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A CANONICAL FORM FOR THE INCLUSION PRINCIPLE OF DYNAMIC SYSTEMS*

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Abstract. The inclusion principle provides a mathematical framework for comparing behavior of dynamic systems having different dimensions. Our main objective is to derive a canonical form for larger systems (expansions) that are obtained by expanding smaller systems (contractions). The form offers full freedom in selecting appropriate matrices for the expansion-contraction process. We will broaden the form to include feedback and propose an explicit characterization of contractible control laws subject to overlapping information structure constraints.

Key words. inclusion principle, expansion, contraction, canonical form, decentralized control

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1. Introduction.

The inclusion principle for dynamic systems [1], which was developed in the 1980s, is now a well-established mathematical framework for comparing systems having different dimensions (for a self-contained presentation of the early results, see [2]). In particular, the principle has been established as a useful tool in formulation of control laws for systems with overlapping information structure constraints [3, 4, 5, 6, 7, 8, 9, 10]. In the past decade, the research on the inclusion principle has been focused on providing a wide variety of conditions for expansion and contraction of continuous, discrete-time, and stochastic dynamic systems [11, 12, 13, 14, 15, 16, 17, 18, 19, 20], which helped resolve both theoretical aspects and practical benefits of the principle in control designs.

Expansion, being an intersection of aggregation [21] and restriction [1, 2, 3], raises a question: What system properties are retained after the expansion-contraction process has been completed [22]? Much progress has been made in identifying the conditions that ensure the invariancy of controllability, observability, and stabilizability in the expanded systems [23, 24, 25].

A central issue in the framework of overlapping decentralized control has been the problem of contractibility of feedback control laws [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. When a system contains overlapping subsystems, it is natural to add the locally available overlapping states to decentralized control in order to improve the performance of the overall system. This fact gives rise to the control design under overlapping information structure constraints, which is handled by expanding the systems into a larger space where the overlapping subsystems appear as disjoint. As a result of the expansion, overlapping decentralized control in the expanded space can be chosen by standard methods which are available for disjoint subsystems. After the selection is made, the expanded control law is contracted to the original space for implementation. While flexibility of the inclusion principle has been

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greatly improved by the new conditions guiding the expansion-contraction process [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20], the contractibility problem has not been satisfactorily resolved. A failure of a condition to provide the required contractibility of a control law is often hard to interpret; one is not sure if the choice of condition or selection of control law is inappropriate, or if contractibility is not possible due to an inherent structure of the system.

Our objective in this paper is to derive a canonical form for the inclusion principle in the spirit of canonical forms for linear dynamical systems [26, 27, 28, 29, 30]. By providing an explicit characterization of expanded systems, the form, as expected, simplifies the study of invariant properties in the expansion-contraction process. The proposed form involves expansion of inputs, outputs, and feedback control laws, thus broadening in an essential way the scope of the canonical form derived previously for state expansion only [1, 2]. A by-product of this fact is a complete resolution of the contractibility problem of expanded control laws for both static and dynamic controllers, which has a special significance in formulations of decentralized control for complex systems under overlapping information structure constraints.

The present paper is organized as follows: In section 2 the inclusion and contractibility of dynamic systems are formulated. Canonical forms for the inclusion principle are established in section 3. In section 4, a problem related to overlapping decentralized control is solved. Next the contractibility of dynamic controllers is discussed in section 5. Finally, in section 6, we offer a few concluding remarks.

2. Inclusion and contractibility. Consider a pair of linear time-invariant systems

$$\mathbf{S}: \quad \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx \end{cases} \tag{2.1}$$

and

$$\tilde{\mathbf{S}}: \quad \begin{cases} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}, \\ \tilde{y} = \tilde{C}\tilde{x}, \end{cases} \tag{2.2}$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, $y(t) \in \mathbf{R}^l$ are the state, input, and output of system $\mathbf{S}$ at time $t \geq 0$, and $\tilde{x}(t) \in \mathbf{R}^{\tilde{n}}$, $\tilde{u}(t) \in \mathbf{R}^{\tilde{m}}$, $\tilde{y}(t) \in \mathbf{R}^{\tilde{l}}$ are those of $\tilde{\mathbf{S}}$, and $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$, $C \in \mathbf{R}^{l \times n}$, $\tilde{A} \in \mathbf{R}^{\tilde{n} \times \tilde{n}}$, $\tilde{B} \in \mathbf{R}^{\tilde{n} \times \tilde{m}}$, $\tilde{C} \in \mathbf{R}^{\tilde{l} \times \tilde{n}}$ are constant matrices. Suppose

$$n \leq \tilde{n}, \quad m \leq \tilde{m}, \quad l \leq \tilde{l},$$

that is, $\mathbf{S}$ is smaller than $\tilde{\mathbf{S}}$. Denote by $x(t; x_0, u)$ and $y[x(t)]$ the state behavior and the corresponding output of system $\mathbf{S}$ for a fixed input $u(t)$ and for an initial state $x(0) = x_0$, respectively. Similar notation $\tilde{x}(t; \tilde{x}_0, \tilde{u})$ and $\tilde{y}[\tilde{x}(t)]$ are used for the state behavior and output of system $\tilde{\mathbf{S}}$.

Let us link systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$ through the following transformations:

$$V : \mathbf{R}^n \longrightarrow \mathbf{R}^{\tilde{n}}, \quad L : \mathbf{R}^m \longrightarrow \mathbf{R}^{\tilde{m}}, \quad T : \mathbf{R}^l \longrightarrow \mathbf{R}^{\tilde{l}}, \tag{2.3}$$

where

$$\operatorname{rank}(V) = n, \quad \operatorname{rank}(L) = m, \quad \operatorname{rank}(T) = l. \tag{2.4}$$

Denote the unique pseudoinverses of V, L, and T by V^+, L^+, and T^+, respectively, and recall the definition of the inclusion principle [1, 2].

DEFINITION 2.1. *The system* $\tilde{\mathbf{S}}$ *includes the system* $\mathbf{S}$*, that is,* $\mathbf{S}$ *is included by* $\tilde{\mathbf{S}}$*, if there exists a triplet* (V, L, T) *satisfying* (2.3) *and* (2.4) *such that, for any initial state* x_0 *and any fixed* $u(t)$ *of system* S*, the choice*

$$\tilde{x}_0 = Vx_0, \quad \tilde{u}(t) = Lu(t) \quad \forall t \geq 0 \tag{2.5}$$

of the initial state $\tilde{x}_0$ *and input* $\tilde{u}(t)$ *of the system* $\tilde{\mathrm{S}}$ *implies*

$$x(t; x_0, u) = V^+\tilde{x}(t; \tilde{x}_0, \tilde{u}), \quad y[x(t)] = T^+\tilde{y}[\tilde{x}(t)] \quad \forall t \geq 0. \tag{2.6}$$

If the system $\tilde{\mathbf{S}}$ *includes the system* $\mathbf{S}$*, then system* $\tilde{\mathbf{S}}$ *is said to be an expansion of the system* $\mathbf{S}$ *and system* $\mathbf{S}$ *is a contraction of system* $\tilde{\mathbf{S}}$*.*

The inclusion principle has been used to expand overlapping decentralized control laws into a larger space, where they appear disjoint, design disjoint laws by known methods, and *contract* them to the original space for implementation (see, e.g., [2]). The central issue in the expansion-contraction process is the problem of contractibility defined as follows [1, 4].

DEFINITION 2.2. *The control law*

$$\tilde{u} = -\tilde{K}\tilde{x} + \tilde{v}$$

given for system $\tilde{\mathbf{S}}$ *is contractible to the control law*

$$u = -Kx + v$$

for implementation in system $\mathbf{S}$ *if one of the following two statements holds:*

(a) *The choice*

$$\tilde{x}_0 = Vx_0, \quad \tilde{u}(t) = Lu(t)$$

implies

$$x(t; x_0, u) = V^+\tilde{x}(t; \tilde{x}_0, \tilde{u}), \quad LKx(t; x_0, u) = \tilde{K}\tilde{x}(t; \tilde{x}_0, \tilde{u}) \tag{2.7}$$

for all $t \geq 0$*, any initial state* x_0*, and any fixed input* $u(t)$ *of system* $\mathbf{S}$*.*

(b) *The choice*

$$\tilde{x}_0 = Vx_0, \quad u = L^+\tilde{u}$$

implies

$$x(t; x_0, u) = V^+\tilde{x}(t; \tilde{x}_0, \tilde{u}), \quad Kx(t; x_0, u) = L^+\tilde{K}\tilde{x}(t; \tilde{x}_0, \tilde{u}) \tag{2.8}$$

for all $t \geq 0$*, any initial state* x_0 *of system* $\mathbf{S}$*, and any fixed input* $\tilde{u}$ *of system* $\tilde{\mathbf{S}}$*.*

It should be pointed out that both conditions in (a) and (b) above ensure that the closed-loop system

$$\dot{\tilde{x}} = (\tilde{A} + \tilde{B}\tilde{K})\tilde{x} + \tilde{B}\tilde{v}$$

includes the closed-loop system

$$\dot{x} = (A + BK)x + Bv.$$

This property plays an important role in the application of the inclusion principle to overlapping decentralized control.

For the expansion-contraction and contractibility between systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$, the conditions are provided in the following theorem [4].

THEOREM 2.3. *Given systems* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$*, and transformations* V*,* L*, and* T *satisfying* (2.3) *and* (2.4).

(i) *System* $\tilde{\mathbf{S}}$ *is an expansion of system* $\mathbf{S}$ *if and only if for all* $i = 1, 2, \ldots, \tilde{n}$

$$\begin{cases} V^+(\tilde{A} - VAV^+)^i V = 0, \\ V^+(\tilde{A} - VAV^+)^{i-1}(\tilde{B}L - VB) = 0, \\ (T^+\tilde{C} - CV^+)(\tilde{A} - VAV^+)^{i-1}V = 0, \\ (T^+\tilde{C} - CV^+)(\tilde{A} - VAV^+)^{i-1}(\tilde{B}L - VB) = 0. \end{cases} \tag{2.9}$$

(ii) *The control law* $-\tilde{K}\tilde{x}$ *is contractible to the control law* $-Kx$ *if and only if either*

$$\begin{cases} V^+(\tilde{A} - VAV^+)^i V = 0, \\ V^+(\tilde{A} - VAV^+)^{i-1}(\tilde{B}L - VB) = 0, \quad i = 1, 2, \ldots, \tilde{n}, \\ (LKV^+ - \tilde{K})\tilde{A}^{i-1}\begin{bmatrix} V & \tilde{B}L \end{bmatrix} = 0 \end{cases} \tag{2.10}$$

or

$$\begin{cases} V^+(\tilde{A} - VAV^+)^i V = 0, \\ V^+(\tilde{A} - VAV^+)^{i-1}(\tilde{B} - VBL^+) = 0, \quad i = 1, 2, \ldots, \tilde{n}, \\ (KV^+ - L^+\tilde{K})\tilde{A}^{i-1}\begin{bmatrix} V & \tilde{B} \end{bmatrix} = 0. \end{cases} \tag{2.11}$$

In applications, the inclusion principle relies heavily on the proper choice of expanded matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{K}$ which are restricted by the expandability and contractibility conditions of Theorem 2.3. In a variety of situations, the conditions have been hard to use since there are no simple rules for their interpretation, nor systematic procedures for utilizing the conditions in the computation of expanded matrices. For this reason, there are a few standard choices [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] that have been repeatedly used in applications, while the full freedom offered by the conditions has remained unexplored. Recently, to broaden the scope of applications of the inclusion principle, new expansion-contraction conditions have been proposed, which involve additional flexibility provided by the choice of complementary matrices [14, 15, 16, 17, 18]. Even in this case, the conditions involve an intricate relationship between powers of matrices that obscures the full flexibility of the proposed choice.

