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No. 9

A BRIEF SURVEY ON HOMOGENIZATION WITH A PHYSICAL APPLICATION

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Abstract. It is well known that electric potentials can be used in diagnostic devices to investigate the properties of biological tissues. Such techniques are essentially based on the possibility of determining the physiological properties of a living body by means of the knowledge of its electrical resistence. However, it has been observed that, applying high frequency alternating potentials to the body, a capacitive behaviour takes place, due to the electric polarization at the interface of the cell membranes, which act as capacitors. In this regards, the biological tissue is modelled as a composite media with a periodic microscopic structure with two finely mixed phases (intra and extra cellular) separated by an imperfect interface (cellular membrane). Hence, the homogenization theory seems to be the appropriate tool in order to discover the macroscopic behaviour of the tissue. In these notes we present the basic ideas of homogenization theory with the purpose of enabling the reader to apply this theory in physical problems. An example of such applications is given in the framework of electrical conduction in living tissues.

Resumen. Es bien conocido que los potenciales eléctricos pueden ser usados en los dispositivos de diagnósticos para investigar algunas de las propiedades de los tejidos biológicos. Tales técnicas son basadas esencialmente sobre la posibilidad de determinar las propiedades fisiológicas de un cuerpo vivo conociendo su resistencia eléctrica. Por otro lado, se ha observado que, aplicando al cuerpo potenciales eléctricos de alta frecuencia, aparece también un comportamiento capacitivo debido a la polarización eléctrica que se crea en la interfase de las dos membranas celulares las cuales reaccionan como capacitores. En tal contexto, el tejido biológico es modelizado como un medio compuesto dotado de una estructura microscópica periódica constituida de dos fases finamente mezcladas (intra y extra celular) separadas de una interfase imperfecta (la membrana celular). Por lo tanto, la teoría de la homogeneización aparece ser el instrumento más apropiado para comprender el comportamiento macroscópico del tejido. En estas notas, se presentan las ideas básicas de la teoría de la homogeneización a fin de permitir al lector de aplicarla a problemas físicos. Se da un ejemplo de tales aplicaciones en el ámbito de la conducción eléctrica en tejidos vivos.

Riassunto. È ben noto che i potenziali elettrici possono essere usati nei dispositivi diagnostici per investigare alcune proprietà dei tessuti biologici. Tali tecniche sono basate essenzialmente sulla possibilità di determinare le proprietà fisiologiche di un corpo vivente, conoscendone la sua resistenza elettrica. Tuttavia, è stato osservato che, applicando al corpo potenziali elettrici ad alte frequenze, appare anche un comportamento capacitivo, dovuto alla polarizzazione elettrica che si crea alle interfaccie delle membrane cellulari, le quali agiscono come dei capacitori. In tale contesto, il tessuto biologico è modellizzato come un mezzo composito dotato di una struttura microscopica periodica, a sua volta costituita da due fasi finemente miscelate (quella intra ed extra-cellulare) separate da un'interfaccia imperfetta (la membrana cellulare). Pertanto, la teoria dell'omogeneizzazione sembra essere lo strumento più appropriato per comprendere il comportamento macroscopico del tessuto. In queste note, presentiamo le idee di base della teoria dell'omogeneizzazione, al fine di permettere al lettore di appropriarsene al punto da essere in grado di applicarla a problemi fisici. Forniamo, inoltre, un esempio di questo genere di applicazioni nell'ambito della suddetta conduzione elettrica nei tessuti viventi.

Keywords: Homogenization, asymptotic expansion, evolution equation with memory, dynamical conditions, electrical conduction in biological tissues.

Parabras claves: Homogeneización, expansión asintótica, ecuación de evolución con memoria, condiciones dinámicas, conducción eléctrica en tejidos biológicos.

AMS Subject Classification: 365B27, 78A70, 45K05, 35J65.

These Notes are the enlarged content of some seminars given by Prof. M. Amar and Prof. R. Gianni at the Department of Mathematics of FCE-UA, Rosario, on July 2004. They contain the basic ideas of the homogenization theory and an example of its application in the framework of electrical conduction in living tissues.

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A BRIEF SURVEY ON HOMOGENIZATION WITH A PHYSICAL APPLICATION

M. Amar[1] – R. Gianni[1]

1. Introduction

These Notes contain the basic ideas of homogenization theory, with the purpose of enabling the reader to apply this theory in physical problems. An example of such applications is given in the framework of electrical conduction in living tissues.
These Notes are not exhaustive and are intended to be just a survey on these topics. For this reason, most of the proofs are skipped; nevertheless, references in which they can be found are given.
The Notes are organized as follows: in Section 2 the relevant function spaces are introduced, together with their main properties; in Section 3 the classical homogenization theory for elliptic equations is shown; finally, in Section 4 the application of the homogenization technique to a non standard framework is performed. Specifically, we study electrical conduction in a biological tissue, which is regarded as a periodic array, due to its microscopic structure composed of the intra and extra-cellular space, separated by an interface given by the cellular membrane.

2. Lebesgue and Sobolev spaces.

Let $\Omega \subseteq \boldsymbol{R}^N$ be an open set, $1 \leq p < +\infty$ and $f : \Omega \to \boldsymbol{R}^M$ be a measurable function. We say that $f \in L^p(\Omega; \boldsymbol{R}^M)$ if

$$\|f\|_p := \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p} < +\infty \, .$$

Analogously, we say that $f \in L^\infty(\Omega; \boldsymbol{R}^M)$ if

$$\|f\|_\infty := \inf\{\alpha \ : \ |f(x)| \leq \alpha \text{ a.e. in } \Omega\} < +\infty \, .$$

In particular, $f \in L^p_{loc}(\Omega; \boldsymbol{R}^M)$, $1 \leq p \leq +\infty$, if $f \in L^p(A; \boldsymbol{R}^M)$, for every open set $A \subset\subset \Omega$.
Such spaces endowed with the norms previously defined are Banach spaces and, for $1 \leq p < +\infty$, they are separable. A sequence $\{f_n\} \subseteq L^p(\Omega, \boldsymbol{R}^M)$ converges strongly to $f \in L^p(\Omega; \boldsymbol{R}^M)$ if $\|f_n - f\|_p \to 0$, for $n \to +\infty$.

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Given $1 \le p \le +\infty$, we denote with p' the conjugate exponent of p, i.e. $1/p + 1/p' = 1$ if $1 < p < +\infty$, $p' = +\infty$ if $p = 1$ and $p' = 1$ if $p = +\infty$. If $1 \le p < +\infty$, $L^{p'}$ is the dual space of L^p, while L^1 is strictly contained in the dual of L^∞. Finally, if $1 < p < +\infty$, the space L^p is a reflexive space.

Let us introduce the notion of weak and weak*-convergence. For the sake of simplicity $M = 1$ (hence we will write $L^p(\Omega)$ instead of $L^p(\Omega; \boldsymbol{R})$).

Let firstly $1 \le p < +\infty$, then $\{f_n\}$ weakly converges to f (and we will write $f_n \rightharpoonup f$) in $L^p(\Omega)$ if

$$\int_\Omega f_n(x) g(x)\, dx \to \int_\Omega f(x) g(x)\, dx \qquad \forall g \in L^{p'}(\Omega) .$$

Analogously, $\{f_n\}$ weakly* converges to f (and we will write $f_n \overset{*}{\rightharpoonup} f$) in $L^\infty(\Omega)$ if

$$\int_\Omega f_n(x) g(x)\, dx \to \int_\Omega f(x) g(x)\, dx \qquad \forall g \in L^1(\Omega) .$$

Theorem 2.1. *Let $\{f_n\}$ be a sequence of functions belonging to $L^p(\Omega)$ strongly converging to $f \in L^p(\Omega)$, $1 \le p \le +\infty$. Then*
(i) $f_n \rightharpoonup f$ in $L^p(\Omega)$ if $1 \le p < +\infty$ and $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$;
(ii) $\|f_n\|_p \to \|f\|_p$, for $1 \le p \le +\infty$.

When $1 < p < +\infty$, it is possible to reverse the previous result, thus obtaining the following theorem.

