

# A Calabi–Yau threefold with non-Abelian fundamental group

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## Abstract

This note, written in 1994, answers a question of Dolgachev by constructing a Calabi–Yau threefold whose fundamental group is the quaternion group  $H_8$ . The construction is reminiscent of Reid’s unpublished construction of a surface with  $p_g = 0$ ,  $K^2 = 2$  and  $\pi_1 = H_8$ ; I explain below the link between the two problems.

## 1 The example

Let  $H_8 = \{\pm 1, \pm i, \pm j, \pm k\}$  be the quaternion group of order 8, and  $V$  its regular representation. We denote by  $\widehat{H}_8$  the group of characters  $\chi: H_8 \rightarrow \mathbf{C}^*$ , which is isomorphic to  $\mathbf{Z}_2 \times \mathbf{Z}_2$ . The group  $H_8$  acts on  $\mathbb{P}(V)$  and<sup>1</sup> on  $S^2 V$ ; for each  $\chi \in \widehat{H}_8$ , we denote by  $(S^2 V)_\chi$  the eigensubspace of  $S^2 V$  with respect to  $\chi$ , that is, the space of quadratic forms  $Q$  on  $\mathbb{P}(V)$  such that  $h \cdot Q = \chi(h)Q$  for all  $h \in H_8$ .

**Theorem 1.1** *For each  $\chi \in \widehat{H}_8$ , let  $Q_\chi$  be a general element of  $(S^2 V)_\chi$ . The subvariety  $\widetilde{X}$  of  $\mathbb{P}(V)$  defined by the 4 equations*

$$Q_\chi = 0 \quad \text{for all } \chi \in \widehat{H}_8$$

*is a smooth threefold, on which the group  $H_8$  acts freely. The quotient  $X := \widetilde{X}/H_8$  is a Calabi–Yau threefold with  $\pi_1(X) = H_8$ .*

Let me observe first that the last assertion is an immediate consequence of the others. Indeed, since  $\widetilde{X}$  is a Calabi–Yau threefold, we have  $h^{1,0}(\widetilde{X}) = h^{2,0}(\widetilde{X}) = \chi(\mathcal{O}_{\widetilde{X}}) = 0$ , hence  $h^{1,0}(X) = h^{2,0}(X) = \chi(\mathcal{O}_X) = 0$ . This implies

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<sup>1</sup>I use Grothendieck’s notation, that is,  $\mathbb{P}(V)$  is the space of hyperplanes in  $V$ .

$h^{3,0}(X) = 1$ , so there exists a nonzero holomorphic 3-form  $\omega$  on  $X$ ; since its pullback to  $\tilde{X}$  is everywhere nonzero,  $\omega$  has the same property, hence  $X$  is a Calabi–Yau threefold. Finally  $\tilde{X}$  is a complete intersection in  $\mathbb{P}(V)$ , hence simply connected by the Lefschetz theorem, so the fundamental group of  $X$  is isomorphic to  $H_8$ .

So the problem is to prove that  $H_8$  acts freely and  $\tilde{X}$  is smooth. To do this, we will need to write down explicit elements of  $(S^2 V)_X$ . As an  $H_8$ -module,  $V$  is the direct sum of the 4 one-dimensional representations of  $H_8$  and twice the irreducible two-dimensional representation  $\rho$ . Thus there exists a system of homogeneous coordinates  $(X_1, X_\alpha, X_\beta, X_\gamma; Y, Z; Y', Z')$  such that

$$g \cdot (X_1, X_\alpha, X_\beta, X_\gamma; Y, Z; Y', Z') = (X_1, \alpha(g)X_\alpha, \beta(g)X_\beta, \gamma(g)X_\gamma; \rho(g)(Y, Z); \rho(g)(Y', Z')).$$

To be more precise, I denote by  $\alpha$  (respectively  $\beta, \gamma$ ) the nontrivial character which is  $+1$  on  $i$  (respectively  $j, k$ ), and I take for  $\rho$  the usual representation via Pauli matrices:

$$\begin{aligned} \rho(i)(Y, Z) &= (\sqrt{-1}Y, -\sqrt{-1}Z), & \rho(j)(Y, Z) &= (-Z, Y), \\ \rho(k)(Y, Z) &= (-\sqrt{-1}Z, -\sqrt{-1}Y). \end{aligned}$$

Then the general element  $Q_X$  of  $(S^2 V)_X$  can be written

$$\begin{aligned} Q_1 &= t_1^1 X_1^2 + t_2^1 X_\alpha^2 + t_3^1 X_\beta^2 + t_4^1 X_\gamma^2 + t_5^1 (Y Z' - Y' Z), \\ Q_\alpha &= t_1^\alpha X_1 X_\alpha + t_2^\alpha X_\beta X_\gamma + t_3^\alpha Y Z + t_4^\alpha Y' Z' + t_5^\alpha (Y Z' + Z Y'), \\ Q_\beta &= t_1^\beta X_1 X_\beta + t_2^\beta X_\alpha X_\gamma + t_3^\beta (Y^2 + Z^2) + t_4^\beta (Y'^2 + Z'^2) + t_5^\beta (Y Y' + Z Z'), \\ Q_\gamma &= t_1^\gamma X_1 X_\gamma + t_2^\gamma X_\alpha X_\beta + t_3^\gamma (Y^2 - Z^2) + t_4^\gamma (Y'^2 - Z'^2) + t_5^\gamma (Y Y' - Z Z'). \end{aligned}$$