In the next section we will establish a canonical form for the inclusion principle of dynamic system $\mathbf{S}$. The canonical form parameterizes explicitly all expansion-contraction matrices in the general setting of transformations V, L, T. Therefore, full freedom of the inclusion principle is readily available for control design.

3. Canonical form. Motivated by the difficulties in characterizing expansion matrices, we propose to derive a canonical form for the inclusion principle. The form resolves the difficulties by providing an explicit parameterization of the expanded system within the framework of expansion-contraction process. To show this, we need the following two lemmas [31, 32].

LEMMA 3.1. *Given* $\mathcal{A} \in \mathbf{R}^{n\times n}, \mathcal{B} \in \mathbf{R}^{n\times m}, \mathcal{C} \in \mathbf{R}^{l\times n}$*, and* $\mathcal{D} \in \mathbf{R}^{l\times m}$*.*

(i)

$$\max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI - \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = n$$

if and only if

$$\mathcal{D} = 0$$

and

$$\max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI - \mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{bmatrix} = n.$$

(ii) *Assume that* $(\mathcal{A}, \mathcal{B})$ *is controllable, i.e.,*

$$\operatorname{rank} \begin{bmatrix} sI - \mathcal{A} & \mathcal{B} \end{bmatrix} = n \quad \forall s \in \mathbf{C}.$$

Then

$$\max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI - \mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{bmatrix} = n$$

if and only if

$$\mathcal{C} = 0.$$

LEMMA 3.2. *Given* $\mathcal{A} \in \mathbf{R}^{n\times n}, \mathcal{B} \in \mathbf{R}^{n\times m}, \mathcal{C} \in \mathbf{R}^{l\times n}$. *Then*

$$\mathcal{C}\mathcal{A}^i\mathcal{B} = 0 \quad \text{for } i = 0, 1, \dots, n-1$$

if and only if

$$\max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI - \mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{bmatrix} = n.$$

Proof. From the Kalman decomposition of a linear time-invariant system [26], there exists nonsingular matrix $\mathcal{X} \in \mathbf{R}^{n\times n}$ such that

$$\mathcal{X}\mathcal{A}\mathcal{X}^{-1} = \begin{array}{c} \begin{array}{cc} \tau_1 & n-\tau_1 \end{array} \\ \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ 0 & \mathcal{A}_{22} \end{bmatrix} \end{array} \begin{array}{l} \}\tau_1 \\ \}n-\tau_1 \end{array}, \quad \mathcal{X}\mathcal{B} = \begin{bmatrix} \mathcal{B}_1 \\ 0 \end{bmatrix} \begin{array}{l} \}\tau_1 \\ \}n-\tau_1 \end{array}, \quad \mathcal{C}\mathcal{X}^{-1} = \begin{array}{c} \begin{array}{cc} \tau_1 & n-\tau_1 \end{array} \\ \begin{bmatrix} \mathcal{C}_1 & \mathcal{C}_2 \end{bmatrix} \end{array},$$

where $(\mathcal{A}_{11}, \mathcal{B}_1)$ is controllable, which implies

$$\operatorname{rank} \begin{bmatrix} \mathcal{B}_1 & \mathcal{A}_{11}\mathcal{B}_1 & \cdots & \mathcal{A}_{11}^{\tau_1-1}\mathcal{B}_1 \end{bmatrix} = \tau_1. \tag{3.1}$$

Since

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} \mathcal{C}\mathcal{B} & \mathcal{C}\mathcal{A}\mathcal{B} & \cdots & \mathcal{C}\mathcal{A}^{n-1}\mathcal{B} \end{bmatrix} &= \operatorname{rank}(\mathcal{C}_1 \begin{bmatrix} \mathcal{B}_1 & \mathcal{A}_{11}\mathcal{B}_1 & \cdots & \mathcal{A}_{11}^{n-1}\mathcal{B}_1 \end{bmatrix}) \\ &= \operatorname{rank}(\mathcal{C}_1 \begin{bmatrix} \mathcal{B}_1 & \mathcal{A}_{11}\mathcal{B}_1 & \cdots & \mathcal{A}_{11}^{\tau_1-1}\mathcal{B}_1 \end{bmatrix}), \end{aligned}$$

so, the property (3.1) gives that $\mathcal{C}\mathcal{A}^i\mathcal{B} = 0$ for all $i = 0, 1, \dots, n-1$ if and only if $\mathcal{C}_1 = 0$.

On the other hand,

$$\begin{aligned} \max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI - \mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{bmatrix} &= (n - \tau_1) + \max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI - \mathcal{A}_{11} & \mathcal{B}_1 \\ \mathcal{C}_1 & 0 \end{bmatrix} \\ &= n + \left(\max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI - \mathcal{A}_{11} & \mathcal{B}_1 \\ \mathcal{C}_1 & 0 \end{bmatrix} - \tau_1 \right); \end{aligned}$$

thus, we have by using Lemma 3.1 that $\max_{s\in\mathbf{C}} \operatorname{rank} \left[\begin{smallmatrix} sI-\mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{smallmatrix}\right] = n$ if and only if $\mathcal{C}_1 = 0$. Hence, Lemma 3.2 follows. □

Now we are ready to present a canonical form for the expansion-contraction triplet $(\tilde{A}, \tilde{B}, \tilde{C})$ under the inclusion principle as follows.

Theorem 3.3. *Given systems* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$*, and transformations* V*,* L*,* T *satisfying* (2.3) *and* (2.4)*, let the QR factorizations of* V*,* L*, and* T *be given by*

$$(3.2)\qquad \begin{cases} \begin{bmatrix} \mathcal{U} & U \end{bmatrix}^T V = \begin{bmatrix} V_{11} \\ 0 \end{bmatrix} \begin{matrix} \}n \\ \}\tilde{n}-n \end{matrix}, & \mathcal{U} \in \mathbf{R}^{\tilde{n}\times n}, \quad U \in \mathbf{R}^{\tilde{n}\times(\tilde{n}-n)}, \\ \\ \begin{bmatrix} \mathcal{P} & P \end{bmatrix}^T L = \begin{bmatrix} L_{11} \\ 0 \end{bmatrix} \begin{matrix} \}m \\ \}\tilde{m}-m \end{matrix}, & \mathcal{P} \in \mathbf{R}^{\tilde{m}\times m}, \quad P \in \mathbf{R}^{\tilde{m}\times(\tilde{m}-m)}, \\ \\ \begin{bmatrix} \mathcal{S} & S \end{bmatrix}^T T = \begin{bmatrix} T_{11} \\ 0 \end{bmatrix} \begin{matrix} \}l \\ \}\tilde{l}-l \end{matrix}, & \mathcal{S} \in \mathbf{R}^{\tilde{l}\times l}, \quad S \in \mathbf{R}^{\tilde{l}\times(\tilde{l}-l)}, \end{cases}$$

where $\begin{bmatrix} \mathcal{U} & U \end{bmatrix}$*,* $\begin{bmatrix} \mathcal{P} & P \end{bmatrix}$*, and* $\begin{bmatrix} \mathcal{S} & S \end{bmatrix}$ *are orthogonal, and* V_{11}*,* L_{11}*, and* T_{11} *are nonsingular. Then, system* $\tilde{\mathbf{S}}$ *is an expansion of the system* $\mathbf{S}$ *if and only if*

$$(3.3)\qquad \begin{cases} \tilde{A} = \begin{bmatrix} V & UW \end{bmatrix} \begin{bmatrix} A & 0 & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & 0 & \tilde{A}_{33} \end{bmatrix} \begin{bmatrix} V^+ \\ (UW)^T \end{bmatrix}, \\ \tilde{B} = \begin{bmatrix} V & UW \end{bmatrix} \begin{bmatrix} B & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \\ 0 & \tilde{B}_{32} \end{bmatrix} \begin{bmatrix} L^+ \\ P^T \end{bmatrix}, \\ \tilde{C} = \begin{bmatrix} T & S \end{bmatrix} \begin{bmatrix} C & 0 & \tilde{C}_{13} \\ \tilde{C}_{21} & \tilde{C}_{22} & \tilde{C}_{23} \end{bmatrix} \begin{bmatrix} V^+ \\ (UW)^T \end{bmatrix}, \end{cases}$$

where $W \in \mathbf{R}^{(\tilde{n}-n)\times(\tilde{n}-n)}$ *is an arbitrary orthogonal matrix,* μ *is an arbitrary integer between* 0 *and* $\tilde{n}-n$*, and* $\tilde{A}_{13} \in \mathbf{R}^{n\times(\tilde{n}-n-\mu)}$*,* $\tilde{A}_{21} \in \mathbf{R}^{\mu\times n}$*,* $\tilde{A}_{22} \in \mathbf{R}^{\mu\times\mu}$*,* $\tilde{A}_{23} \in \mathbf{R}^{\mu\times(\tilde{n}-n-\mu)}$*,* $\tilde{A}_{33} \in \mathbf{R}^{(\tilde{n}-n-\mu)\times(\tilde{n}-n-\mu)}$*,* $\tilde{B}_{12} \in \mathbf{R}^{n\times(\tilde{m}-m)}$*,* $\tilde{B}_{21} \in \mathbf{R}^{\mu\times m}$*,* $\tilde{B}_{22} \in \mathbf{R}^{\mu\times(\tilde{m}-m)}$*,* $\tilde{B}_{32} \in \mathbf{R}^{(\tilde{n}-n-\mu)\times(\tilde{m}-m)}$*,* $\tilde{C}_{13} \in \mathbf{R}^{l\times(\tilde{n}-n-\mu)}$*,* $\tilde{C}_{21} \in \mathbf{R}^{(\tilde{l}-l)\times n}$*,* $\tilde{C}_{22} \in \mathbf{R}^{(\tilde{l}-l)\times\mu}$*, and* $\tilde{C}_{23} \in \mathbf{R}^{(\tilde{l}-l)\times(\tilde{n}-n-\mu)}$ *are constant matrices with arbitrary elements.*

Proof. It is easy to see that

$$V^+ = \begin{bmatrix} V_{11}^{-1} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{U} & U \end{bmatrix}^T, \quad L^+ = \begin{bmatrix} L_{11}^{-1} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{P} & P \end{bmatrix}^T, \quad T^+ = \begin{bmatrix} T_{11}^{-1} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{S} & S \end{bmatrix}^T.$$

In the following we prove the necessity first and then sufficiency.