Theorem 2.2. *Let $1 < p < +\infty$ and let $\{f_n\}$ be a sequence of functions belonging to $L^p(\Omega)$. Let us assume that $f_n \rightharpoonup f$ in $L^p(\Omega)$ and $\|f_n\|_p \to \|f\|_p$. Then $f_n \to f$ in $L^p(\Omega)$.*

Theorem 2.3. *Let $\{f_n\}$ be a sequence of functions belonging to $L^p(\Omega)$ strongly convergent to $f \in L^p(\Omega)$, $1 \le p \le +\infty$. Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a set $N \subset \Omega$ of null measure, such that $f_{n_k}(x) \to f(x)$ for every $x \in \Omega \setminus N$, which implies that the subsequence $\{f_{n_k}\}$ pointwise converges to f almost everywhere.*

Theorem 2.4. *(i) Let $\{f_n\}$ be a sequence of measurable functions defined in Ω, such that $f_n(x) \ge 0$ almost everywhere, for every $n \in \boldsymbol{N}$. Then*

$$\int_\Omega [\liminf_{n\to+\infty} f_n(x)]\, dx \le \liminf_{n\to+\infty} \int_\Omega f_n(x)\, dx .$$

(ii) Let $\{f_n\}$ be a sequence of measurable functions defined in Ω, such that $0 \le f_n(x) \le f_{n+1}(x)$ almost everywhere, for every $n \in \boldsymbol{N}$. Then

$$\int_\Omega f_n(x)\, dx \to \int_\Omega f(x)\, dx \qquad \text{dove} \qquad f(x) = \sup_{n\in\boldsymbol{N}} f_n(x) .$$

(iii) Let $\{f_n\}$ be a sequence of measurable functions defined in Ω, pointwise converging almost everywhere to a function f. Let us also assume that, for every $x \in \Omega$ and every $n \in \boldsymbol{N}$, $|f_n(x)| \le g(x)$, with $g \in L^1(\Omega)$. Then, $f \in L^1(\Omega)$ and $\|f_n - f\|_1 \to 0$, for $n \to +\infty$.

Theorem 2.5. *Let $\{f_n\}$ be a weakly convergent sequence in $L^p(\Omega)$, if $1 \le p < +\infty$, (respectively, weakly* convergent in $L^\infty(\Omega)$, if $p = +\infty$). Then it is equibounded.*

Let us assume that $\{f_n\}$ *be an equibounded sequence in* $L^p(\Omega)$, $1 < p \leq +\infty$. *Then there exists a function* $f \in L^p(\Omega)$ *and a subsequence of* $\{f_n\}$ *weakly converging to* f *for* $1 < p < +\infty$ *(respectively, weakly** *converging to* f *for* $p = +\infty$*).*

For $p = 2$, the Banach space L^p is in fact an Hilbert space, endowed with the scalar product $(f, g) := \int_\Omega f(x)g(x)\ dx$. Therefore, the Cauchy-Schwarz inequality for Hilbert spaces implies

$$\left| \int_\Omega f(x)g(x)\ dx \right| \leq \|f\|_2 \|g\|_2 \qquad \forall f, g \in L^2(\Omega)\ .$$

In particular, we have that the product of two functions f and g in L^2 belongs to L^1. These result can be generalized to the case in which $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, where p' is the conjugate exponent of p. In fact we have the following theorem.

Theorem 2.6. *(Hölder Inequality) Let* $\Omega \subset \boldsymbol{R}^N$ *be an open set, let* $1 \leq p \leq +\infty$ *and* p' *be the conjugate exponent of* p. *Then for every* $u \in L^p(\Omega)$ *and* $v \in L^{p'}(\Omega)$, *we have* $uv \in L^1(\Omega)$ *and*

$$\int_\Omega |u(x)v(x)|\ dx \leq \|u\|_p \|v\|_{p'}\ .$$

We now state a useful result concerning the weak convergence of periodic functions. We assume, for the sake of simplicity, f to be $(0,1)^N$-periodic.

Theorem 2.7. *Set* $Y = (0,1)^N$ *and let* $f \in L^p(Y)$, $1 \leq p \leq +\infty$. *We still denote by* f *its periodic extension to the whole* $\boldsymbol{R}^N$. *Set* $f_\varepsilon(x) = f(x/\varepsilon)$; *then, for* $\varepsilon \to 0^+$, *we have*

$$f_\varepsilon \rightharpoonup \overline{f} := \int_Y f(x)\ dx if\ 1 \leq p < +\infty\ ;$$

$$f_\varepsilon \overset{*}{\rightharpoonup} \overline{f} := \int_Y f(x)\ dx if\ p = +\infty\ .$$

For example, if $f(x) = \sin x$, we obtain $\sin(x/\varepsilon) \overset{*}{\rightharpoonup} 0$ in $L^\infty(0, 2\pi)$, which is the well known Riemann-Lebesgue Lemma. In particular, this implies that the sequence $\{\sin(nx)\}$ is an example of a weakly convergent sequence, which does not strongly converges.

Let us recall now a well known property of summable functions.

Lemma 2.8. *Let* $\Omega \subseteq \boldsymbol{R}^N$ *be an open set and* $f \in L^1_{loc}(\Omega)$. *We assume that*

$$\int_\Omega f(x)\phi(x)\ dx = 0 \qquad \forall \phi \in \boldsymbol{C}_0^\infty(\Omega)\ .$$

Then $f = 0$ *almost everywhere in* Ω.

Corollary 2.9. *Let* $\Omega \subseteq \boldsymbol{R}^N$ *be an open set and* $f \in L^1_{loc}(\Omega)$. *We assume that*

$$\int_\Omega f(x)\phi(x)\ dx = 0 \tag{2.1}$$

for every functions $\phi \in \boldsymbol{C}_0^\infty(\Omega)$ *such that* $⨍_\Omega \phi\ dx = \frac{1}{|\Omega|}\int_\Omega \phi\ dx = 0$. *Then there exists a constant* $c \in \boldsymbol{R}$ *such that* $f = c$ *almost everywhere in* Ω.

Proof. Let $f \in L^1_{loc}(\Omega)$, $\psi \in \boldsymbol{C}^\infty_0(\Omega)$ and $\Psi \in \boldsymbol{C}^\infty_0(\Omega)$ wiht $⨍_\Omega \Psi \, dx = 1$. Then

$$\int_\Omega [f - ⨍_\Omega f\Psi \, dy]\psi \, dx = \int_\Omega f\psi \, dx - \left(⨍_\Omega f\Psi \, dy\right)\left(\int_\Omega \psi \, dx\right)$$
$$= \int_\Omega f[\psi - \Psi ⨍_\Omega \psi \, dy] \, dx = 0$$

since, if we set $\phi = \psi - \Psi ⨍_\Omega \psi \, dy$, we obtain $\phi \in C^\infty_0(\Omega)$ and $⨍_\Omega \phi \, dx = 0$. Hence, from Lemma 2.8 we get $f - ⨍_\Omega f\Psi \, dy = 0$ almost everywhere in Ω; i.e., the thesis holds with $c = ⨍_\Omega f\Psi \, dy$. □

We end up recalling the main properties of the functions which admits weak derivatives. Let $k \in \boldsymbol{N}$ and $1 \leq p \leq +\infty$, we denote with $W^{k,p}(\Omega; \boldsymbol{R}^M)$ the space of the measurable functions which admit k-derivatives in the sense of distributions which belong to L^p. This is a Banach space endowed with the norm

$$\|f\|_{k,p} := \left(\sum_{\alpha=0}^{k} \|\nabla^\alpha f\|_p^p\right)^{1/p} \qquad \text{if } 1 \leq p < +\infty \; ;$$
$$\|f\|_{k,\infty} := \max_{0\leq\alpha\leq k} \|\nabla^\alpha f\|_\infty \qquad \text{if } p = +\infty \; ; \tag{2.2}$$

where $\nabla^\alpha f$ denotes the tensor of the weak derivatives of f of order α (and $\nabla^0 f \equiv f$).
As usual, we denote with $H^k(\Omega; \boldsymbol{R}^M)$ the space $W^{k,2}(\Omega; \boldsymbol{R}^M)$. Moreover, for $1 \leq p < +\infty$, $W^{k,p}_0(\Omega; \boldsymbol{R}^M)$ is the closure of $\boldsymbol{C}^\infty_0(\Omega; \boldsymbol{R}^M)$ with respect to the norm of $W^{k,p}$ and $H^k_0(\Omega; \boldsymbol{R}^M) = W^{k,2}_0(\Omega; \boldsymbol{R}^M)$. The space $W^{k,p}_0(\Omega; \boldsymbol{R}^M)$ is a Banach space endowed with the norm of $W^{k,p}$. In the case where $\Omega = (a,b)$ is a real interval, $W^{1,1}(a,b)$ can be identified with the space of the absolute continuous functions on (a,b).
We recall that, if p' is the conjugate exponent of p, $W^{-k,p'}(\Omega; \boldsymbol{R}^M)$ is the dual space of $W^{k,p}_0(\Omega; \boldsymbol{R}^M)$; in particular, for $k = 1$ and $M = 1$, $W^{-1,p'}(\Omega)$ is the dual space of $W^{1,p}_0(\Omega)$ and every element $\overline{g}$ belonging to $W^{-1,p'}(\Omega)$ can be represented as follows:

$$\overline{g} = g - \sum_{j=1}^{N} \partial_j g_j$$

where $g, g_j \in L^{p'}(\Omega)$ and the $\partial_j g_j$ are distributional derivatives. Moreover, if Ω is bounded, $h \leq k$ and $q < p$, $W^{k,p}(\Omega; \boldsymbol{R}^M) \subset W^{h,q}(\Omega; \boldsymbol{R}^M)$.
We recall that, for $1 \leq p < +\infty$, $W^{k,p}(\Omega; \boldsymbol{R}^M)$ is a separable space and for $1 < p < +\infty$ it is a reflexive space. Finally, if Ω is sufficiently smooth (i.e., $\partial\Omega \in \boldsymbol{C}^1$) and $1 \leq p < +\infty$, it follows that $\boldsymbol{C}^\infty(\Omega; \boldsymbol{R}^M)$ is dense in $W^{k,p}(\Omega; \boldsymbol{R}^M)$, with respect to the norm previously defined. If Ω is bounded, $W^{1,\infty}(\Omega; \boldsymbol{R}^M)$ can be identified with the space of Lipschitz continuous functions.

