form Q vanishing on d general reduced points, so Q^2 vanishes on the double points. Uniqueness is easy to check for $n = 2$ and $n = 3$; the case $n = 4$ is in [C3].

Dimension 5. Our final step is to construct a collection of 21 double points in $\mathbb{P}^5$ imposing $126 = \binom{5+4}{4}$ conditions on quartics.

(The argument is quite similar to that of [AH1]. Since, unfortunately, the numerics prohibit the use of lemma 5, we perform a minor variation.)

The idea is as follows. Using the uniqueness of a quartic through 9 general points of $\mathbb{P}^3$, we find a set $\Phi \cup \{p,q\}$ of 15 points of $\mathbb{P}^4$ whose double does not lie on a quartic but yields a base locus scheme Υ supported on $\{p,q\}$. We arrange, in particular, that the scheme Υ does not depend on all of the points of Φ. Then we may find a set $\Sigma \subset \mathbb{P}^4$ of 6 points so that $\Sigma^2 \cup \Phi \cup \Upsilon$ is 4-independent. Arguing along the lines of lemma 5, we produce two points p', q' of $\mathbb{P}^5$ so that there is no quartic that is double on $\Sigma \cup \Phi \cup \{p', q'\}$.

Choose $\mathbb{P}^3 \subset \mathbb{P}^4 \subset \mathbb{P}^5$.

In $\mathbb{P}^3$, choose a general collection $\Phi_3 \cup \{p,q\}$ of 10 points. We have

$$h_{\mathbb{P}^3}(\Phi_3^2 \cup \{p\}, 4) = 33 \text{ and } h_{\mathbb{P}^3}(\Phi_3^2 \cup \{p\}^2, 4) = 34 = 4 \cdot 9 - 2;$$

hence there is a unique plane S so that $\{p\}^2|_S$ is in the base locus of the system of quartics through $\Phi_3^2 \cup \{p\}$. Further, $h_{\mathbb{P}^3}(\Phi_3^2 \cup \{p\}^2 \cup \{q\}, 4) = 35$.

Next, by lemma 8 and lemma 3 we may choose a set $\Phi_4 \subset \mathbb{P}^4$ of 5 points satisfying

$$h_{\mathbb{P}^4}(\Phi_4^2 \cup \Phi_3 \cup \{p\} \cup \{q\}, 3) = 5 \cdot 5 + 10 = 35.$$

Hence

$$h_{\mathbb{P}^4}(\Phi_4^2 \cup \Phi_3^2 \cup \{p,q\}^2, 4) = 70;$$

in particular, for any subscheme $\Lambda \subset \{p,q\}^2$ having $\Lambda \cap (p^2|_S \cup q^2|_{\mathbb{P}^3}) \subseteq \{p,q\}$ we have

$$h_{\mathbb{P}^4}(\Phi_4^2 \cup \Phi_3^2 \cup \Lambda, 4) = 5 \cdot 13 + \deg \Lambda.$$

Let $\Phi = \Phi_3 \cup \Phi_4$ and $\Upsilon = p^2|_S \cup q^2|_{\mathbb{P}^3}$.

We seek a collection $\Sigma \subset \mathbb{P}^5$ of 6 points to satisfy $h_{\mathbb{P}^5}(\Sigma^2 \cup \Phi \cup \Upsilon, 3) = 56$. Namely, take $\Sigma \subset \mathbb{P}^5 - \mathbb{P}^4$ in linearly general position. By lemma 8 we have

$$\begin{aligned} h_{\mathbb{P}^5}(\Sigma^2 \cup \Phi_3 \cup \{p,q\}^2|_{\mathbb{P}^3}, 3) &\geq h_{\mathbb{P}^5}(\Sigma^2, 2) + h_{\mathbb{P}^4}(\operatorname{Sec}\Sigma \cap \mathbb{P}^4, 2) + h_{\mathbb{P}^3}(\Phi_3 \cup \{p,q\}^2, 3) \\ &= 52; \end{aligned}$$

hence $h_{\mathbb{P}^5}(\Sigma^2 \cup \Phi_3 \cup \Upsilon, 3) = 51$. Since Υ depends only on Φ_3, we may assume (by lemma 3 and lemma 8) that Φ_4 is sufficiently general to ensure that

$$h_{\mathbb{P}^5}(\Sigma^2 \cup \Phi \cup \Upsilon, 3) = 56.$$

Choose general points $p', q' \in \mathbb{P}^5$ together with a flat family degenerating p' to p, q' to q, and a subscheme $\Upsilon' \subset \{p,q\}^2$ to Υ with respect to the linear system under consideration. We shall show that $\Sigma^2 \cup \Phi^2 \cup \{p', q'\}^2$ does not lie on a quartic of $\mathbb{P}^5$.

Suppose $\xi' \subset \{p', q'\}^2$ is any curvilinear subscheme. Write $\xi' = \gamma' \cup \zeta'$, where $\gamma' \subset \Upsilon'$ and $\zeta' \cap \Upsilon' \subseteq \{p', q'\}$. Let ξ, ζ, γ be the corresponding flat limits at $\{p, q\}$.

Then by semicontinuity we have

$$\begin{aligned} h_{\mathbb{P}^5}(\Sigma^2 \cup \Phi^2 \cup \xi', 4) &\geq h_{\mathbb{P}^5}(\Sigma^2 \cup \Phi^2 \cup \zeta \cup \gamma', 4) \\ &\geq h_{\mathbb{P}^5}(\Sigma^2 \cup \Phi \cup \tilde{\zeta} \cup \gamma', 3) + h_{\mathbb{P}^4}(\Phi^2 \cup (\zeta \cap \mathbb{P}^4), 3) \\ &\geq h_{\mathbb{P}^5}(\Sigma^2 \cup \Phi \cup \tilde{\zeta} \cup \gamma, 3) + 65 + \deg(\zeta \cap \mathbb{P}^4) \\ &= 114 + \deg\tilde{\zeta} + \deg(\zeta \cap \mathbb{P}^4) + \deg\gamma \\ &= 6 \cdot 19 + \deg\xi'. \end{aligned}$$

Hence by lemma 4 we have

$$h_{\mathbb{P}^5}(\Sigma^2 \cup \Phi^2 \cup \{p', q'\}^2, 4) = 126.$$

This completes the proof of theorem 2. □

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