Necessity. For any $\tilde{A} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$, $\tilde{B} \in \mathbf{R}^{\tilde{n}\times\tilde{m}}$ and $\tilde{C} \in \mathbf{R}^{\tilde{l}\times\tilde{n}}$, define

$$(3.4)\qquad \begin{cases} \begin{bmatrix} \mathcal{U} & U \end{bmatrix}^T \tilde{A} \begin{bmatrix} \mathcal{U} & U \end{bmatrix} = \begin{array}{c} \begin{matrix} n & \tilde{n}-n \end{matrix} \\ \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \end{array} \begin{matrix} \}n \\ \}\tilde{n}-n \end{matrix}, \\ \begin{bmatrix} \mathcal{U} & U \end{bmatrix}^T \tilde{B} \begin{bmatrix} \mathcal{P} & P \end{bmatrix} = \begin{array}{c} \begin{matrix} m & \tilde{m}-m \end{matrix} \\ \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{bmatrix} \end{array} \begin{matrix} \}n \\ \}\tilde{n}-n \end{matrix}, \\ \begin{bmatrix} \mathcal{S} & S \end{bmatrix}^T \tilde{C} \begin{bmatrix} \mathcal{U} & U \end{bmatrix} = \begin{array}{c} \begin{matrix} n & \tilde{n}-n \end{matrix} \\ \begin{bmatrix} \hat{C}_{11} & \hat{C}_{12} \\ \hat{C}_{21} & \hat{C}_{22} \end{bmatrix} \end{array} \begin{matrix} \}l \\ \}\tilde{l}-l \end{matrix}. \end{cases}$$

Let the system $\tilde{\mathbf{S}}$ be an expansion of the system $\mathbf{S}$. By Theorem 2.3 and Lemma 3.2, we have that

$$\max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI-\tilde{A}+VAV^{+} & (\tilde{A}-VAV^{+})V & \tilde{B}L-VB \\ V^{+} & 0 & 0 \end{bmatrix} = \tilde{n},$$
$$\max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI-\tilde{A}+VAV^{+} & V & \tilde{B}L-VB \\ T^{+}\tilde{C}-CV^{+} & 0 & 0 \end{bmatrix} = \tilde{n},$$

which gives

$$\max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI-\tilde{A} & \tilde{A}V-VA & \tilde{B}L-VB \\ V^{+} & 0 & 0 \end{bmatrix} = \max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI-\tilde{A} & V & \tilde{B}L \\ T^{+}\tilde{C}-CV^{+} & 0 & 0 \end{bmatrix} = \tilde{n}.$$

Hence, by using (3.4) we get

$$\max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI-\hat{A}_{11} & -\hat{A}_{12} & \hat{A}_{11}V_{11}-V_{11}A & \hat{B}_{11}L_{11}-V_{11}B \\ -\hat{A}_{21} & sI-\hat{A}_{22} & \hat{A}_{21}V_{11} & \hat{B}_{21}L_{11} \\ V_{11}^{-1} & 0 & 0 & 0 \end{bmatrix}$$

$$= \max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI-\hat{A}_{11} & -\hat{A}_{12} & V_{11} & \hat{B}_{11}L_{11} \\ -\hat{A}_{21} & sI-\hat{A}_{22} & 0 & \hat{B}_{21}L_{11} \\ T_{11}^{-1}\hat{C}_{11}-CV_{11}^{-1} & T_{11}^{-1}\hat{C}_{12} & 0 & 0 \end{bmatrix} = \tilde{n},$$

that is,

$$\max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI-\hat{A}_{22} & \hat{A}_{21}V_{11} & \hat{B}_{21}L_{11} \\ -\hat{A}_{12} & \hat{A}_{11}V_{11}-V_{11}A & \hat{B}_{11}L_{11}-V_{11}B \end{bmatrix}$$

$$= \max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI-\hat{A}_{22} & -\hat{A}_{21} & \hat{B}_{21}L_{11} \\ T_{11}^{-1}\hat{C}_{12} & T_{11}^{-1}\hat{C}_{11}-CV_{11}^{-1} & 0 \end{bmatrix} = \tilde{n}-n,$$

which, by means of Lemma 3.1, is equivalent to

$$\begin{bmatrix} \hat{A}_{11}V_{11}-V_{11}A & \hat{B}_{11}L_{11}-V_{11}B \end{bmatrix} = 0, \quad T_{11}^{-1}\hat{C}_{11}-CV_{11}^{-1} = 0$$

and

$$\max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI-\hat{A}_{22} & \hat{A}_{21} & \hat{B}_{21} \\ \hat{A}_{12} & 0 & 0 \end{bmatrix} = \max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI-\hat{A}_{22} & \hat{A}_{21} & \hat{B}_{21} \\ \hat{C}_{12} & 0 & 0 \end{bmatrix} = \tilde{n}-n$$

or, equivalently,

$$\hat{A}_{11} = V_{11}AV_{11}^{-1}, \quad \hat{B}_{11} = V_{11}BR_{11}^{-1}, \quad \hat{C}_{11} = T_{11}CV_{11}^{-1}, \tag{3.5}$$

and

$$\max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI-\hat{A}_{22} & \hat{A}_{21} & \hat{B}_{21} \\ \hat{A}_{12} & 0 & 0 \\ \hat{C}_{12} & 0 & 0 \end{bmatrix} = \tilde{n}-n. \tag{3.6}$$

Now we only need to characterize $\hat{A}_{22}$, $\hat{A}_{21}$, $\hat{A}_{12}$, $\hat{B}_{21}$, and $\hat{C}_{12}$ in (3.6). It is well known [33] that there is an orthogonal matrix $W \in \mathbf{R}^{(\tilde{n}-n)\times(\tilde{n}-n)}$ and an integer μ between 0 and $\tilde{n}-n$ such that

$$\left\{\begin{array}{l} W^T \hat{A}_{22} W = \begin{array}{cc} \mu & \tilde{n}-n-\mu \\ \left[\begin{array}{cc} \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & \tilde{A}_{33} \end{array}\right] & \begin{array}{l} \}\mu \\ \}\tilde{n}-n-\mu \end{array} \end{array}, \\ W^T \left[\hat{A}_{21} V_{11} \;\middle|\; \hat{B}_{21} L_{11}\right] = \begin{array}{cc} n \quad m & \\ \left[\begin{array}{cc} \tilde{A}_{21} & \tilde{B}_{21} \\ 0 & 0 \end{array}\right] & \begin{array}{l} \}\mu \\ \}\tilde{n}-n-\mu \end{array} \end{array}, \\ (\tilde{A}_{22}, \left[\tilde{A}_{21} \quad \tilde{B}_{21}\right]) \text{ is controllable.} \end{array}\right. \tag{3.7}$$

Set

$$\left[\begin{array}{c} \hat{A}_{12} \\ \hline \hat{C}_{12} \end{array}\right] W = \begin{array}{cc} \mu \quad \tilde{n}-n-\mu & \\ \left[\begin{array}{cc} \tilde{A}_{12} & V_{11}\tilde{A}_{13} \\ \tilde{C}_{12} & T_{11}\tilde{C}_{13} \end{array}\right] & \begin{array}{l} \}n \\ \}l \end{array} \end{array}.$$

Then, (3.6) and Lemma 3.1 imply

$$\tilde{A}_{12} = 0, \quad \tilde{C}_{12} = 0,$$

that is,

$$\left[\begin{array}{c} \hat{A}_{12} \\ \hline \hat{C}_{12} \end{array}\right] W = \left[\begin{array}{cc} 0 & V_{11}\tilde{A}_{13} \\ \hline 0 & T_{11}\tilde{C}_{13} \end{array}\right]. \tag{3.8}$$

Hence, (3.3) follows directly from a simple calculation using (3.4), (3.5), (3.7), and (3.8).

Sufficiency. Let (3.3) hold for an arbitrary orthogonal matrix $W \in \mathbf{R}^{(\tilde{n}-n)\times(\tilde{n}-n)}$, an arbitrary integer μ between 0 and $\tilde{n}-n$, and arbitrary matrices $\tilde{A}_{13} \in \mathbf{R}^{n\times(\tilde{n}-n-\mu)}$, $\tilde{A}_{21} \in \mathbf{R}^{\mu\times n}$, $\tilde{A}_{22} \in \mathbf{R}^{\mu\times\mu}$, $\tilde{A}_{23} \in \mathbf{R}^{\mu\times(\tilde{n}-n-\mu)}$, $\tilde{A}_{33} \in \mathbf{R}^{(\tilde{n}-n-\mu)\times(\tilde{n}-n-\mu)}$, $\tilde{B}_{12} \in \mathbf{R}^{n\times(\tilde{m}-m)}$, $\tilde{B}_{21} \in \mathbf{R}^{\mu\times m}$, $\tilde{B}_{22} \in \mathbf{R}^{\mu\times(\tilde{m}-m)}$, $\tilde{B}_{32} \in \mathbf{R}^{(\tilde{n}-n-\mu)\times(\tilde{m}-m)}$, $\tilde{C}_{13} \in \mathbf{R}^{l\times(\tilde{n}-n-\mu)}$, $\tilde{C}_{21} \in \mathbf{R}^{(\tilde{l}-l)\times n}$, $\tilde{C}_{22} \in \mathbf{R}^{(\tilde{l}-l)\times\mu}$, and $\tilde{C}_{23} \in \mathbf{R}^{(\tilde{l}-l)\times(\tilde{n}-n-\mu)}$. A direct calculation yields that

$$\begin{aligned} &\max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI-\tilde{A} & \tilde{A}V-VA & \tilde{B}L-VB \\ V^+ & 0 & 0 \end{bmatrix} \\ &= \max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI-A & 0 & -\tilde{A}_{13} & 0 & 0 \\ -\tilde{A}_{21} & sI-\tilde{A}_{22} & -\tilde{A}_{23} & \tilde{A}_{21} & \tilde{B}_{21} \\ 0 & 0 & sI-\tilde{A}_{33} & 0 & 0 \\ I & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= n+\mu+(\tilde{n}-n-\mu) = \tilde{n} \end{aligned}$$

and

$$\begin{aligned} &\max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI-\tilde{A} & V & \tilde{B}L \\ T^+\tilde{C}-CV^+ & 0 & 0 \end{bmatrix} \\ &= \max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI-A & 0 & -\tilde{A}_{13} & I & 0 \\ -\tilde{A}_{21} & sI-\tilde{A}_{22} & -\tilde{A}_{23} & 0 & \tilde{B}_{21} \\ 0 & 0 & sI-\tilde{A}_{33} & 0 & 0 \\ 0 & 0 & \tilde{C}_{13} & 0 & 0 \end{bmatrix} \\ &= n+\mu+(\tilde{n}-n-\mu) = \tilde{n}. \end{aligned}$$

Consequently, we obtain that

$$\max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI-\tilde{A}+VAV^+ & (\tilde{A}-VAV^+)V & \tilde{B}L-VB \\ V^+ & 0 & 0 \end{bmatrix}$$
$$= \max_{s\in\mathbf{C}} \operatorname{rank} \begin{bmatrix} sI-\tilde{A}+VAV^+ & V & \tilde{B}L-VB \\ T^+\tilde{C}-CV^+ & 0 & 0 \end{bmatrix} = \tilde{n}.$$

Therefore, by Theorem 2.3 and Lemma 3.2 the system $\tilde{\mathbf{S}}$ is an expansion of the system $\mathbf{S}$. ∎

Since the expansion process underlying the above canonical form (3.3) involves the inputs and outputs, it includes the canonical form obtained in [1] (see also [2]).