Theorem 2.10. *(Embedding Theorem) Let $\Omega \subset \boldsymbol{R}^N$ be a bounded open set. We have:*
(i) if $1 \leq p < N$, then $W^{1,p}_0(\Omega; \boldsymbol{R}^M) \subset L^q(\Omega; \boldsymbol{R}^M)$, for every $1 \leq q \leq \frac{Np}{N-p}$ and the embedding is compact for $1 \leq q < \frac{Np}{N-p}$;
(ii) if $p = N$, then $W^{1,p}_0(\Omega; \boldsymbol{R}^M) \subset L^q(\Omega; \boldsymbol{R}^M)$, for every $1 \leq q < +\infty$ and the embedding is compact;

(iii) if $p > N$, then $W_0^{1,p}(\Omega;\boldsymbol{R}^M) \subset \boldsymbol{C}_0(\overline{\Omega};\boldsymbol{R}^M)$ and the embedding is compact.

The previous theorem still holds if $W_0^{1,p}(\Omega;\boldsymbol{R}^M)$ is replaced by $W^{1,p}(\Omega;\boldsymbol{R}^M)$, provided that $\partial\Omega$ is sufficiently smooth; i.e., $\partial\Omega$ is of class $\boldsymbol{C}^1$.
We note that Theorem 2.10 implies that, if $1 \le p \le +\infty$ and $f_n \rightharpoonup f$ in $W_0^{1,p}(\Omega;\boldsymbol{R}^M)$, then $f_n \to f$ in $L^p(\Omega;\boldsymbol{R}^M)$.

Theorem 2.11. *(Poincaré Inequality) Let $\Omega \subset \boldsymbol{R}^N$ be a bounded, open, connected set, having the boundary of class $\boldsymbol{C}^1$ and $1 \le p < +\infty$. Then,*
(i) there exists a constant $c > 0$ such that, for every $f \in W_0^{1,p}(\Omega;\boldsymbol{R}^M)$, we have

$$\|f\|_p \le c\|\nabla f\|_p \ ;$$

(ii) there exists a constant $c > 0$ such that, for every $f \in W^{1,p}(\Omega;\boldsymbol{R}^M)$, we have

$$\|f - \overline{f}\|_p \le c\|\nabla f\|_p \ ,$$

where $\overline{f} = \frac{1}{|\Omega|}\int_\Omega f(x)\ dx$.

Remark 2.12. More in general property (i) holds also for open set which are neither connected nor smooth, while property (ii) remains true also for $p = +\infty$. □

In particular Poincaré inequality implies that, for $W_0^{1,p}(\Omega;\boldsymbol{R}^M)$, the L^p-norm of the gradient is equivalent to the norm of $W^{1,p}$ defined in (2.2).

Finally, we recall this useful existence result for periodic solution of elliptic equation (see [16], Lemma 2.1).

Lemma 2.13. *Let $Y = (0,1)^N$, $F \in L^2(Y)$ and, for every $i,j = 1,\dots,N$, let a_{ij} be measurable functions such that*

$$\begin{array}{lll} i) & a_{ij}(y) = a_{ji}(y) & \text{a.e. in } Y\,,\ \forall i,j = 1,\dots,N\,; \\ ii) & \lambda|\xi|^2 \le a_{ij}(y)\xi_i\xi_j \le \Lambda|\xi|^2 & \forall \xi \in \boldsymbol{R}^N \end{array}$$

with $0 < \lambda \le \Lambda < +\infty$. Then, the equation

$$\begin{cases} -\operatorname{div}(a_{ij}(y)\nabla\phi) = F & \text{in } Y \\ \phi & Y\text{-periodic} \end{cases}$$

admits a solution in $H^1(Y)$, which is defined up to an additive constant, iff

$$\int_Y F(y)\ dy = 0\,.$$

For more details on these subjects, see [1] and [12].

3. Asymptotic expansions and classical homogenization.

Let Ω be a bounded open subset of $\boldsymbol{R}^N$ with $n \ge 1$. Let $Y = (0,1)^n$ be the unit cell in $\boldsymbol{R}^N$. A function $f(x)$, defined on $\boldsymbol{R}^N$, is said to be Y-periodic if it is periodic of period 1 with respect to each variable x_i, with $1 \le i \le n$. We denote by $L^2_\#(Y)$ and $H^1_\#(Y)$ the spaces of functions in $L^2(\boldsymbol{R}^N)$ and $H^1(\boldsymbol{R}^N)$, respectively, which are Y-periodic.
Let $A(y)$ be a square matrix of order n with entries $a_{ij}(y)$ which are measurable Y-periodic functions. We assume that there exist two constants $0 < \lambda < \Lambda < +\infty$ such that, for a.e. $y \in Y$,

$$\lambda|\xi|^2 \le a_{ij}(y)\xi_i\xi_j \le \Lambda|\xi|^2 \qquad \forall \xi \in \boldsymbol{R}^N. \tag{3.1}$$

Let $A_\varepsilon(x) = A(\frac{x}{\varepsilon})$ be a periodically oscillating matrix of coefficients. For a given function $f \in L^2(\Omega)$ we consider the following equation

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

which admits a unique solution in $H_0^1(\Omega)$. The homogenization of equation (3.2) is by now a classical matter (see e.g. [6], [7], [8], [18]). We briefly recall the main ingredients of this process that we shall use later (we mainly follow the exposition of Ch. I §2 in [8]). Firstly we note that, by standard existence and uniqueness results, (3.2) is shown to be well-posed in $H_0^1(\Omega)$ (see, for instance, Theorem 8.3 in [14]) and, using standard energy estimate (obtained multiplying the first equation in (3.2) by u_ε and integrating by parts) it can be easily proved that there exists a function $u \in H_0^1(\Omega)$ such that, up to a subsequence,

$$u_\varepsilon \rightharpoonup u \qquad \text{weakly in } H_0^1(\Omega). \tag{3.3}$$

It remains to identify the limit function u. To this purpose, we assume that the solution u_ε admits the following ansatz (or asymptotic expansion)

$$u_\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \varepsilon^3 u_3(x, \frac{x}{\varepsilon}) + \dots \tag{3.4}$$

where each function $u_i(x, y)$ is Y-periodic with respect to the fast variable y. Plugging this ansatz in equation (3.2) and identifying different powers of ε yields a cascade of equations. Defining an operator L_ε by $L_\varepsilon \phi = -\operatorname{div} A_\varepsilon \nabla \phi$, we may write $L_\varepsilon = \varepsilon^{-2} L_0 + \varepsilon^{-1} L_1 + L_2$, where

$$\begin{aligned} L_0 &= -\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right) \\ L_1 &= -\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right) \\ L_2 &= -\frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial}{\partial x_j}\right). \end{aligned}$$

The two space variables x and y are taken as independent, and at the end of the computation y is replaced by $\frac{x}{\varepsilon}$. Equation (3.2) is therefore equivalent to the following system

$$\begin{aligned} &L_0 u_0 = 0 \\ &L_0 u_1 + L_1 u_0 = 0 \\ &L_0 u_2 + L_1 u_1 + L_2 u_0 = f \\ &L_0 u_3 + L_1 u_2 + L_2 u_1 = 0 \\ &\dots\dots \end{aligned} \tag{3.5}$$

the solutions of which are easily computed. The first equation in (3.5); i.e.,

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left(a_{ij}(y)\dfrac{\partial u_0(x,y)}{\partial y_j}\right) = 0 & \text{in } Y \\ y \mapsto u_0(x,y) & Y\text{-periodic} \end{cases}$$

where y is the independent variable, while x plays the role of a parameter, has a constant solution (where the constant clearly depends on x), due to Lemma 2.13; so that $u_0(x, y) \equiv u_0(x)$, which does not depend on y. The second equation of (3.5) can be explicitly solved in term of u_0; indeed, it can be easily verified that the solution u_1 is given by

$$u_1(x, \frac{x}{\varepsilon}) = -\chi^j(\frac{x}{\varepsilon})\frac{\partial u_0}{\partial x_j}(x) + \tilde{u}_1(x) \tag{3.6}$$

where, thanks to Lemma 2.13, $\chi^j(y)$, $j = 1, \dots, n$, are the unique solutions in $H^1_\#(Y)$ with zero average of the cell equation

$$\begin{cases} L_0\chi^j = -\dfrac{\partial a_{ij}}{\partial y_i}(y) & \text{in } Y; \\ \displaystyle\int_Y \chi^j(y)dy = 0 \qquad y \to \chi^j(y) & Y\text{-periodic}\,. \end{cases} \tag{3.7}$$