For fixed  $\mathbf{t} := (t_i^j)$ , let  $\mathcal{X}_\mathbf{t}$  be the subvariety of  $\mathbb{P}(V)$  defined by the equations  $Q_X = 0$ . Let us check first that the action of  $H_8$  on  $\mathcal{X}_\mathbf{t}$  has no fixed points for  $\mathbf{t}$  general enough. Since a point fixed by an element  $h$  of  $H_8$  is also fixed by  $h^2$ , it is sufficient to check that the element  $-1 \in H_8$  acts without fixed point, that is, that  $\mathcal{X}_\mathbf{t}$  does not meet the linear subspaces  $L_+$  and  $L_-$  defined by  $Y = Z = Y' = Z' = 0$  and  $X_1 = X_\alpha = X_\beta = X_\gamma = 0$  respectively.

Let  $x = (0, 0, 0, 0; Y, Z; Y', Z') \in \mathcal{X}_\mathbf{t} \cap L_-$ . One of the coordinates, say  $Z$ , is nonzero; since  $Q_1(x) = 0$ , there exists  $k \in \mathbb{C}$  such that  $Y' = kY$ ,  $Z' = kZ$ . Substituting in the equations  $Q_\alpha(x) = Q_\beta(x) = Q_\gamma(x) = 0$  gives

$$\begin{aligned} (t_3^\alpha + t_5^\alpha k + t_4^\alpha k^2) Y Z &= (t_3^\beta + t_5^\beta k + t_4^\beta k^2) (Y^2 + Z^2) = \\ &= (t_3^\gamma + t_5^\gamma k + t_4^\gamma k^2) (Y^2 - Z^2) = 0 \end{aligned}$$

which has no nonzero solutions for a generic choice of  $\mathbf{t}$ .

Now let  $x = (X_1, X_\alpha, X_\beta, X_\gamma; 0, 0; 0, 0) \in \mathcal{X}_t \cap L_+$ . As soon as the  $t_i^x$  are nonzero, two of the  $X$ -coordinates cannot vanish, otherwise all the coordinates would be zero. Expressing that  $Q_\beta = Q_\gamma = 0$  has a nontrivial solution in  $(X_\beta, X_\gamma)$  gives  $X_\alpha^2$  as a multiple of  $X_1^2$ , and similarly for  $X_\beta^2$  and  $X_\gamma^2$ . But then  $Q_1(x) = 0$  is impossible for a general choice of  $t$ .

Now we want to prove that  $\mathcal{X}_t$  is smooth for  $t$  general enough. Let  $\mathcal{Q} = \bigoplus_{\chi \in \widehat{H}_8} (S^2 V)_\chi$ ; then  $t := (t_i^x)$  is a system of coordinates on  $\mathcal{Q}$ . The equations  $Q_\chi = 0$  define a subvariety  $\mathcal{X}$  in  $\mathcal{Q} \times \mathbb{P}(V)$ , whose fibre above a point  $t \in \mathcal{Q}$  is  $\mathcal{X}_t$ . Consider the second projection  $p: \mathcal{X} \rightarrow \mathbb{P}(V)$ . For  $x \in \mathbb{P}(V)$ , the fibre  $p^{-1}(x)$  is the linear subspace of  $\mathcal{Q}$  defined by the vanishing of the  $Q_\chi$ , viewed as linear forms in  $t$ . These forms are clearly linearly independent as soon as they do not vanish. In other words, if we denote by  $B_\chi$  the base locus of the quadrics in  $(S^2 V)_\chi$  and put  $B = \bigcup B_\chi$ , the map  $p: \mathcal{X} \rightarrow \mathbb{P}(V)$  is a vector bundle fibration above  $\mathbb{P}(V) \setminus B$ ; in particular  $\mathcal{X}$  is nonsingular outside  $p^{-1}(B)$ . Therefore it is enough to prove that  $\mathcal{X}_t$  is smooth at the points of  $B \cap \mathcal{X}_t$ .