Remark 1. Let

$$M=\tilde{A}-VAV^+, \quad N=\tilde{B}-VBL^+, \quad G=\tilde{C}-TCV^+.$$

Matrices M, N, G defined above are complementary matrices [1, 15]. Obviously, using the same notation as in Theorem 3.3, we conclude that system $\tilde{\mathbf{S}}$ is an expansion of $\mathbf{S}$ if and only if

$$\begin{cases} M=\begin{bmatrix} V & UW \end{bmatrix} \begin{bmatrix} 0 & 0 & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & 0 & \tilde{A}_{33} \end{bmatrix} \begin{bmatrix} V^+ \\ (UW)^T \end{bmatrix}, \\ N=\begin{bmatrix} V & UW \end{bmatrix} \begin{bmatrix} 0 & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \\ 0 & \tilde{B}_{32} \end{bmatrix} \begin{bmatrix} L^+ \\ P^T \end{bmatrix}, \\ G=\begin{bmatrix} T & S \end{bmatrix} \begin{bmatrix} 0 & 0 & \tilde{C}_{13} \\ \tilde{C}_{21} & \tilde{C}_{22} & \tilde{C}_{23} \end{bmatrix} \begin{bmatrix} V^+ \\ (UW)^T \end{bmatrix}, \end{cases}$$

that is, Theorem 3.3 established a canonical form for complementary matrices as well.

Remark 2. In the case that matrices V, L, and T are defined as

$$(3.9)\quad V=\begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \end{bmatrix}, \quad L=\begin{bmatrix} I_{m_1} & 0 & 0 \\ 0 & I_{m_2} & 0 \\ 0 & I_{m_2} & 0 \\ 0 & 0 & I_{m_3} \end{bmatrix}, \quad T=\begin{bmatrix} I_{l_1} & 0 & 0 \\ 0 & I_{l_2} & 0 \\ 0 & I_{l_2} & 0 \\ 0 & 0 & I_{l_3} \end{bmatrix}$$

with

$$n_1+n_2+n_3=n, \quad m_1+m_2+m_3=m, \quad l_1+l_2+l_3=l,$$

$$n_1+2n_2+n_3=\tilde{n}, \quad m_1+2m_2+m_3=\tilde{m}, \quad l_1+2l_2+l_3=\tilde{l},$$

two classes of complementary matrices have been identified in [14, 15] such that system $\tilde{\mathbf{S}}$ includes system $\mathbf{S}$; see (3.30) and (3.31) in [15]. These classes can be obtained by choosing $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ in (3.3) as follows:

$$\begin{cases} \tilde{A}=\begin{bmatrix} V & U \end{bmatrix} \begin{bmatrix} A & 0 \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} V^+ \\ U^T \end{bmatrix}, \\ \tilde{B}=\begin{bmatrix} V & U \end{bmatrix} \begin{bmatrix} B & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} L^+ \\ P^T \end{bmatrix}, \\ \tilde{C}=\begin{bmatrix} T & S \end{bmatrix} \begin{bmatrix} C & 0 \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} V^+ \\ U^T \end{bmatrix}, \end{cases} \quad \text{or} \quad \begin{cases} \tilde{A}=\begin{bmatrix} V & U \end{bmatrix} \begin{bmatrix} A & X_{12} \\ 0 & X_{22} \end{bmatrix} \begin{bmatrix} V^+ \\ U^T \end{bmatrix}, \\ \tilde{B}=\begin{bmatrix} V & U \end{bmatrix} \begin{bmatrix} B & Y_{12} \\ 0 & Y_{22} \end{bmatrix} \begin{bmatrix} L^+ \\ P^T \end{bmatrix}, \\ \tilde{C}=\begin{bmatrix} T & S \end{bmatrix} \begin{bmatrix} C & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} V^+ \\ U^T \end{bmatrix}. \end{cases}$$

Remark 3. The two special cases of aggregation and restriction, which have been used extensively in the existing literature, can now by easily characterized by the canonical form of Theorem 3.3.

- System $\mathbf{S}$ is an aggregation of system $\tilde{\mathbf{S}}$ if and only if

$$\begin{cases} \tilde{A} = \begin{bmatrix} V & U \end{bmatrix} \begin{bmatrix} A & X_{12} \\ 0 & X_{22} \end{bmatrix} \begin{bmatrix} V^+ \\ U^T \end{bmatrix}, \\ \tilde{B} = \begin{bmatrix} V & U \end{bmatrix} \begin{bmatrix} B & Y_{12} \\ 0 & Y_{22} \end{bmatrix} \begin{bmatrix} L^+ \\ P^T \end{bmatrix}, \\ \tilde{C} = \begin{bmatrix} T & S \end{bmatrix} \begin{bmatrix} C & Z_{12} \\ 0 & Z_{22} \end{bmatrix} \begin{bmatrix} V^+ \\ U^T \end{bmatrix}. \end{cases}$$

- System $\mathbf{S}$ is a restriction of system $\tilde{\mathbf{S}}$ if and only if

$$\begin{cases} \tilde{A} = \begin{bmatrix} V & U \end{bmatrix} \begin{bmatrix} A & 0 \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} V^+ \\ U^T \end{bmatrix}, \\ \tilde{B} = \begin{bmatrix} V & U \end{bmatrix} \begin{bmatrix} B & 0 \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} L^+ \\ P^T \end{bmatrix}, \\ \tilde{C} = \begin{bmatrix} T & S \end{bmatrix} \begin{bmatrix} C & 0 \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} V^+ \\ U^T \end{bmatrix}. \end{cases}$$

When the underlying space of an expansion is used to design control with information structure constraints, then problems arise with control laws when they have to be contracted for implementation in the original space. The explicit contractibility conditions are provided by the following control law canonical form.

THEOREM 3.4. *Given systems* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$, *and transformations* V, L, T *satisfying* (2.3) *and* (2.4), *the control law*

$$\tilde{u} = -\tilde{K}\tilde{x}$$

for system $\tilde{\mathbf{S}}$ *is contractible to the control law*

$$u = -Kx$$

for system $\mathbf{S}$ *if and only if one of the following two statements holds:*

(a) *Matrices* $\tilde{A}$ *and* $\tilde{B}$ *of system* $\tilde{\mathbf{S}}$ *are given by* (3.3) *and*

$$\tilde{K} = \begin{bmatrix} L & P \end{bmatrix} \begin{bmatrix} K & 0 & \tilde{K}_{13} \\ 0 & 0 & \tilde{K}_{23} \end{bmatrix} \begin{bmatrix} V^+ \\ (UW)^T \end{bmatrix}, \tag{3.10}$$

where W *is orthogonal and is the same as that in* (3.3), *and matrices* $\tilde{K}_{13} \in \mathbf{R}^{m\times(\tilde{n}-n-\mu)}$ *and* $\tilde{K}_{23} \in \mathbf{R}^{(\tilde{m}-m)\times(\tilde{n}-n-\mu)}$ *have arbitrary elements.*

(b) *Matrices* $\tilde{A}$ *and* $\tilde{B}$ *of system* $\tilde{\mathbf{S}}$ *are given by* (3.3) *with*

$$\tilde{B}_{12} = 0, \quad \tilde{B}_{32} = 0$$

and

$$\tilde{K} = \begin{bmatrix} L & P \end{bmatrix} \begin{bmatrix} K & 0 & \tilde{K}_{13} \\ \tilde{K}_{21} & \tilde{K}_{22} & \tilde{K}_{23} \end{bmatrix} \begin{bmatrix} V^+ \\ (UW)^T \end{bmatrix}, \tag{3.11}$$

where W is orthogonal and is the same as that in (3.3)*, and matrices* $\tilde{K}_{13} \in \mathbf{R}^{m\times(\tilde{n}-n-\mu)}$ *and* $\begin{bmatrix}\tilde{K}_{21} & \tilde{K}_{22} & \tilde{K}_{23}\end{bmatrix} \in \mathbf{R}^{(\tilde{m}-m)\times\tilde{n}}$ *have arbitrary elements.*

Proof. The proof is similar to that of Theorem 3.3 and hence is omitted. □

A corollary to Theorems 3.3 and 3.4, which delineates an important class of contractible control laws [17], is now automatic.

COROLLARY 3.5. *Given a system* $\mathbf{S}$ *and transformations* V*,* L*,* T *satisfying* (2.3) *and* (2.4)*, if matrices* $\tilde{A}$ *and* $\tilde{B}$ *are given by* (3.3) *with* $\mu = 0$*, then any control law* $\tilde{u} = -\tilde{K}\tilde{x}$ *for system* $\tilde{\mathbf{S}}$ *is contractible to the control law* $u = -Kx$ *with* $K = L^{+}\tilde{K}V$ *for system* $\mathbf{S}$*.*

Remark 4. The definition in [15, 16] for the contractibility is different from that given in [4, 17, 19]. In [15, 16] it is defined that the control law $\tilde{u} = -\tilde{K}\tilde{x}$ for the expanded system $\tilde{\mathbf{S}}$ is contractible to the control law $u = -Kx$ for system $\mathbf{S}$ if the choice $\tilde{x}_0 = Vx_0$ and $\tilde{u} = Lu$ implies

$$Kx(t; x_0, u) = L^{+}\tilde{K}\tilde{x}(t; \tilde{x}_0, \tilde{u})$$

for any $t \geq 0$, any initial state x_0, and any fixed input u of system $\mathbf{S}$. If such a definition is used, then we can show that the control law $\tilde{u} = -\tilde{K}\tilde{x}$ for the expanded system $\tilde{\mathbf{S}}$ is contractible to the control law $u = -Kx$ for system $\mathbf{S}$ if and only if matrices $\tilde{A}$ and $\tilde{B}$ of system $\tilde{\mathbf{S}}$ are given by (3.3) and $\tilde{K}$ is given by (3.11).

It was observed in [15] that our ability to use generalized (system) decompositions depends crucially not only on the choice of the transformation matrices V, R, and T, but also on the selection of the expansion-contraction matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{K}$ of expanded system $\tilde{\mathbf{S}}$. All previous results enable such selection only partially because of the usage of the forms of matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{K}$ in system $\tilde{\mathbf{S}}$ corresponding only with some particular cases. Theorems 3.3 and 3.4 have established a canonical form for the inclusion principle of dynamic system $\mathbf{S}$, which explicitly parameterizes all admissible expansion-contraction matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{K}$ in system $\tilde{\mathbf{S}}$ and thus provides full freedom under the inclusion principle. Therefore, the significance of Theorems 3.3 and 3.4 is obvious. We hasten to add, however, that in choosing suitable expansions in applications of the inclusion principle, the role of complementary matrices [16] is indispensable.

An important issue in the expansion-contraction process has been the conditions under which structural properties of expansions and contractions, such as controllability, observability, and stabilizability, remain invariant in the process. This issue has been raised in [22, 23, 24] regarding controllability and observability, and general conditions for their invariance have been formulated in [25]. To provide a comprehensive relationship between expansions and contractions using the present canonical forms, let us state the following definitions [34].