By Lemma 2.13, the third equation in (3.5), which can be written as the system

$$\begin{cases} L_0 u_2 = f - L_1 u_1 - L_2 u_0 & \text{in } Y \\ y \mapsto u_2(x,y) & Y\text{-periodic}\,, \end{cases}$$

where u_0 and u_1 play the role of known functions, is solvable if and only if

$$\int_Y [f(x) - (L_1u_1)(x,y) - (L_2u_0)(x,y)]\ dy = 0\,.$$

After simple calculations, the previous condition can be read as

$$-\operatorname{div}(A^*\nabla u_0) = f$$

where the homogenized matrix A^* is defined by its constant entries a^*_{ij} given by

$$a^*_{ij} = \int_Y [a_{ij}(y) - a_{ik}(y)\frac{\partial \chi^j}{\partial y_k}(y)]\ dy.$$

The homogenized problem for $u_0(x)$ is just the previous compatibility condition, complemented with the natural boundary condition obtained by (3.2); i.e.,

$$\begin{cases} -\operatorname{div}(A^*\nabla u_0) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.8}$$

It is important to observe that the homogenized matrix A^* it is not obtained simply by averaging the original one, even if this could be considered more natural. Obviously, this implies that the convergence of the solutions $\{u_\varepsilon\}$ cannot take place in $H^1_0(\Omega)$ strongly. Equation (3.8) is well-posed in $H^1_0(\Omega)$ since it is easily seen that A^* is bounded and coercive (see Remark 2.6 in [8]).
Moreover, also the solution u_2 can be explicitly given in terms of u_0; in fact,

$$u_2(x, \frac{x}{\varepsilon}) = \chi^{ij}(\frac{x}{\varepsilon})\frac{\partial^2 u}{\partial x_i \partial x_j}(x) - \chi^j(\frac{x}{\varepsilon})\frac{\partial \tilde{u}_1}{\partial x_j}(x) + \tilde{u}_2(x) \tag{3.9}$$

where $\chi^{ij} \in H^1_\#(Y)$, for $i,j = 1, \dots, n$, are the solutions of another cell problem (see (2.42) and (2.39) in [8])

$$\begin{cases} L_0\chi^{ij} = b_{ij} - \displaystyle\int_Y b_{ij}(y)\ dy & \text{in } Y; \\ \displaystyle\int_Y \chi^{ij}(y)dy = 0 \qquad y \to \chi^{ij}(y) & Y\text{-periodic.} \end{cases} \tag{3.10}$$

with

$$b_{ij}(y) = a_{ij}(y) - a_{ik}(y)\frac{\partial \chi^j}{\partial y_k} - \frac{\partial}{\partial y_k}(a_{ki}(y)\chi^j).$$

Remark that, so far (i.e. if we do not look at higher order equations in (3.5)), the functions $\tilde{u}_1$ in (3.6) and $\tilde{u}_2$ in (3.9) are non-oscillating functions that are not determined. As pointed out in [8], if we stop expansion (3.4) at the first order, the function $\tilde{u}_1$ (and a fortiori $\tilde{u}_2$) does not play any role, and so we may choose $\tilde{u}_1 \equiv 0$. However, if higher order terms are considered, then $\tilde{u}_1$ must satisfy some equation. More precisely, the compatibility condition of the fourth equation of (3.5) leads to (see [8], equation (2.45))

$$-\operatorname{div} A^*\nabla\tilde{u}_1 = c_{ijk}\frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} \tag{3.11}$$

with

$$c_{ijk} = \int_Y [a_{kl}(y)\frac{\partial \chi^{ij}}{\partial y_l}(y) - a_{ij}(y)\chi^k(y)]\ dy.$$

Similar considerations hold for $\tilde{u}_2$, but we shall not need it in the sequel. At this point we emphasize that the above method of two-scale asymptotic expansion is formal. However, a well-known theorem states that the two first terms of (3.4) are correct.

Theorem 3.1. *For every $\varepsilon > 0$, let u_ε be the solution of (3.2) and u_0 be the solution of (3.5). Then $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1_0(\Omega)$.*

As consequence of previous theorem, we have been able to identify the limit function u in (3.3), which is equal to u_0, solution of (3.8).

Theorem 3.2. *Let u_ε and u be the unique solutions of (3.2) and (**??**) respectively. Then, u_ε converges weakly to u in $H^1_0(\Omega)$. If furthermore $u \in W^{2,\infty}(\Omega)$, then*

$$\|u_\varepsilon(x) - u_0(x) - \varepsilon u_1(x, \frac{x}{\varepsilon})\|_{H^1(\Omega)} \leq C\sqrt{\varepsilon}$$

where u_1 is given by (3.6).

The proof of Theorems 3.1 and 3.2 are completely standard (see e.g. [8] and [16]). Remark that it holds whatever the choice of $\tilde{u}_1$ is, since the term $\varepsilon\tilde{u}_1(x)$ is smaller than $\sqrt{\varepsilon}$ in the $H^1(\Omega)$-norm. The error estimate of order $\sqrt{\varepsilon}$ in Theorem 3.2, although generically optimal, is a little surprising since one could expect to get ε if the next order term in the ansatz was truly $\varepsilon^2 u_2(x, \frac{x}{\varepsilon})$. As it is well known, this worse-than-expected result is due to the appearance of boundary layers (see [7], [9], [16]).

The previous method can be easily generalized to the case in which the matrix A depends also on the "slow variable" x; i.e., $A = A(x, y)$ and $A_\varepsilon(x) = A(x, x/\varepsilon)$, provided that it continuously depends on x (see, for instance, [8] and [16]).
Similarly, with the previous technique and without relevant changes, we can prove the same results for the case in which the source term depends on ε; i.e., $f = f_\varepsilon$, provided that the sequence $\{f_\varepsilon\}$ converges strongly in $L^2(\Omega)$.

For the sake of completeness, we recall that homogenization is often performed for integral functionals. This leads to an interesting and useful theory (the Γ-convergence theory, see for instance [10], [11], [13]), which however is out of the aims of these Notes.

4. An application of homogenization theory to a biological problem.

It is well known that electric potentials can be used in diagnostic devices to investigate the properties of biological tissues. Such techniques are essentially based on the possibility of determining the physiological properties of a living body by means of the knowledge of its electrical resistence. From a mathematical point of view, this leads to an inverse problem for an elliptic equation, usually the Laplacian, which is the equation satisfied by the electrical potential, when the body is assumed to display only a resistive behavior. However, it has been observed that, applying high frequency potentials to the body, a capacitive behavior appears, due to the electric polarization at the interface of the cell membranes, which act as capacitors. This phenomenon (known in physics as Maxwell-Wagner effect) is studied modelling the biological tissue as a composite media with a periodic microscopic structure where two finely mixed conductive phases (intra and extra cellular) are separated by a dielectric interface (the cellular membrane). From the mathematical point of view, the electrical current flow through the tissue is described by means of a system of decoupled elliptic equations in the two conductive phases (obtained from the Maxwell equations, under the quasi-static assumption; i.e., we assume that the magnetic effects are negligible). The solutions of this system are coupled because of the interface conditions at the membrane, whose physical behaviour is described by means of a dynamical boundary condition, together with the flux-continuity assumption.
In this regard, different models are obtained corresponding to different scaling with respect to ε (where ε denotes the length of the periodicity cell) of the relevant physical quantity α, entering in the dynamical boundary condition given by

$$\frac{\alpha}{\varepsilon^k}\frac{\partial}{\partial t}[u_\varepsilon] = \sigma\nabla u_\varepsilon^{\text{out}}\cdot\nu\,, \qquad \text{on the membrane interface,} \tag{4.1}$$

with $k \in \boldsymbol{Z}$. Because of the complex geometry of the domain, these models are not easily handled, for example from the numerical point of view. This justify the need of the homogenization approach, with the aim of producing macroscopic models for the whole medium as $\varepsilon \to 0$. The macroscopic equations thus obtained are, in some cases, different from the elliptic one used up to now. For instance, case $k = 1$, the one considered in these Notes (see [2] and [3]), leads to an elliptic equation with memory, as it could be expected in any electrical circuit in which a capacitor is present; in turn, case $k = -1$ (see [15]) leads to a degenerate parabolic system, the well known bidomain model for the cardiac syncithial tissue, where however, in the left hand side of (4.1) an extra term depending on $[u_\varepsilon]$ appears, modelling the nonlinear conductive behaviour of the membrane.

Let Ω be an open connected bounded subset of $\boldsymbol{R}^N$, and let $\Omega = \Omega^\varepsilon_{\text{int}} \cup \Omega^\varepsilon_{\text{out}} \cup \Gamma^\varepsilon$, where $\Omega^\varepsilon_{\text{int}}$ and $\Omega^\varepsilon_{\text{out}}$ are two disjoint open subsets of Ω, and $\Gamma^\varepsilon = \partial\Omega^\varepsilon_{\text{int}} \cap \Omega = \partial\Omega^\varepsilon_{\text{out}} \cap \Omega$. Let also $T > 0$ be a given time.