Observe that an element  $x$  in  $B$  has two of its  $X$ -coordinates zero. Since the equations are symmetric in the  $X$ -coordinates we may assume  $X_\beta = X_\gamma = 0$ . Then the Jacobian matrix

$$\left( \frac{\partial Q_\chi}{\partial X_\psi}(x) \right) \quad \text{takes the form} \quad \begin{pmatrix} 2t_1^1 X_1 & 2t_2^1 X_\alpha & 0 & 0 \\ t_1^\alpha X_\alpha & t_1^\alpha X_1 & 0 & 0 \\ 0 & 0 & t_1^\beta X_1 & t_2^\beta X_\alpha \\ 0 & 0 & t_2^\gamma X_\alpha & t_1^\gamma X_1 \end{pmatrix}.$$

For generic  $t$ , this matrix is of rank 4 except when all the  $X$ -coordinates of  $x$  vanish; but we have seen that this is impossible when  $t$  is general enough.  $\square$

## 2 Some comments

As mentioned in the introduction, the construction is inspired by Reid's example [R] of a surface of general type with  $p_g = 0$ ,  $K^2 = 2$ ,  $\pi_1 = H_8$ . This is more than a coincidence. In fact, let  $\tilde{S}$  be the hyperplane section  $X_1 = 0$  of  $\tilde{X}$ . It is stable under the action of  $H_8$  (so that  $H_8$  acts freely on  $\tilde{S}$ ), and we can prove as above that it is smooth for a generic choice of the parameters. The surface  $S := \tilde{S}/H_8$  is a Reid surface, embedded in  $X$  as an ample divisor, with  $h^0(X, \mathcal{O}_X(S)) = 1$ . In general, let us consider a Calabi–Yau threefold  $X$  which contains a *rigid ample surface*, that is, a smooth ample divisor  $S$  such that  $h^0(\mathcal{O}_X(S)) = 1$ . Put  $L := \mathcal{O}_X(S)$ . Then  $S$  is a minimal surface of general type (because  $K_S = L|_S$  is ample); by the Lefschetz theorem, the natural map  $\pi_1(S) \rightarrow \pi_1(X)$  is an isomorphism. Because of the exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow L \rightarrow K_S \rightarrow 0,$$

the geometric genus  $p_g(S) := h^0(K_S)$  is zero.

We have  $K_S^2 = L^3$ ; the Riemann–Roch theorem on  $X$  yields

$$1 = h^0(L) = \frac{L^3}{6} + \frac{L \cdot c_2}{12}.$$

Since  $L \cdot c_2 > 0$  as a consequence of Yau’s theorem (see for instance [B], Cor. 2), we obtain  $K_S^2 \leq 5$ .

For surfaces with  $p_g = 0$  and  $K_S^2 = 1$  or  $2$ , we have a great deal of information about the *algebraic* fundamental group, that is the profinite completion of the fundamental group (see [B-P-V] for an overview). In the case  $K_S^2 = 1$ , the algebraic fundamental group is cyclic of order  $\leq 5$ ; if  $K_S^2 = 2$ , it is of order  $\leq 9$ ; moreover the dihedral group  $D_8$  cannot occur. D. Naie [N] has recently proved that the symmetric group  $\mathfrak{S}_3$  can also not occur; therefore the quaternion group  $H_8$  is the only non-Abelian group which occurs in this range.

On the other hand, little is known about surfaces with  $p_g = 0$  and  $K_S^2 = 3, 4$  or  $5$ . Inoue has constructed examples with  $\pi_1 = H_8 \times (\mathbb{Z}_2)^n$ , with  $n = K^2 - 2$  (*loc. cit.*); I do not know if they can appear as rigid ample surfaces in a Calabi–Yau threefold.

Let us denote by  $\tilde{X}$  the universal cover of  $X$ , by  $\tilde{L}$  the pullback of  $L$  to  $\tilde{X}$ , and by  $\rho$  the representation of  $G$  on  $H^0(\tilde{X}, \tilde{L})$ . We have  $\text{Tr } \rho(g) = 0$  for  $g \neq 1$  by the holomorphic Lefschetz formula, and  $\text{Tr } \rho(1) = \chi(\tilde{L}) = |G| \chi(L) = |G|$ . Therefore  $\rho$  is isomorphic to the regular representation. Looking at the list in *loc. cit.* we get a few examples of this situation, for instance:

- $G = \mathbb{Z}_5$ ,  $\tilde{X}$  = a quintic hypersurface in  $\mathbb{P}^4$ ;
- $G = (\mathbb{Z}_2)^3$  or  $\mathbb{Z}_4 \times \mathbb{Z}_2$ ,  $\tilde{X}$  = an intersection of 4 quadrics in  $\mathbb{P}^7$  as above;
- $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ ,  $\tilde{X}$  = a hypersurface of bidegree  $(3, 3)$  in  $\mathbb{P}^2 \times \mathbb{P}^2$ .

Of course, when looking for Calabi–Yau threefolds with interesting  $\pi_1$ , there is no reason to assume that it contains an ample rigid surface. Observe however that if we want to use the preceding method, in other words, to find a projective space  $\mathbb{P}(V)$  with an action of  $G$  and a smooth invariant linearly normal Calabi–Yau threefold  $\tilde{X} \subset \mathbb{P}(V)$ , then the line bundle  $\mathcal{O}_{\tilde{X}}(1)$  will be the pullback of an ample line bundle  $L$  on  $X$ , and by the above argument the representation of  $G$  on  $V$  will be  $h^0(L)$  times the regular representation. This leaves little hope to find an invariant Calabi–Yau threefold when the product  $h^0(L)|G|$  becomes large.

## References

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