DEFINITION 3.6. *Given a system* $\mathbf{S}$*. The sets of the uncontrollable modes, the unobservable modes, and the invariant zeros of system* $\mathbf{S}$ *are defined, respectively, by*

$$\Sigma_c(A, B) := \{\lambda \in \mathbf{C} : \ \operatorname{rank}[\lambda I - A \quad B] < n\},$$

$$\Sigma_o(C, A) := \left\{\lambda \in \mathbf{C} : \ \operatorname{rank}\begin{bmatrix}\lambda I - A \\ C\end{bmatrix} < n\right\},$$

and

$$\Sigma_z(C, A, B) := \left\{\lambda \in \mathbf{C} : \ \operatorname{rank}\begin{bmatrix}\lambda I - A & B \\ C & 0\end{bmatrix} < \max_{s\in\mathbf{C}} \operatorname{rank}\begin{bmatrix}sI - A & B \\ C & 0\end{bmatrix}\right\}.$$

DEFINITION 3.7. *Given $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times m}$, let X be a nonsingular matrix such that $(X^{-1}AX, X^{-1}B)$ is in its controllability canonical form, i.e.,*

$$\begin{cases} X^{-1}AX = \begin{array}{cc} \mu & n-\mu \\ \end{array} \\ \qquad\qquad \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{array}{l} \}\mu \\ \}n-\mu \end{array}, \quad X^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{array}{l} \}\mu \\ \}n-\mu \end{array}, \\ (A_{11}, B_1) \text{ is controllable.} \end{cases}$$

Then the controllability subspace $\mathcal{C}(A, B)$ of (A, B) is defined as

$$\mathcal{C}(A, B) = \text{Range}\left(X \begin{bmatrix} I_\mu \\ 0 \end{bmatrix} \right).$$

The desired result relating stability, controllability, observability, detectability, and stability of the invariant zeros is provided by the following.

THEOREM 3.8. *Given a system $\mathbf{S}$ and transformations V, L, and T satisfying (2.3) and (2.4), assume $n < \tilde{n}$, $m < \tilde{m}$, and $l < \tilde{l}$. Let $\bar{\mathbf{C}}^+$ denote the closed right half complex plane. Then, there exist matrices $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ such that the following properties hold simultaneously:*

(3.12) *System $\tilde{\mathbf{S}}$ is an expansion of system $\mathbf{S}$,*

(3.13) $\sigma(A) \subset \sigma(\tilde{A}), \quad \sigma(\tilde{A}) \cap \bar{\mathbf{C}}^+ = \sigma(A) \cap \bar{\mathbf{C}}^+,$

(3.14) $\Sigma_c(\tilde{A}, \tilde{B}) = \Sigma_c(A, B),$

(3.15) $\Sigma_o(\tilde{C}, \tilde{A}) = \Sigma_0(C, A),$

(3.16) $\Sigma_z(C, A, B) \subset \Sigma_z(\tilde{C}, \tilde{A}, \tilde{B}), \quad \Sigma_z(\tilde{C}, \tilde{A}, \tilde{B}) \cap \bar{\mathbf{C}}^+ = \Sigma_z(C, A, B) \cap \bar{\mathbf{C}}^+.$

Hence, stability, controllability, stabilizability, observability, detectability, and the stability of the invariant zeros can be transmitted simultaneously from system $\mathbf{S}$ to system $\tilde{\mathbf{S}}$ under the inclusion principle.

Proof. Let U, P, and Q be the same as those in Theorem 3.3. Take $\mu = 0$ in (3.3) and define

$$\begin{cases} \tilde{A} = \begin{bmatrix} V & U \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & \mathcal{A} \end{bmatrix} \begin{bmatrix} V^+ \\ U^T \end{bmatrix}, \\ \tilde{B} = \begin{bmatrix} V & U \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & \mathcal{B} \end{bmatrix} \begin{bmatrix} L^+ \\ P^T \end{bmatrix}, \\ \tilde{C} = \begin{bmatrix} T & S \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & \mathcal{C} \end{bmatrix} \begin{bmatrix} V^+ \\ U^T \end{bmatrix}, \end{cases}$$

where

$$\mathcal{A} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{\tilde{n}-n} \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} b_1 & 0 \\ b_2 & 0 \\ \vdots & 0 \\ b_{\tilde{n}-n} & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} c_1 & c_2 & \cdots & c_{\tilde{n}-n} \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{\tilde{n}-n} < 0, \quad b_1 c_1 > 0, \quad b_2 c_2 > 0, \ldots, b_{\tilde{n}-n} c_{\tilde{n}-n} > 0.$$

It is easy to see that

$$\sigma(\mathcal{A}) \subset \mathbf{C}/\bar{\mathbf{C}}^{+}, \quad \Sigma_c(\mathcal{A}, \mathcal{B}) = \Sigma_o(\mathcal{C}, \mathcal{A}) = \emptyset, \quad \Sigma_z(\mathcal{C}, \mathcal{A}, \mathcal{B}) \subset \mathbf{C}/\bar{\mathbf{C}}^{+}. \tag{3.17}$$

For $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ above, Theorem 3.3 implies that system $\tilde{\mathbf{S}}$ is an expansion of system $\mathbf{S}$, the property (3.13) is obvious, and properties (3.14), (3.15), and (3.16) follow directly from (3.17) and the following facts:

$$\begin{cases} \sigma(\tilde{A}) = \sigma(A) \cup \sigma(\mathcal{A}), & \Sigma_c(\tilde{A}, \tilde{B}) = \Sigma_c(A, B) \cup \Sigma_c(\mathcal{A}, \mathcal{B}), \\ \Sigma_o(\tilde{C}, \tilde{A}) = \Sigma(C, A) \cup \Sigma(\mathcal{C}, \mathcal{A}), & \Sigma_z(\tilde{C}, \tilde{A}, \tilde{B}) = \Sigma_z(C, A, B) \cup \Sigma_z(\mathcal{C}, \mathcal{A}, \mathcal{B}). \end{cases} \qquad \square$$

Remark 5. The result in [22] states that when using well-known particular forms of aggregations and restrictions, controllability or observability of the original system carries over to the expanded system, but not both. This result has been shown to be false in [24], which is confirmed by Theorem 3.8. However, it is obvious from Theorem 3.3 that the result of [22] is true when $\tilde{m} = m$ and $\tilde{l} = l$.

4. Overlapping decentralized control. A wide variety of applications of the expansion-contraction concept relies on decentralized control with overlapping information structure constraints. When a plant is composed of interconnected subsystems that share common parts, decentralized control laws, which utilize the state variables of the overlapping parts, are superior to disjoint decentralized control laws. This has been the case in the platooning of vehicles on highways and in the air where state variables are shared between adjacent vehicles [4, 9, 10, 35, 36]. Similarly, in electric power systems tie-line information is used to control each individual power area by decentralized control [2, 3, 7]. Another example is a plant which is overlapped by two controllers for reliability enhancement. The controllers either simultaneously stabilize the plant or individually, whenever one of them has failed [2, 37].

Assume that the system $\mathbf{S}$ is composed of two overlapping subsystems and is represented by the matrices

$$\begin{cases} A = \begin{array}{cc} \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} & \\ \left[\begin{array}{ccc} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{array}\right] & \begin{array}{l} \}n_1 \\ \}n_2 \\ \}n_3 \end{array} \end{array}, \\ B = \begin{array}{cc} \begin{array}{ccc} m_1 & m_2 & m_3 \end{array} & \\ \left[\begin{array}{ccc} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{array}\right] & \begin{array}{l} \}n_1 \\ \}n_2 \\ \}n_3 \end{array} \end{array}, \\ C = \begin{array}{cc} \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} & \\ \left[\begin{array}{ccc} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{array}\right] & \begin{array}{l} \}l_1 \\ \}l_2 \\ \}l_3 \end{array} \end{array}, \end{cases} \tag{4.1}$$

where the lines delineate the subsystems. Using standard linear transformations defined by matrices (3.9), we obtain the expanded matrices as

$$
(4.2)\qquad
\left\{
\begin{array}{l}
\tilde{A} = \begin{array}{cc}
\begin{array}{ccccc} n_1 & n_2 & & n_2 & n_3 \end{array} & \\
\left[\begin{array}{cc|cc}
\tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} \\
\tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} \\
\hline
\tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} & \tilde{A}_{34} \\
\tilde{A}_{41} & \tilde{A}_{42} & \tilde{A}_{43} & \tilde{A}_{44}
\end{array}\right] &
\begin{array}{l} \}n_1 \\ \}n_2 \\ \}n_2 \\ \}n_3 \end{array}
\end{array}, \\
\tilde{B} = \begin{array}{cc}
\begin{array}{ccccc} m_1 & m_2 & & m_2 & m_3 \end{array} & \\
\left[\begin{array}{cc|cc}
\tilde{B}_{11} & \tilde{B}_{12} & \tilde{B}_{13} & \tilde{B}_{14} \\
\tilde{B}_{21} & \tilde{B}_{22} & \tilde{B}_{23} & \tilde{B}_{24} \\
\hline
\tilde{B}_{31} & \tilde{B}_{32} & \tilde{B}_{33} & \tilde{B}_{34} \\
\tilde{B}_{41} & \tilde{B}_{42} & \tilde{B}_{43} & \tilde{B}_{44}
\end{array}\right] &
\begin{array}{l} \}n_1 \\ \}n_2 \\ \} \\ n_2 \\ \}n_3 \end{array}
\end{array}, \\
\tilde{C} = \begin{array}{cc}
\begin{array}{ccccc} n_1 & n_2 & & n_2 & n_3 \end{array} & \\
\left[\begin{array}{cc|cc}
\tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} & \tilde{C}_{14} \\
\tilde{C}_{21} & \tilde{C}_{22} & \tilde{C}_{23} & \tilde{C}_{24} \\
\hline
\tilde{C}_{31} & \tilde{C}_{32} & \tilde{C}_{33} & \tilde{C}_{34} \\
\tilde{C}_{41} & \tilde{C}_{42} & \tilde{C}_{43} & \tilde{C}_{44}
\end{array}\right] &
\begin{array}{l} \}l_1 \\ \}l_2 \\ \}l_2 \\ \}l_3 \end{array}
\end{array},
\end{array}
\right.
$$

where the overlapping subsystems appear as disjoint.