We are interested in the homogenization limit as $\varepsilon \searrow 0$ of the problem for $u_\varepsilon(x,t)$ (here the operators div and ∇ act only with respect to the space variable x)

$$-\operatorname{div}(\sigma_{\text{int}}\nabla u_\varepsilon) = 0\,, \qquad \text{in } \Omega^\varepsilon_{\text{int}}; \tag{4.2}$$

$$-\operatorname{div}(\sigma_{\text{out}}\nabla u_\varepsilon) = 0\,, \qquad \text{in } \Omega^\varepsilon_{\text{out}}; \tag{4.3}$$

$$\sigma_{\text{int}}\nabla u_\varepsilon^{(\text{int})}\cdot\nu = \sigma_{\text{out}}\nabla u_\varepsilon^{(\text{out})}\cdot\nu\,, \qquad \text{on } \Gamma^\varepsilon; \tag{4.4}$$

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u_\varepsilon] + \frac{1}{\varepsilon}f([u_\varepsilon]) = \sigma_{\text{out}}\nabla u_\varepsilon^{(\text{out})}\cdot\nu\,, \qquad \text{on } \Gamma^\varepsilon; \tag{4.5}$$

$$[u_\varepsilon](x,0) = S_\varepsilon(x)\,, \qquad \text{on } \Gamma^\varepsilon; \tag{4.6}$$

$$u_\varepsilon(x,t) = 0\,, \qquad \text{on } \partial\Omega. \tag{4.7}$$

The notation in (4.2)–(4.5), (4.7), means that the indicated equations are in force in the relevant spatial domain for $0 < t < T$.
Here σ_{int}, σ_{out} and α are positive constants, and ν is the normal unit vector to Γ^ε pointing into $\Omega^\varepsilon_{\text{out}}$. Since u_ε is not in general continuous across Γ^ε we have set

$$u_\varepsilon^{(\text{int})} := \text{ trace of } u_{\varepsilon|\Omega^\varepsilon_{\text{int}}} \text{ on } \Gamma^\varepsilon; \quad u_\varepsilon^{(\text{out})} := \text{ trace of } u_{\varepsilon|\Omega^\varepsilon_{\text{out}}} \text{ on } \Gamma^\varepsilon.$$

Indeed we refer conventionally to $\Omega^\varepsilon_{\text{int}}$ as to the *interior domain*, and to $\Omega^\varepsilon_{\text{out}}$ as to the *outer domain*. We also denote

$$[u_\varepsilon] := u_\varepsilon^{(\text{out})} - u_\varepsilon^{(\text{int})}\,.$$

Similar conventions are employed for other quantities; for example (4.4) can be rewritten as

$$[\sigma\nabla u_\varepsilon\cdot\nu] = 0\,, \qquad \text{on } \Gamma^\varepsilon,$$

where

$$\sigma = \sigma_{\text{int}} \quad \text{in } \Omega^\varepsilon_{\text{int}}, \qquad \sigma = \sigma_{\text{out}} \quad \text{in } \Omega^\varepsilon_{\text{out}}.$$

The function f fulfils

$$f \in C^2(\boldsymbol{R})\,, \quad f'\,, f'' \in L^\infty(\boldsymbol{R})\,, \qquad f(0) = 0\,. \tag{4.8}$$

The initial data S_ε will be discussed below.

The well posedness of problem (4.2)–(4.7) is a direct consequence of the following theorem (see [5]). A similar result holds true also for all the other problems which we will encounter in these Notes, which differ just for the nature of the boundary conditions.

Theorem 4.1. *Let Ω be an open connected bounded subset of $\boldsymbol{R}^N$ such that $\Omega = \overline{\Omega_1 \cup \Omega_2 \cup \Gamma}$, where Ω_1 and Ω_2 are two disjoint open subset of Ω, $\Gamma = \overline{\partial\Omega_1 \cap \Omega} = \overline{\partial\Omega_2 \cap \Omega}$ is a compact regular set, and $|\Gamma \cap \partial\Omega|_{N-1} = 0$. Assume also that Ω, Ω_1 and Ω_2 have Lipschitz boundaries. Let f be as in (4.8). Let $g \in L^2(\Omega\times(0,T))$, $q, h \in L^2(0,T;L^2(\Gamma))$, and $S \in H^{1/2}_o(\Gamma,\Omega)$. Therefore, problem*

$$-\sigma\,\Delta\, v = g(t)\,, \qquad \text{in } \Omega_{\text{int}},\ \Omega_{\text{out}}; \tag{4.9}$$

$$[\sigma\nabla v\cdot\nu] = q(t)\,, \qquad \text{on } \Gamma; \tag{4.10}$$

$$\alpha\frac{\partial}{\partial t}[v] + f([v]) = \sigma_{\text{out}}\nabla v^{(\text{out})}\cdot\nu + h(t)\,, \qquad \text{on } \Gamma; \tag{4.11}$$

$$[v](x,0) = S\,, \qquad \text{on } \Gamma; \tag{4.12}$$

$$v(x,t) = 0\,, \qquad \text{on } \partial\Omega; \tag{4.13}$$

admits a unique solution $v \in L^2(0,T;\mathcal{H}^1_o(\Omega))$ *with* $[v] \in C(0,T;L^2(\Gamma))$*, where* $\mathcal{H}^1_o(\Omega) = \{u=(u_1,u_2) \mid u_1 := u_{|\Omega_{\rm int}}\,,\ u_2 := u_{|\Omega_{\rm out}} \text{ with } u_1,u_2 \in H^1_o(\Omega)\}$.

The technique employed to prove this theorem relies on a result of existence and uniqueness for abstract parabolic equations (see [19], Chapter 23), to which problem (4.9)–(4.13) can be reduce, by means of a suitable identification of the function spaces there involved.

In order to be more specific about the geometry of the domains of interest, let us introduce a periodic open subset E of $\boldsymbol{R}^N$, so that $E+z=E$ for all $z \in \boldsymbol{Z}^N$. For all $\varepsilon>0$ define $\Omega^\varepsilon_{\rm int} = \Omega \cap \varepsilon E$, $\Omega^\varepsilon_{\rm out} = \Omega \setminus \overline{\varepsilon E}$. We assume that Ω, E have regular boundary, say of class C^∞ for the sake of simplicity. We also employ the notation $Y=(0,1)^N$, and $E_{\rm int} = E \cap Y$, $E_{\rm out} = Y \setminus \overline{E}$, $\Gamma = \partial E \cap \overline{Y}$. As a simplifying assumption, we stipulate that $|\Gamma \cap \partial Y|_{N-1} = 0$.
Essentially, we will show that, if $\gamma^{-1}\varepsilon \le S_\varepsilon(x) \le \gamma\varepsilon$, where S_ε is the initial jump prescribed in (4.6), for a fixed constant $\gamma>1$, then u_ε becomes stable as $\varepsilon \to 0$ (i.e., it converges to a nonvanishing bounded function). Therefore, let us stipulate that

$$S_\varepsilon(x) = \varepsilon S_1\big(x,\frac{x}{\varepsilon}\big) + \varepsilon R_\varepsilon(x)\,, \tag{4.14}$$

where $S_1 : \Omega \times \partial E \to \boldsymbol{R}$, and

$$\begin{aligned} &\|S_1\|_{L^\infty(\Omega\times\partial E)} < \infty\,, \qquad \|R_\varepsilon\|_{L^\infty(\Omega)} \to 0\,,\ \text{as } \varepsilon \to 0;\\ &S_1(x,y) \text{ is continuous in } x, \text{ uniformly over } y \in \partial E,\\ &\text{and periodic in } y, \text{ for each } x \in \Omega. \end{aligned} \tag{4.15}$$

In [5], under the assumptions above, we prove existence and uniqueness of a weak solution to (4.2)–(4.7), in the class

$$u_{\varepsilon|\Omega^\varepsilon_i} \in L^2(0,T;H^1(\Omega^\varepsilon_i))\,, \qquad i=1\,,2\,, \tag{4.16}$$

and $u_{\varepsilon|\partial\Omega} = 0$ in the sense of traces. The weak formulation of the problem is

$$\begin{aligned} \int_0^T\!\!\int_\Omega \sigma\nabla u_\varepsilon \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_0^T\!\!\int_{\Gamma^\varepsilon} f([u_\varepsilon])[\psi]\,\mathrm{d}\sigma\,\mathrm{d}t \\ - \frac{\alpha}{\varepsilon}\int_0^T\!\!\int_{\Gamma^\varepsilon} [u_\varepsilon]\frac{\partial}{\partial t}[\psi]\,\mathrm{d}\sigma\,\mathrm{d}t - \frac{\alpha}{\varepsilon}\int_{\Gamma^\varepsilon}[u_\varepsilon](0)[\psi](0)\,\mathrm{d}\sigma = 0\,, \end{aligned} \tag{4.17}$$

for each $\psi \in L^2(\Omega\times(0,T))$ such that ψ is in the class (4.16), $[\psi] \in H^1(0,T;L^2(\Gamma^\varepsilon))$, and ψ vanishes on $\partial\Omega\times(0,T)$, as well as at $t=T$.
In this section we aim at identifying the form of the homogenized equation, via the two-scale method (see [8], [16], [18]). Introduce the microscopic variables $y \in Y$, $y = x/\varepsilon$, assuming

$$u_\varepsilon = u_\varepsilon(x,y,t) = u_0(x,y,t) + \varepsilon u_1(x,y,t) + \varepsilon^2 u_2(x,y,t) + \dots\,. \tag{4.18}$$

Note that u_0, u_1, u_2 are periodic in y, and u_1, u_2 are assumed to have zero integral average over Y. Recalling that

$$\mathrm{div} = \frac{1}{\varepsilon}\,\mathrm{div}_y + \mathrm{div}_x\,, \qquad \nabla = \frac{1}{\varepsilon}\nabla_y + \nabla_x\,, \tag{4.19}$$

we compute

$$\Delta u_\varepsilon = \frac{1}{\varepsilon^2} A_0 u_0 + \frac{1}{\varepsilon}(A_0 u_1 + A_1 u_0) + (A_0 u_2 + A_1 u_1 + A_2 u_0) + \dots\,, \tag{4.20}$$

Here

$$A_0 = \Delta_y\,, \quad A_1 = \operatorname{div}_y \nabla_x + \operatorname{div}_x \nabla_y\,, \quad A_2 = \Delta_x\,. \tag{4.21}$$

Let us recall explicitly that

$$\nabla u_\varepsilon = \frac{1}{\varepsilon}\nabla_y u_0 + \big(\nabla_x u_0 + \nabla_y u_1\big) + \varepsilon\big(\nabla_y u_2 + \nabla_x u_1\big) + \dots\,, \tag{4.22}$$

and stipulate, in addition to (4.14),

$$S_\varepsilon = S_\varepsilon(x,y) = \varepsilon S_1(x,y) + \varepsilon^2 S_2(x,y) + \dots\,. \tag{4.23}$$

Finally we expand

$$f([u_\varepsilon]) = f([u_0]) + \varepsilon f'([u_0])[u_1] + \varepsilon^2\big\{f'([u_0])[u_2] + \frac{1}{2} f''([u_0])[u_1]^2\big\} + \dots\,.$$

4.1. **The term of order ε^{-2}.** Equating the first term on the right hand side of (4.20) to zero, and applying (4.18), (4.22) to (4.2)–(4.6) we find

$$-\sigma\,\Delta_y u_0 = 0\,, \qquad \text{in } E_{\text{int}},\, E_{\text{out}}; \tag{4.24}$$

$$\sigma_{\text{int}} \nabla_y u_0^{(\text{int})} \cdot \nu = \sigma_{\text{out}} \nabla_y u_0^{(\text{out})} \cdot \nu\,, \qquad \text{on } \Gamma; \tag{4.25}$$

$$\alpha \frac{\partial}{\partial t}[u_0] + f([u_0]) = \sigma_{\text{out}} \nabla_y u_0^{(\text{out})} \cdot \nu\,, \qquad \text{on } \Gamma; \tag{4.26}$$

$$[u_0](x,y,0) = 0\,, \qquad \text{on } \Gamma. \tag{4.27}$$

In (4.27) we have also exploited the expansion (4.23). It follows (see [5], and recall that $f(0)=0$) that

$$u_0 = u_0(x,t)\,. \tag{4.28}$$

4.2. **The term of order ε^{-1}.** Proceeding as above, but taking into consideration the second term on the right hand side of (4.20) we obtain

$$-\sigma\,\Delta_y u_1 = \sigma A_1 u_0 = 0\,, \qquad \text{in } E_{\text{int}},\, E_{\text{out}}; \tag{4.29}$$

$$[\sigma \nabla_y u_1 \cdot \nu] = -[\sigma \nabla_x u_0 \cdot \nu]\,, \qquad \text{on } \Gamma; \tag{4.30}$$

$$\alpha \frac{\partial}{\partial t}[u_1] + f'(0)[u_1] = \sigma_{\text{out}} \nabla_y u_1^{(\text{out})} \cdot \nu + \sigma_{\text{out}} \nabla_x u_0 \cdot \nu\,, \qquad \text{on } \Gamma; \tag{4.31}$$

$$[u_1](x,y,0) = S_1(x,y)\,, \qquad \text{on } \Gamma. \tag{4.32}$$

In (4.29) and in (4.31) we have made use of (4.28), and of its consequence $[u_0]=0$.

4.3. **The $\mathcal{T}$ transform. Cell functions.** Let $s:\Gamma \to \boldsymbol{R}$ be a jump function. Consider the problem

$$-\sigma\,\Delta_y v = 0\,, \qquad \text{in } E_{\text{int}},\, E_{\text{out}}; \tag{4.33}$$

$$[\sigma \nabla_y v \cdot \nu] = 0\,, \qquad \text{on } \Gamma; \tag{4.34}$$

$$\alpha \frac{\partial}{\partial t}[v] + f'(0)[v] = \sigma_{\text{out}} \nabla_y v^{(\text{out})} \cdot \nu\,, \qquad \text{on } \Gamma; \tag{4.35}$$

$$[v](y,0) = s(y)\,, \qquad \text{on } \Gamma, \tag{4.36}$$

where v is a periodic function in Y, such that $\int_Y v = 0$. Define the transform $\mathcal{T}$ by

$$\mathcal{T}(s)(y,t) = v(y,t)\,, \qquad y \in Y\,, t > 0\,,$$

and extend the definition of $\mathcal{T}$ to vector (jump) functions, by letting it act componentwise on its argument.
Introduce also the functions $\chi^0 : Y \to \boldsymbol{R}^N$, $\chi^1 : Y \times (0,T) \to \boldsymbol{R}^N$ as follows. The components χ^0_h, $h = 1, \dots, N$, satisfy

$$-\sigma\,\Delta_y\,\chi^0_h = 0\,, \qquad \text{in } E_{\text{int}},\ E_{\text{out}}; \tag{4.37}$$

$$[\sigma(\nabla_y\chi^0_h - \boldsymbol{e}_h)\cdot\nu] = 0\,, \qquad \text{on } \Gamma; \tag{4.38}$$

$$[\chi^0_h] = 0\,, \qquad \text{on } \Gamma. \tag{4.39}$$

We also require χ^0_h to be a periodic function with vanishing integral average over Y. Moreover χ^1_h is defined from

$$\alpha\chi^1_h = \mathcal{T}\big(\sigma_{\text{out}}(\nabla_y\chi_h^{0\,(\text{out})} - \boldsymbol{e}_h)\cdot\nu\big)\,. \tag{4.40}$$

Let us stipulate that u_1 may be written in the form

$$u_1(x,y,t) = -\chi^0(y)\cdot\nabla_x u_0(x,t) + \mathcal{T}(S_1(x,\cdot))(y,t) - \int_0^t \nabla_x u_0(x,\tau)\cdot\chi^1(y,t-\tau)\,\mathrm{d}\tau\,. \tag{4.41}$$

It is worthwhile making some remarks on the structure of u_1, which is made of three parts: the first one is the standard one (see (3.6)); while the second one keeps into account the effect of the initial datum. The real novelty is the third integral term, which is a non local memory term, due to the capacitive behavior of the membrane; i.e., to the dynamical nature of condition (4.5).

4.4. **Reconciling** (4.41) **with** (4.29)–(4.32)**.** Equations (4.29) are equivalent to (4.37), when we remember that the terms χ^1_h and $\mathcal{T}(S_1(x,\cdot))$ in (4.41) fulfil (4.33). Next, let us impose (4.30) in (4.41). We get, on recalling (4.34)

$$[\sigma\nabla_y u_1\cdot\nu] = -[\sigma\nabla_y\chi^0_h(y)\cdot\nu]u_{0x_h}(x,t) = -[\sigma\nabla_x u_0\cdot\nu] = -[\sigma u_{0x_h}(x,t)\nu_h]\,.$$

In order to satisfy this requirement, we prescribe (4.38). Note that (4.32) is obviously satisfied, owing to the definition of $\mathcal{T}$.
Finally we get to (4.31), which we combine with (4.41) obtaining

$$\begin{aligned}\alpha\frac{\partial}{\partial t}[u_1] + f'(0)[u_1] = {}& -\alpha[\chi^0(y)\cdot\frac{\partial}{\partial t}\nabla_x u_0(x,t)] + \alpha\frac{\partial}{\partial t}[\mathcal{T}(S_1(x,\cdot))](y,t)\\ & - \alpha\nabla_x u_0(x,t)\cdot[\chi^1](y,0) - \alpha\int_0^t \nabla_x u_0(x,\tau)\cdot\frac{\partial}{\partial t}[\chi^1](y,t-\tau)\,\mathrm{d}\tau\\ & - f'(0)[\chi^0(y)\cdot\nabla_x u_0(x,t)] + f'(0)[\mathcal{T}(S_1(x,\cdot))](y,t)\\ & - f'(0)\int_0^t \nabla_x u_0(x,\tau)\cdot[\chi^1](y,t-\tau)\,\mathrm{d}\tau\,.\end{aligned}$$

On the other hand,

$$\sigma_{\rm out}\nabla_y u_1^{\rm (out)}\cdot\nu+\sigma_{\rm out}\nabla_x u_0\cdot\nu=-\sigma_{\rm out}\nabla_y\chi_h^{0\,{\rm (out)}}(y)\cdot\nu u_{0x_h}(x,t)$$
$$+\sigma_{\rm out}\nabla_y\mathcal{T}(S_1(x,\cdot))^{\rm (out)}\cdot\nu-\int_0^t u_{0x_h}(x,\tau)\sigma_{\rm out}\nabla_y\chi_h^1(y,t-\tau)\cdot\nu\,{\rm d}\tau+\sigma_{\rm out}u_{0x_h}(x,t)\nu_h\,.$$

Hence, (4.39)–(4.40) follow, on equating the quantities above.