An interesting idea was recently proposed in [15, 16] to use complementary matrices in order to make the interconnection (off-diagonal) block matrices as sparse as possible, thus enhancing decentralized control strategies for stabilization of the overall system. Note that V, L, and T are given by (3.9), so, the matrices V^+, L^+, T^+, U, P, and S in Theorem 3.3 are given by

$$
V^+ = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & I_{n_2}/2 & I_{n_2}/2 & 0 \\ 0 & 0 & 0 & I_{n_3} \end{bmatrix}, \quad
L^+ = \begin{bmatrix} I_{m_1} & 0 & 0 & 0 \\ 0 & I_{m_2}/2 & I_{m_2}/2 & 0 \\ 0 & 0 & 0 & I_{m_3} \end{bmatrix},
$$

$$
T^+ = \begin{bmatrix} I_{l_1} & 0 & 0 & 0 \\ 0 & I_{l_2}/2 & I_{l_2}/2 & 0 \\ 0 & 0 & 0 & I_{l_3} \end{bmatrix}
$$

and

$$
U = \begin{bmatrix} 0_{n_1\times n_2} \\ I_{n_2}/\sqrt{2} \\ -I_{n_2}/\sqrt{2} \\ 0_{n_3\times n_2} \end{bmatrix}, \quad
P = \begin{bmatrix} 0_{m_1\times m_2} \\ I_{m_2}/\sqrt{2} \\ -I_{m_2}/\sqrt{2} \\ 0_{m_3\times m_2} \end{bmatrix}, \quad
S = \begin{bmatrix} 0_{l_1\times l_2} \\ I_{l_2}/\sqrt{2} \\ -I_{l_2}/\sqrt{2} \\ 0_{l_3\times l_2} \end{bmatrix}.
$$

From Theorem 3.3 we have that all expansion matrices $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ of system $\mathbf{S}$ are of the forms

(4.3)

$$
\tilde{A} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & I_{n_2} & 0 & W/\sqrt{2} \\ 0 & I_{n_2} & 0 & -W/\sqrt{2} \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix}
\begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & X_{15} \\ A_{21} & A_{22} & A_{23} & 0 & X_{25} \\ A_{31} & A_{32} & A_{33} & 0 & X_{35} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} \\ 0 & 0 & 0 & 0 & X_{55} \end{bmatrix}
\begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & I_{n_2}/2 & I_{n_2}/2 & 0 \\ 0 & 0 & 0 & I_{n_3} \\ 0 & W^T/\sqrt{2} & -W^T/\sqrt{2} & 0 \end{bmatrix},
$$

$$\tilde{B} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & I_{n_2} & 0 & W/\sqrt{2} \\ 0 & I_{n_2} & 0 & -W/\sqrt{2} \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} & Y_{14} \\ B_{21} & B_{22} & B_{23} & Y_{24} \\ B_{31} & B_{32} & B_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \\ 0 & 0 & 0 & Y_{54} \end{bmatrix} \begin{bmatrix} I_{m_1} & 0 & 0 & 0 \\ 0 & I_{m_2}/2 & I_{m_2}/2 & 0 \\ 0 & 0 & 0 & I_{m_3} \\ 0 & I_{m_2}/\sqrt{2} & -I_{m_2}/\sqrt{2} & 0 \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} I_{l_1} & 0 & 0 & 0 \\ 0 & I_{l_2} & 0 & I_{l_2}/\sqrt{2} \\ 0 & I_{l_2} & 0 & -I_{l_2}/\sqrt{2} \\ 0 & 0 & I_{l_3} & 0 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & Z_{15} \\ C_{21} & C_{22} & C_{23} & 0 & Z_{25} \\ C_{31} & C_{32} & C_{33} & 0 & Z_{35} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & I_{n_2}/2 & I_{n_2}/2 & 0 \\ 0 & 0 & 0 & I_{n_3} \\ 0 & W^T/\sqrt{2} & -W^T/\sqrt{2} & 0 \end{bmatrix},$$

where $W \in \mathbf{R}^{n_2 \times n_2}$ is orthogonal, X_{44}, X_{55}, X_{i5} $(i = 1, \ldots, 4)$, Y_{j4} and Z_{4j} $(j = 1, \ldots, 5)$, X_{4k}, and Y_{4k} and Z_{k5} $(k = 1, 2, 3)$ are arbitrary matrices with appropriate dimensions, and in particular $X_{44} \in \mathbf{R}^{\mu \times \mu}$, $X_{55} \in \mathbf{R}^{(n_2-\mu)\times(n_2-\mu)}$, μ is an integer between 0 and n_2. Thus, by a direct computation using (4.3) we obtain

$$\tilde{A}_{14} = A_{13}, \quad \tilde{A}_{41} = A_{31}, \quad \tilde{B}_{14} = B_{13}, \quad \tilde{B}_{41} = B_{31}, \quad \tilde{C}_{14} = C_{13}, \quad \tilde{C}_{41} = C_{13}.$$

Consequently, system $\tilde{\mathbf{S}}$ is maximally sparsified if and only if

$$\begin{cases} \tilde{A}_{31} = 0, \ \tilde{A}_{32} = 0, \ \tilde{A}_{42} = 0, \ \tilde{A}_{23} = 0, \ \tilde{A}_{24} = 0, \ \tilde{A}_{13} = 0, \\ \tilde{B}_{31} = 0, \ \tilde{B}_{32} = 0, \ \tilde{B}_{42} = 0, \ \tilde{B}_{23} = 0, \ \tilde{B}_{24} = 0, \ \tilde{B}_{13} = 0, \\ \tilde{C}_{31} = 0, \ \tilde{C}_{32} = 0, \ \tilde{C}_{42} = 0, \ \tilde{C}_{23} = 0, \ \tilde{C}_{24} = 0, \ \tilde{C}_{13} = 0. \end{cases} \tag{4.4}$$

Now, the following problem is of interest.

Problem 1. Under what conditions does there exist an expansion $\tilde{\mathbf{S}}$ of system $\mathbf{S}$ having matrices (4.4)?

It has been mentioned in [15] that in some situation Problem 1 is solvable, but no solvability conditions have been stated; Problem 1 cannot be solved simply by setting $\tilde{A} := VAV^+$, $\tilde{B} := VBL^+$, and $\tilde{C} := TCV^+$, because

$$\begin{cases} VAV^+ = \left[\begin{array}{cc|cc} A_{11} & A_{12}/2 & A_{12}/2 & A_{13} \\ A_{21} & A_{22}/2 & A_{22}/2 & A_{23} \\ \hline A_{21} & A_{22}/2 & A_{22}/2 & A_{23} \\ A_{31} & A_{32}/2 & A_{32}/2 & A_{33} \end{array}\right], \\ VBL^+ = \left[\begin{array}{cc|cc} B_{11} & B_{12}/2 & B_{12}/2 & B_{13} \\ B_{21} & B_{22}/2 & B_{22}/2 & B_{23} \\ \hline B_{21} & B_{22}/2 & B_{22}/2 & B_{23} \\ B_{31} & B_{32}/2 & B_{32}/2 & B_{33} \end{array}\right], \\ TCV^+ = \left[\begin{array}{cc|cc} C_{11} & C_{12}/2 & C_{12}/2 & C_{13} \\ C_{21} & C_{22}/2 & C_{22}/2 & C_{23} \\ \hline C_{21} & C_{22}/2 & C_{22}/2 & C_{23} \\ C_{31} & C_{32}/2 & C_{32}/2 & C_{33} \end{array}\right]; \end{cases}$$

in fact, there are no *general* algorithms for producing such systems. We provide these conditions by the following.

THEOREM 4.1. *Let the triplet* (A, B, C) *of system* $\mathbf{S}$ *be as in* (4.1) *and let matrices* V*,* L*, and* T *be those of* (3.9)*. Then, there exists an expansion* $\tilde{\mathbf{S}}$ *of system* $\mathbf{S}$ *such* *that* (4.4) *holds if and only if*

$$\mathcal{C}(A_{22}, \begin{bmatrix} A_{21} & -A_{23} & B_{21} & -B_{23} \end{bmatrix}) \subset \ker\left(\begin{bmatrix} A_{12} \\ -A_{32} \\ C_{12} \\ -C_{32} \end{bmatrix}\right). \tag{4.5}$$

Furthermore, in the case that condition (4.5) *is true, triplet* $(\tilde{A}, \tilde{B}, \tilde{C})$ *of the expanded system* $\tilde{\mathbf{S}}$ *is given by*

$$\begin{cases} \tilde{A} = \begin{bmatrix} A_{11} & A_{12} & | & 0 & A_{13} \\ 2A_{21} & A_{22} & | & 0 & 0 \\ --- & --- & - & --- & --- \\ 0 & 0 & | & A_{22} & 2A_{23} \\ A_{31} & 0 & | & A_{32} & A_{33} \end{bmatrix}, \\ \\ \tilde{B} = \begin{bmatrix} B_{11} & B_{12} & | & 0 & B_{13} \\ 2B_{21} & B_{22} & | & 0 & 0 \\ --- & --- & - & --- & --- \\ 0 & 0 & | & B_{22} & 2B_{23} \\ B_{31} & 0 & | & B_{32} & B_{33} \end{bmatrix}, \\ \\ \tilde{C} = \begin{bmatrix} C_{11} & C_{12} & | & 0 & C_{13} \\ 2C_{21} & C_{22} & | & 0 & 0 \\ --- & --- & - & --- & --- \\ 0 & 0 & | & C_{22} & 2C_{23} \\ C_{31} & 0 & | & C_{32} & C_{33} \end{bmatrix}. \end{cases} \tag{4.6}$$

Proof. Since (4.3) holds, hence $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ satisfy (4.4) if and only if

$$\begin{cases} \begin{bmatrix} 0 & I_{n_2} & 0 & -W/\sqrt{2} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & X_{15} \\ A_{21} & A_{22} & A_{23} & 0 & X_{25} \\ A_{31} & A_{32} & A_{33} & 0 & X_{35} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} \\ 0 & 0 & 0 & 0 & X_{55} \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2}/2 \\ 0 & 0 \\ 0 & W^T/\sqrt{2} \end{bmatrix} = 0, \\ \begin{bmatrix} 0 & 0 & I_{n_3} & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & X_{15} \\ A_{21} & A_{22} & A_{23} & 0 & X_{25} \\ A_{31} & A_{32} & A_{33} & 0 & X_{35} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} \\ 0 & 0 & 0 & 0 & X_{55} \end{bmatrix} \begin{bmatrix} 0 \\ I_{n_2}/2 \\ 0 \\ W^T/\sqrt{2} \end{bmatrix} = 0, \\ \begin{bmatrix} 0 & I_{n_2} & 0 & W/\sqrt{2} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & X_{15} \\ A_{21} & A_{22} & A_{23} & 0 & X_{25} \\ A_{31} & A_{32} & A_{33} & 0 & X_{35} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} \\ 0 & 0 & 0 & 0 & X_{55} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ I_{n_2}/2 & 0 \\ 0 & I_{n_3} \\ -W^T/\sqrt{2} & 0 \end{bmatrix} = 0, \\ \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & X_{15} \\ A_{21} & A_{22} & A_{23} & 0 & X_{25} \\ A_{31} & A_{32} & A_{33} & 0 & X_{35} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} \\ 0 & 0 & 0 & 0 & X_{55} \end{bmatrix} \begin{bmatrix} 0 \\ I_{n_2}/2 \\ 0 \\ -W^T/\sqrt{2} \end{bmatrix} = 0, \end{cases}$$

$$\begin{cases} \begin{bmatrix} 0 & I & 0 & -W/\sqrt{2} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} & Y_{14} \\ B_{21} & B_{22} & B_{23} & Y_{24} \\ B_{31} & B_{32} & B_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \\ 0 & 0 & 0 & Y_{54} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I/2 \\ 0 & 0 \\ 0 & I/\sqrt{2} \end{bmatrix} = 0, \\ \begin{bmatrix} 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} & Y_{14} \\ B_{21} & B_{22} & B_{23} & Y_{24} \\ B_{31} & B_{32} & B_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \\ 0 & 0 & 0 & Y_{54} \end{bmatrix} \begin{bmatrix} 0 \\ I/2 \\ 0 \\ I/\sqrt{2} \end{bmatrix} = 0, \\ \begin{bmatrix} 0 & I & 0 & W/\sqrt{2} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} & Y_{14} \\ B_{21} & B_{22} & B_{23} & Y_{24} \\ B_{31} & B_{32} & B_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \\ 0 & 0 & 0 & Y_{54} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ I/2 & 0 \\ 0 & I \\ -I/\sqrt{2} & 0 \end{bmatrix} = 0, \\ \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} & Y_{14} \\ B_{21} & B_{22} & B_{23} & Y_{24} \\ B_{31} & B_{32} & B_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \\ 0 & 0 & 0 & Y_{54} \end{bmatrix} \begin{bmatrix} 0 \\ I/2 \\ 0 \\ -I/\sqrt{2} \end{bmatrix} = 0, \end{cases}$$

and

$$\begin{cases} \begin{bmatrix} 0 & I & 0 & -I/\sqrt{2} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & Z_{15} \\ C_{21} & C_{22} & C_{23} & 0 & Z_{25} \\ C_{31} & C_{32} & C_{33} & 0 & Z_{35} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I/2 \\ 0 & 0 \\ 0 & W^T/\sqrt{2} \end{bmatrix} = 0, \\ \begin{bmatrix} 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & Z_{15} \\ C_{21} & C_{22} & C_{23} & 0 & Z_{25} \\ C_{31} & C_{32} & C_{33} & 0 & Z_{35} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} \end{bmatrix} \begin{bmatrix} 0 \\ I/2 \\ 0 \\ W^T/\sqrt{2} \end{bmatrix} = 0, \\ \begin{bmatrix} 0 & I & 0 & I/\sqrt{2} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & Z_{15} \\ C_{21} & C_{22} & C_{23} & 0 & Z_{25} \\ C_{31} & C_{32} & C_{33} & 0 & Z_{35} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ I/2 & 0 \\ 0 & I \\ -W^T/\sqrt{2} & 0 \end{bmatrix} = 0, \\ \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & Z_{15} \\ C_{21} & C_{22} & C_{23} & 0 & Z_{25} \\ C_{31} & C_{32} & C_{33} & 0 & Z_{35} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} \end{bmatrix} \begin{bmatrix} 0 \\ I/2 \\ 0 \\ -W^T/\sqrt{2} \end{bmatrix} = 0. \end{cases}$$

Thus, a simple calculation yields that there exists a triplet $(\tilde{A}, \tilde{B}, \tilde{C})$ of the form (4.3) such that (4.4) holds if and only if

$$\begin{cases} A_{22} = W \begin{bmatrix} X_{44} & X_{45} \\ 0 & X_{55} \end{bmatrix} W^T, \\ \begin{bmatrix} A_{21} & -A_{23} & B_{21} & -B_{23} \end{bmatrix} = W \begin{bmatrix} X_{41} & X_{43} & Y_{41} & Y_{43} \\ 0 & 0 & 0 & 0 \end{bmatrix} /\sqrt{2} \end{cases} \tag{4.7}$$

and

$$\begin{bmatrix} A_{12} \\ -A_{32} \\ C_{12} \\ -C_{32} \end{bmatrix} = \sqrt{2} \begin{bmatrix} 0 & X_{15} \\ 0 & X_{35} \\ 0 & Z_{15} \\ 0 & Z_{35} \end{bmatrix} W^T, \tag{4.8}$$

which is equivalent to condition (4.5).

Conversely, if condition (4.5) holds, then in (4.3) we can choose an orthogonal matrix W such that

$$(W^T A_{22} W, W^T \sqrt{2} \begin{bmatrix} A_{21} & -A_{23} & B_{21} & -B_{23} \end{bmatrix})$$

is in the controllability staircase form (4.7) [33] of $(A_{22}, \sqrt{2} \begin{bmatrix} A_{21} & -A_{23} & B_{21} & -B_{23} \end{bmatrix})$, let μ be the dimension of its controllability subspace, and define

$$\begin{bmatrix} X_{44} & X_{45} \\ 0 & X_{55} \end{bmatrix}, \quad \begin{bmatrix} X_{41} & X_{43} & Y_{41} & Y_{43} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} X_{15} \\ X_{35} \\ Z_{15} \\ Z_{35} \end{bmatrix}$$

by equations (4.7) and (4.8) with $X_{44} \in \mathbf{R}^{\mu \times \mu}$ and $X_{55} \in \mathbf{R}^{(n_2-\mu)\times(n_2-\mu)}$. Now $(X_{44}, [X_{41} \quad X_{43} \quad Y_{41} \quad Y_{43}])$ is controllable. In addition, define

$$\begin{cases} X_{25} = 0, \quad X_{42} = 0, \quad Y_{24} = 0, \quad Y_{42} = 0, \quad Z_{25} = 0, \quad Z_{42} = 0, \\ Y_{14} = B_{12}/\sqrt{2}, \quad Y_{34} = -B_{32}/\sqrt{2}, \quad Z_{41} = \sqrt{2} C_{21}, \quad Z_{43} = -\sqrt{2} C_{23}, \\ \begin{bmatrix} Y_{44} \\ Y_{54} \end{bmatrix} = W^T B_{22}, \quad \begin{bmatrix} Z_{44} & Z_{45} \end{bmatrix} = C_{22} W. \end{cases} \tag{4.9}$$

Then (4.6) follows. □

Condition (4.5) can be verified easily using the well-known controllability staircase form of linear systems (see, e.g., [33]). Theorem 4.1 defines a numerically stable method for solving Problem 1.

5. Contractibility of dynamic controllers. Now, by capitalizing on the canonical form for state feedback laws, we want to present explicit solvability conditions for contractibility of dynamic controllers. They are exhaustive and include the sufficient conditions obtained in [17, 18].

Let us consider a dynamic controller for system $\mathbf{S}$:

$$\mathbf{C}: \quad \begin{cases} \dot{w} = Fw + Gu + Jy, \ w(0) = w_0, \\ u = Kw + Hy + v, \end{cases} \tag{5.1}$$

where $w \in \mathbf{R}^\tau$, $u \in \mathbf{R}^m$, and $y \in \mathbf{R}^l$ are the state, input, and output of $\mathbf{C}$. An expansion $\tilde{\mathbf{C}}$ of controller $\mathbf{C}$ is defined as

$$\tilde{\mathbf{C}}: \quad \begin{cases} \dot{\tilde{w}} = \tilde{F}\tilde{w} + \tilde{G}\tilde{u} + \tilde{J}\tilde{y}, \ \tilde{w}(0) = \tilde{w}_0, \\ \tilde{u} = \tilde{K}w + \tilde{H}y + \tilde{v}, \end{cases} \tag{5.2}$$

where $\tilde{w} \in \mathbf{R}^{\tilde{\tau}}$, $\tilde{u} \in \mathbf{R}^{\tilde{m}}$, and $\tilde{y} \in \mathbf{R}^{\tilde{l}}$. We recall the following [18].

Definition 5.1. *The controller $\tilde{\mathbf{C}}$ for system $\tilde{\mathbf{S}}$ is contractible to the controller $\mathbf{C}$ for system $\mathbf{S}$ if there exist matrices V, L, T, D, and E satisfying* (2.3) *and* (2.4) *and*

$$\operatorname{rank}(E) = \tau, \quad \operatorname{rank}(D) = m \tag{5.3}$$

such that one of the following two statements holds:

(a) *For any initial states* x_0 *and* w_0 *and any input* u, *the choice*

$$\tilde{x}_0 = V x_0, \quad \tilde{w}_0 = E w_0, \quad \tilde{u} = L u$$

implies that

$$\begin{cases} x(t; x_0, u) = V^+ \tilde{x}(t; \tilde{x}_0, \tilde{u}), \quad y[x(t)] = T^+ \tilde{y}[\tilde{x}(t)], \\ w(t; w_0, u) = E^+ \tilde{w}(t; \tilde{w}_0, \tilde{u}), \quad D(Kw + Hy) = \tilde{K}\tilde{w} + \tilde{H}\tilde{y} \quad \forall t \geq 0. \end{cases}$$

(b) *For any initial states* x_0 *and* w_0 *and any input* u, *the choice*

$$\tilde{x}_0 = V x_0, \quad \tilde{w}_0 = E w_0, \quad u = L^+ \tilde{u}$$

implies that

$$\begin{cases} x(t; x_0, u) = V^+ \tilde{x}(t; \tilde{x}_0, \tilde{u}), \quad y[x(t)] = T^+ \tilde{y}[\tilde{x}(t)], \\ w(t; w_0, u) = E^+ \tilde{w}(t; \tilde{w}_0, \tilde{u}), \quad Kw + Hy = D^+(\tilde{K}\tilde{w} + \tilde{H}\tilde{y}) \quad \forall t \geq 0. \end{cases}$$

We shall now give an explicit characterization of contractibility of controller $\tilde{\mathbf{C}}$ by the following.

THEOREM 5.2. *Given system* $\mathbf{S}$ *and transformation matrices* V, L, T, D, *and* E *satisfying* (2.3), (2.4), *and* (5.3), *let the QR factorizations of* V, L, *and* T *be given by* (3.2). *Furthermore, let the QR factorizations of matrices* D *and* E *be given by*

$$\begin{cases} \begin{bmatrix} \mathcal{X} & X \end{bmatrix}^T D = \begin{bmatrix} D_{11} \\ 0 \end{bmatrix} \begin{matrix} \}m \\ \}\tilde{m} - m \end{matrix}, \quad \mathcal{X} \in \mathbf{R}^{\tilde{m} \times m}, \quad X \in \mathbf{R}^{\tilde{m} \times (\tilde{m} - m)}, \\ \begin{bmatrix} \mathcal{Y} & Y \end{bmatrix}^T E = \begin{bmatrix} E_{11} \\ 0 \end{bmatrix} \begin{matrix} \}\tau \\ \}\tilde{\tau} - \tau \end{matrix}, \quad \mathcal{Y} \in \mathbf{R}^{\tilde{\tau} \times \tau}, \quad Y \in \mathbf{R}^{\tilde{\tau} \times (\tilde{\tau} - \tau)}, \end{cases} \tag{5.4}$$

where $\begin{bmatrix} \mathcal{X} & X \end{bmatrix}$ *and* $\begin{bmatrix} \mathcal{Y} & Y \end{bmatrix}$ *are orthogonal, and* D_{11} *and* E_{11} *are nonsingular. Then, the controller* $\tilde{\mathbf{C}}$ *for system* $\tilde{\mathbf{S}}$ *is contractible to the controller* $\mathbf{C}$ *for system* $\mathbf{S}$ *if one of the following four statements holds:*