4.5. **The term of order ε^0.** Let us first calculate

$$A_1u_1=2\frac{\partial^2u_1}{\partial x_j\partial y_j}\,,$$

where we employ the summation convention. Therefore, the complete problem involving the third term on the right hand side of (4.20) is

$$-\sigma\,\Delta_y\,u_2=\sigma\,\Delta_x\,u_0+2\sigma\frac{\partial^2u_1}{\partial x_j\partial y_j}\,,\qquad E_{\rm int},E_{\rm out} \tag{4.42}$$

$$[\sigma\nabla_y u_2\cdot\nu]=-[\sigma\nabla_x u_1\cdot\nu]\,,\qquad\text{on }\Gamma; \tag{4.43}$$

$$\alpha\frac{\partial}{\partial t}[u_2]+f'(0)[u_2]+\frac{f''(0)}{2}[u_1]^2=\sigma_{\rm out}\nabla_x u_1^{\rm (out)}\cdot\nu+\sigma_{\rm out}\nabla_y u_2^{\rm (out)}\cdot\nu\,,\qquad\text{on }\Gamma; \tag{4.44}$$

$$[u_2](x,y,0)=S_2(x,y)\,,\qquad\text{on }\Gamma. \tag{4.45}$$

4.6. **Formal derivation of the homogenized equation.** Integrating by parts equation (4.42) both in $E_{\rm int}$ and in $E_{\rm out}$, and adding the two contributions, we get

$$\left[\int_{E_{\rm int}}+\int_{E_{\rm out}}\right]\left\{\sigma\,\Delta_x\,u_0(x,t)+2\sigma\frac{\partial^2u_1}{\partial x_j\partial y_j}\right\}{\rm d}y$$
$$=\int_\Gamma\left\{\sigma_{\rm out}\nabla_y u_2^{\rm (out)}\cdot\nu-\sigma_{\rm int}\nabla_y u_2^{\rm (int)}\cdot\nu\right\}{\rm d}\sigma=\int_\Gamma[\sigma\nabla_y u_2\cdot\nu]\,{\rm d}\sigma=-\int_\Gamma[\sigma\nabla_x u_1\cdot\nu]\,{\rm d}\sigma\,.$$

Thus

$$\sigma_0\,\Delta_x\,u_0=2\int_\Gamma[\sigma\nabla_x u_1\cdot\nu]\,{\rm d}\sigma-\int_\Gamma[\sigma\nabla_x u_1\cdot\nu]\,{\rm d}\sigma=\int_\Gamma[\sigma\nabla_x u_1\cdot\nu]\,{\rm d}\sigma\,,$$

where we denote

$$\sigma_0=\sigma_{\rm int}|E_{\rm int}|+\sigma_{\rm out}|E_{\rm out}|\,. \tag{4.46}$$

We use next the expansion (4.41); namely, we recall that, in it, only last two terms on the right hand side have a non zero jump across Γ. Thus we infer from the equality above

$$\sigma_0\,\Delta_x\,u_0=\int_\Gamma[\sigma\nabla_x u_1\cdot\nu]\,{\rm d}\sigma=-\int_\Gamma[\sigma]\chi_h^0(y)\nu_j\,{\rm d}\sigma u_{0x_hx_j}(x,t)$$
$$+\frac{\partial}{\partial x_j}\int_\Gamma[\sigma\mathcal{T}(S_1(x,\cdot))](y,t)\nu_j\,{\rm d}\sigma-\int_0^t u_{0x_hx_j}(x,\tau)\int_\Gamma[\sigma\chi_h^1](y,t-\tau)\nu_j\,{\rm d}\sigma\ {\rm d}\tau\,.$$

We finally write the PDE for u_0 in $\Omega \times (0,T)$ as

$$-\operatorname{div}\left(\sigma_0 \nabla_x u_0 + A^0 \nabla_x u_0 + \int_0^t A^1(t-\tau)\nabla_x u_0(x,\tau)\,\mathrm{d}\tau - \int_\Gamma [\sigma \mathcal{T}(S_1(x,\cdot))](y,t)\nu\,\mathrm{d}\sigma\right) = 0\,. \quad (4.47)$$

The two matrices A^i are defined by

$$(A^0)_{jh} = \int_\Gamma [\sigma]\chi_h^0(y)\nu_j\,\mathrm{d}\sigma\,, \qquad (A^1(t))_{jh} = \int_\Gamma [\sigma\chi_h^1](y,t)\nu_j\,\mathrm{d}\sigma\,. \quad (4.48)$$

The matrices A^0 and A^1 are symmetric, and $\sigma_0 I + A^0$ is positive definite (see [3], Section 4). Equation (4.47) is different from the standard elliptic equation which are usually employed in bioimpedentiometric tomography. In fact, the appearance of the memory term in the equation (which, from the mathematical point of view, is a consequence of the structure of the first corrector u_1) seems to be in agreement with the fact that a contribution to current flux is produced not only by the potential applied to the boundary but also by the charge and discharge cycles of the membranes; i.e., to their capacitive behaviour.

For equation (4.47), complemented with boundary conditions (e.g. Dirichlet boundary condition), an existence and uniqueness theorem, both for weak and classical solutions, is available (see [4]). Indeed, let us consider the following problem

$$\begin{cases} -\operatorname{div}\left(A(x)\nabla_x u + \int_0^t B(x,t-\tau)\nabla_x u(x,\tau)\,d\tau\right) = g(x,t) & \text{in } \Omega\times(0,T)\,, \\ u = f & \text{in } \partial\Omega\times(0,T)\,, \end{cases} \quad (4.49)$$

where $A(x)$ is a simmetric and positive definite matrix, $B(x,t)$ is a simmetric matrix, $g:\Omega\times(0,T)\to \boldsymbol{R}$ and $f:\overline{\Omega}\times(0,T)\to\boldsymbol{R}$ are given functions. Then the two following results hold true.

Theorem 4.2. *Let $A \in L^\infty(\Omega;\boldsymbol{R}^{N^2})$ be such that $\lambda|\xi|^2 \le A(x)\xi\cdot\xi \le \Lambda|\xi|^2$, for suitable $0<\lambda<\Lambda<+\infty$, for almost every $x\in\Omega$ and every $\xi\in\boldsymbol{R}^N$; let $B \in L^2(0,T;L^\infty(\Omega;\boldsymbol{R}^{N^2}))$, and let $f\in L^2(0,T;H^1(\Omega))$. Assume that $g:\Omega\times(0,T)\to\boldsymbol{R}$ is a Carathéodory function such that $g\in L^2(0,T;H^{-1}(\Omega))$.*
Then, there exists a unique function $u\in L^2(0,T;H^1(\Omega))$ satisfying in the sense of distributions problem (4.49).

Theorem 4.3. *Let $m\ge 0$ be any fixed integer and let also $0<\alpha<1$. Let $A \in C^{1+\alpha}(\overline{\Omega};\boldsymbol{R}^{N^2})$ satisfy the assumption of Theorem 4.2 and*

$$B\in C^0([0,T];C^{1+\alpha}(\overline{\Omega};\boldsymbol{R}^{N^2})) \qquad \textit{be such that} \qquad B'\in L^2(0,T;W^{1,\infty}(\Omega;\boldsymbol{R}^{N^2}))\,.$$

Assume that $g \in C^0([0,T]; C^{m+\alpha}(\overline{\Omega}))$, *and that* $\nabla_x g(x,t)$ *and* $g_t(x,t)$ *exist and are bounded. Let* $f \in C^0([0,T]; C^{m+2+\alpha}(\overline{\Omega}))$, *with* $f_t \in L^\infty(0,T; C^{m+2+\alpha}(\overline{\Omega}))$.
Then there exists a unique function $u \in C^0([0,T]; C^{1+\alpha}(\overline{\Omega})) \cap L^\infty(0,T; C^{m+2+\alpha}(\overline{\Omega}))$ *solving* (4.49) *in the classical sense.*

Both proofs can be obtained, for example, with a standard delay argument or a fixed point theorem, together with an a-priori estimate in the corresponding function spaces. The a-priori estimates are obtained as in standard elliptic equations, using also the Gronwall's Theorem to deal with the memory term.

Remark 4.4. In the case when $f(s) = s$, $g = 0$, one can give the following representation of u_2. For any given pair of jump functions s_1, $s_2 : \Gamma \to \boldsymbol{R}$ define $\widetilde{\mathcal{T}}_j(s_1, s_2) = v$, $j = 1, \dots, N$, where v is a periodic function with zero average in Y, solving

$$-\sigma\,\Delta_y\, v = \frac{\sigma}{\sigma_0}\int_\Gamma [\sigma\mathcal{T}(s_1)]\nu_j\,\mathrm{d}\sigma + 2\sigma\frac{\partial}{\partial y_j}\mathcal{T}(s_1)\,, \qquad \text{in } E_{\text{int}},\ E_{\text{out}}; \tag{4.50}$$

$$[\sigma\nabla_y v\cdot\nu] = -[\sigma\mathcal{T}(s_1)\nu_j]\,, \qquad \text{on } \Gamma; \tag{4.51}$$

$$\alpha\frac{\partial}{\partial t}[v] = \sigma_{\text{out}}\nabla_y v^{(\text{out})}\cdot\nu + \sigma_{\text{out}}\mathcal{T}(s_1)^{(\text{out})}\nu_j \qquad \text{on } \Gamma; \tag{4.52}$$

$$[v](y,0) = s_2(y)\,, \qquad \text{on } \Gamma. \tag{4.53}$$

Note that $\mathcal{T}(s) = \widetilde{\mathcal{T}}_j(0,s)$ for any choice of j. Moreover define the functions $\tilde\chi^0_{hj} : Y \to \boldsymbol{R}$, $\tilde\chi^1_{hj} : \Gamma \to \boldsymbol{R}$, for h, $j = 1, \dots, N$, by means of

$$-\sigma\,\Delta_y\,\tilde\chi^0_{hj} = -\frac{\sigma}{\sigma_0}A^0_{hj} - 2\sigma\frac{\partial}{\partial y_j}\chi^0_h\,, \qquad \text{in } E_{\text{int}},\ E_{\text{out}}; \tag{4.54}$$

$$[\sigma(\nabla_y\tilde\chi^0_{hj} - \chi^0_h\boldsymbol{e}_j)\cdot\nu] = 0\,, \qquad \text{on } \Gamma; \tag{4.55}$$

$$[\tilde\chi^0_{hj}] = 0\,, \qquad \text{on } \Gamma; \tag{4.56}$$

and of

$$\alpha\tilde\chi^1_{hj} = \sigma_{\text{out}}(\nabla_y\tilde\chi^{0\,(\text{out})}_{hj} - \chi^0_h\boldsymbol{e}_j)\cdot\nu\,, \qquad \text{on } \Gamma. \tag{4.57}$$

We require $\tilde\chi^0_{hj}$ to be a periodic function with vanishing integral average over Y. Then one can check that the problem for u_2, (4.42)–(4.45) is equivalent to the representation

$$u_2(x,y,t) = \tilde\chi^0_{hj}(y)\frac{\partial^2 u_0}{\partial x_h\partial x_j}(x,t) + \widetilde{\mathcal{T}}_j\Big(\frac{\partial S_1}{\partial x_j}(x,\cdot), S_2(x,\cdot)\Big)(y,t) + \int_0^t \frac{\partial^2 u_0}{\partial x_h\partial x_j}(x,\tau)\bar{\mathcal{T}}_{jh}(y,t-\tau)\,\mathrm{d}\tau\,, \tag{4.58}$$

where $\bar{\mathcal{T}}_{jh}(y,t) = \widetilde{\mathcal{T}}_j(-\chi^1_h, \tilde\chi^1_{hj})(y,t)$. □

4.7. **Final remarks.** We would like to observe that the model considered in these Notes, together with the one corresponding to $k = -1$ in (4.1), preserves, in the limit, the membrane properties (i.e., the constant α). This is not true for all the other choices of k. Indeed, we expect that, both for case $k = 1$ and $k = -1$, assigning an alternating potential on the boundary will result in a periodic steady state or a limit cycle as $t \to +\infty$, eventually displaying also a phase delay, as it is expected in a resistive-capacitive circuit. It remains open to decide what is the more suitable model between these two. Indeed, it seems that both of them are valid in their respective frequency ranges; also we think that the one treated in these Notes is more apt to describe the response of a biological tissue to an impulse potential.

The applicability of the model to real physical situations is connected to the study of an inverse problem, which for the elliptic equation was traditionally related to the study of the Neumann-Dirichlet map. This problem has been widely studied. On the contrary (a part from some geometrically simple cases), the inverse problem for equation (4.47) is still open; also, the Dirichlet-Neumann map has to replaced by the one in which the Dirichlet boundary condition is assigned together with the condition:

$$\sigma_0 \frac{\partial u_0}{\partial n} + A^0_{ij} \frac{\partial u_0}{\partial x_i} n_j + \int_0^t A^1_{ij}(t-\tau) \frac{\partial u_0}{\partial x_i}(x,\tau) n_j \, \mathrm{d}\tau = h(x,t)\,,$$

where n is the outward normal to $\partial\Omega$ and h is a given function.

References

[1] R. Adams: *Sobolev spaces*, Academic Press, New York, 1975.

[2] M. Amar, D. Andreucci, P. Bisegna, R. Gianni: Homogenization limit for electrical conduction in biological tissues in the radio-frequency range, *C.R. Mecanique, vol. 331, (2003), pp. 503–508.*

[3] M. Amar, D. Andreucci, P. Bisegna, R. Gianni: Evolution and memory effects in the homogenization limit for electrical conduction in biological tissues, *Mathematical Models and Methods in Applied Sciences, Vol. 14(9), (2004), pp. 1261–1295.*

[4] M. Amar, D. Andreucci, P. Bisegna, R. Gianni: An elliptic equation with history, *C.R. Acad. Sci. Paris, Ser. I 338, (2004), pp. 595–598.*

[5]] M. Amar, D. Andreucci, P. Bisegna, R. Gianni: Existence and uniqueness for an elliptic problem with evolution arising in electrodynamics, *Nonlinear Analysis: RealWorld Applications, Vol. 6, (2005), pp. 367-380.*

[6] I. Babuška: Solution of interface problems by homogenization I, II, III, *Siam J. Math. Anal., 7 (1976), pp. 603–634 and pp. 635–645, 8 (1977), pp. 923–937.*

[7] N. Bakhvalov, G. Panasenko: *Homogenization: averaging processes in periodic media*, Mathematics and its Applications, Vol. 36, Kluwer Academic Publishers, Dordrecht, 1990.

[8] A. Bensoussan, J.L. Lions, G.Papanicolaou: *Asymptotic Analysis for Periodic Structures*, North Holland, Amsterdam, 1978.

[9] A. Bensoussan, J.L. Lions, G. Papanicolaou: Boundary layers and homogenization of transport processes, *Publ. RIMS, Kyoto Univ., 15 (1979), pp. 53–157.*

[10] A. Braides: Γ-*convergence for beginners.* Oxford University Press, New York, (2002).

[11] A. Braides, A. Defranceschi: *Homogenization of Multiple Integrals.* Oxford University Press, New York, (1998).

[12] H. Brezis: *Analyse fonctionelle*, Masson, Paris, 1983.

[13] G. Dal Maso: *An introduction to* Γ-*convergence.* Birkhäuser, Boston, (1993).

[14] D. Gilbarg, N.S. Trudinger: *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 1983.

[15] W. Krassowska, J.C. Neu: Homogenization of Syncytial Tissues", *Critical Reviews in Biomedical Engineering, 21 (1993), pp. 137–199.*

[16] J.L. Lions: *Some Methods in the Mathematical Analysis of Systems and Their Controls*, Science Press, Beijing, 1981.

[17] R. Lipton: Heat Conduction in Fine Scale Mixtures with Interfacial Contact Resistance, *SIAM Journal of Applied Mathematics, 58 (1998), pp. 55–72.*

[18] E. Sánchez-Palencia *Non Homogeneous Media and Vibration Theory*, Lecture notes in physics, Vol. 127, Springer Verlag, 1980.

[19] E. Zeidler: *Nonlinear functional analysis and its applications*, vol. II/A, Springer-Verlag, Berlin, 1990.

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[2] FASANO A. & PRIMICERIO M., *Blow-up and regularization for the Hele-Shaw problem,* in *Variational and free boundary problems*, Friedman A. & Spruck J. (Eds.), IMA Math. Appl. Vol. 53, Springer Verlag, New York (1993), pp. 73-85.

[3] RODRIGUES J. F., *Obstacle problems in mathematical physics*, North-Holland Mathematics Studies N. 134, North-Holland, Amsterdam (1987).

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