(a) *Matrices* $\tilde{A}$, $\tilde{B}$, *and* $\tilde{C}$ *of system* $\tilde{\mathbf{S}}$ *are given by* (3.3) *and furthermore,* $\tilde{F}$, $\tilde{G}$, $\tilde{J}$, $\tilde{K}$, *and* $\tilde{H}$ *are given by*

$$\begin{cases} \tilde{F} = \begin{bmatrix} E & YZ \end{bmatrix} \begin{bmatrix} F & 0 & \tilde{F}_{13} \\ \tilde{F}_{21} & \tilde{F}_{22} & \tilde{F}_{23} \\ 0 & 0 & \tilde{F}_{33} \end{bmatrix} \begin{bmatrix} E^+ \\ (YZ)^T \end{bmatrix}, \\ \tilde{G} = \begin{bmatrix} E & YZ \end{bmatrix} \begin{bmatrix} G & \tilde{G}_{12} \\ \tilde{G}_{21} & \tilde{G}_{22} \\ 0 & \tilde{G}_{32} \end{bmatrix} \begin{bmatrix} L^+ \\ P^T \end{bmatrix}, \\ \tilde{J} = \begin{bmatrix} E & YZ \end{bmatrix} \begin{bmatrix} J & 0 \\ \tilde{J}_{21} & \tilde{J}_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^+ \\ S^T \end{bmatrix}, \\ \tilde{K} = \begin{bmatrix} D & X \end{bmatrix} \begin{bmatrix} K & 0 & \tilde{K}_{13} \\ 0 & 0 & \tilde{K}_{23} \end{bmatrix} \begin{bmatrix} E^+ \\ (YZ)^T \end{bmatrix}, \\ \tilde{H} = \begin{bmatrix} D & X \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^+ \\ S^T \end{bmatrix}. \end{cases} \tag{5.5}$$

(b) *Matrices* $\tilde{A}$, $\tilde{B}$, *and* $\tilde{C}$ *of system* $\tilde{\mathbf{S}}$ *are given by* (3.3) *with* $\tilde{C}_{21} = 0$ *and* $\tilde{C}_{22} = 0$. *Furthermore,* $\tilde{F}$, $\tilde{G}$, $\tilde{J}$, $\tilde{K}$, *and* $\tilde{H}$ *are given by*

$$\left\{\begin{array}{l} \tilde{F} = \begin{bmatrix} E & YZ \end{bmatrix} \begin{bmatrix} F & 0 & \tilde{F}_{13} \\ \tilde{F}_{21} & \tilde{F}_{22} & \tilde{F}_{23} \\ 0 & 0 & \tilde{F}_{33} \end{bmatrix} \begin{bmatrix} E^+ \\ (YZ)^T \end{bmatrix}, \\ \tilde{G} = \begin{bmatrix} E & YZ \end{bmatrix} \begin{bmatrix} G & \tilde{G}_{12} \\ \tilde{G}_{21} & \tilde{G}_{22} \\ 0 & \tilde{G}_{32} \end{bmatrix} \begin{bmatrix} L^+ \\ P^T \end{bmatrix}, \\ \tilde{J} = \begin{bmatrix} E & YZ \end{bmatrix} \begin{bmatrix} J & \tilde{J}_{12} \\ \tilde{J}_{21} & \tilde{J}_{22} \\ 0 & \tilde{J}_{32} \end{bmatrix} \begin{bmatrix} T^+ \\ S^T \end{bmatrix}, \\ \tilde{K} = \begin{bmatrix} D & X \end{bmatrix} \begin{bmatrix} K & 0 & \tilde{K}_{13} \\ 0 & 0 & \tilde{K}_{23} \end{bmatrix} \begin{bmatrix} E^+ \\ (YZ)^T \end{bmatrix}, \\ \tilde{H} = \begin{bmatrix} D & X \end{bmatrix} \begin{bmatrix} H & \tilde{H}_{12} \\ 0 & \tilde{H}_{22} \end{bmatrix} \begin{bmatrix} T^+ \\ S^T \end{bmatrix}. \end{array}\right. \tag{5.6}$$

(c) *Matrices* $\tilde{A}$, $\tilde{B}$, *and* $\tilde{C}$ *of system* $\tilde{\mathbf{S}}$ *are given by* (3.3) *with* $\tilde{B}_{12} = 0$ *and* $\tilde{B}_{32} = 0$. *Furthermore, matrices* $\tilde{F}$, $\tilde{G}$, $\tilde{J}$, $\tilde{K}$, *and* $\tilde{H}$ *are given by*

$$\left\{\begin{array}{l} \tilde{F} = \begin{bmatrix} E & YZ \end{bmatrix} \begin{bmatrix} F & 0 & \tilde{F}_{13} \\ \tilde{F}_{21} & \tilde{F}_{22} & \tilde{F}_{23} \\ 0 & 0 & \tilde{F}_{33} \end{bmatrix} \begin{bmatrix} E^+ \\ (YZ)^T \end{bmatrix}, \\ \tilde{G} = \begin{bmatrix} E & YZ \end{bmatrix} \begin{bmatrix} G & 0 \\ \tilde{G}_{21} & \tilde{G}_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} L^+ \\ P^T \end{bmatrix}, \\ \tilde{J} = \begin{bmatrix} E & YZ \end{bmatrix} \begin{bmatrix} J & 0 \\ \tilde{J}_{21} & \tilde{J}_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^+ \\ S^T \end{bmatrix}, \\ \tilde{K} = \begin{bmatrix} D & X \end{bmatrix} \begin{bmatrix} K & 0 & \tilde{K}_{13} \\ \tilde{K}_{21} & \tilde{K}_{22} & \tilde{K}_{23} \end{bmatrix} \begin{bmatrix} E^+ \\ (YZ)^T \end{bmatrix}, \\ \tilde{H} = \begin{bmatrix} D & X \end{bmatrix} \begin{bmatrix} H & 0 \\ \tilde{H}_{21} & \tilde{H}_{22} \end{bmatrix} \begin{bmatrix} T^+ \\ S^T \end{bmatrix}. \end{array}\right. \tag{5.7}$$

(d) *Matrices* $\tilde{A}$, $\tilde{B}$, *and* $\tilde{C}$ *of system* $\tilde{\mathbf{S}}$ *are given by* (3.3) *with* $\tilde{B}_{12} = 0$, $\tilde{B}_{32} = 0$, $\tilde{C}_{21} = 0$, *and* $\tilde{C}_{22} = 0$. *Furthermore, matrices* $\tilde{F}$, $\tilde{G}$, $\tilde{J}$, $\tilde{K}$, *and* $\tilde{H}$ *are given by*

$$\left\{\begin{array}{l} \tilde{F} = \begin{bmatrix} E & YZ \end{bmatrix} \begin{bmatrix} F & 0 & \tilde{F}_{13} \\ \tilde{F}_{21} & \tilde{F}_{22} & \tilde{F}_{23} \\ 0 & 0 & \tilde{F}_{33} \end{bmatrix} \begin{bmatrix} E^+ \\ (YZ)^T \end{bmatrix}, \\ \tilde{G} = \begin{bmatrix} E & YZ \end{bmatrix} \begin{bmatrix} G & 0 \\ \tilde{G}_{21} & \tilde{G}_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} L^+ \\ P^T \end{bmatrix}, \\ \tilde{J} = \begin{bmatrix} E & YZ \end{bmatrix} \begin{bmatrix} J & \tilde{J}_{12} \\ \tilde{J}_{21} & \tilde{J}_{22} \\ 0 & \tilde{J}_{32} \end{bmatrix} \begin{bmatrix} T^+ \\ S^T \end{bmatrix}, \\ \tilde{K} = \begin{bmatrix} D & X \end{bmatrix} \begin{bmatrix} K & 0 & \tilde{K}_{13} \\ \tilde{K}_{21} & \tilde{K}_{22} & \tilde{K}_{23} \end{bmatrix} \begin{bmatrix} E^+ \\ (YZ)^T \end{bmatrix}, \\ \tilde{H} = \begin{bmatrix} D & X \end{bmatrix} \begin{bmatrix} H & \tilde{H}_{12} \\ \tilde{H}_{21} & \tilde{H}_{22} \end{bmatrix} \begin{bmatrix} T^+ \\ S^T \end{bmatrix}. \end{array}\right. \tag{5.8}$$

In (a), (b), (c), *and* (d) *above,* $Z \in \mathbf{R}^{(\tilde{\tau}-\tau)\times(\tilde{\tau}-\tau)}$ *is an arbitrary orthogonal matrix,* ν *is an arbitrary integer between* 0 *and* $\tilde{\tau}-\tau$, *and* $\tilde{F}_{13} \in \mathbf{R}^{\tau\times(\tilde{\tau}-\tau-\nu)}$, $\tilde{F}_{21} \in \mathbf{R}^{\nu\times\tau}$, $\tilde{F}_{22} \in \mathbf{R}^{\nu\times\nu}$, $\tilde{F}_{23} \in \mathbf{R}^{\nu\times(\tilde{\tau}-\tau-\nu)}$, $\tilde{F}_{33} \in \mathbf{R}^{(\tilde{\tau}-\tau-\nu)\times(\tilde{\tau}-\tau-\nu)}$, $\tilde{G}_{21} \in \mathbf{R}^{\nu\times m}$, $\tilde{G}_{22} \in \mathbf{R}^{\nu\times(\tilde{m}-m)}$, $\tilde{J}_{12} \in \mathbf{R}^{\tau\times(\tilde{l}-l)}$, $\tilde{J}_{21} \in \mathbf{R}^{\nu\times l}$, $\tilde{J}_{22} \in \mathbf{R}^{\nu\times(\tilde{l}-l)}$, $\tilde{J}_{32} \in \mathbf{R}^{(\tilde{\tau}-\tau-\nu)\times(\tilde{l}-l)}$, $\tilde{K}_{13} \in \mathbf{R}^{m\times(\tilde{\tau}-\nu-\tau)}$, $\tilde{K}_{21} \in \mathbf{R}^{(\tilde{m}-m)\times\tau}$, $\tilde{K}_{22} \in \mathbf{R}^{(\tilde{m}-m)\times\nu}$, $\tilde{K}_{23} \in \mathbf{R}^{(\tilde{m}-m)\times(\tilde{\tau}-\nu-\tau)}$, $\tilde{H}_{12} \in \mathbf{R}^{m\times(\tilde{l}-l)}$, $\tilde{H}_{21} \in \mathbf{R}^{(\tilde{m}-m)\times l}$, *and* $\tilde{H}_{22} \in \mathbf{R}^{(\tilde{m}-m)\times(\tilde{l}-l)}$ *are matrices with arbitrary elements.*

Proof. Theorem 5.2 can be proved using Definitions 5.1(a) and (b) directly, hence its proof is omitted. □

Similarly, as in Remarks 2 and 3, if in Theorem 5.2, we take $\nu = 0$ or $\nu = \tilde{\tau}-\tau$, then we can obtain some particular solvability conditions for contractibility of dynamic controllers, which contain the results of [17, 18] as special cases.

6. Conclusions. A canonical form for expanded systems is proposed in the inclusion principle for dynamic systems. The main benefits of the form are as follows:

1. In Theorems 3.3 and 3.4 we have established canonical forms for expansion-contraction matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{K}$, which provide an explicit parameterization of all expansion-contraction matrices. As a result, the full freedom in selecting the expansion-contraction matrices can be exploited in system analysis and design.
2. Theorem 3.8 provides a simple way to determine if stability, stabilizability, controllability, detectability, observability, and the stability of the invariant zeros carry over from a system $\mathbf{S}$ to its expansion $\tilde{\mathbf{S}}$.
3. In Theorem 4.1, we solved Problem 1, which is central to overlapping decentralized control and which has not been solved in full generality by existing methods.
4. By Theorem 5.2 we broaden the class of dynamic controllers which are contractible for implementation in the original system.

It is hoped that the proposed canonical form will simplify not only design of overlapping decentralized control, but also design of reduced-order controllers [6, 23], where the laws can be generated in the smaller space and then expanded for implementation in the original system.

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