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A CANONICAL TRANSFORMATION NEAR A BOUNDARY POINT

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ABSTRACT. A local homogeneous canonical transformation is constructed which straightens a curved boundary and freezes the coefficients of the principal part of a pseudo-differential operator in the neighborhood of a nonglancing ray.

Duistermaat and Hörmander [1] have studied the propogation along bicharacteristics of wave front sets of solutions of certain partial differential equations, using Fourier integral operators to effect a canonical transformation taking the given operator (locally) into $\partial/\partial x_1$. Hörmander [2] has also studied the problem with the aid of specially constructed pseudo-differential operators. Lax and Nirenberg [3] have applied the latter method to the study of boundary value problems, but thus far their approach has not handled the glancing ray case. As a first step towards adapting the approach of [1] to deal with boundary value problems, we construct a canonical transformation, away from glancing rays, which simultaneously reduces the boundary and the equation to a convenient form. I wish to thank Ralph Phillips for many helpful conversations.

Let $p(x, t; \xi, \tau)$ be a real symbol which is positive homogeneous of degree $m \geq 0$, m an integer, and with $(x, t, \xi, \tau) \in R^{n-1} \times R \times R^{n-1} \times R$. Let $0 \neq (\xi^0, \tau^0)$ satisfy $\partial p(0, 0; \xi^0, \tau^0)/\partial\tau \neq 0$. Let Γ be a smooth surface in R^n, passing through $(0, 0)$, and such that the normal to Γ at $(0, 0)$ points in the direction of the t axis.

Theorem. *There is a canonical map* $\chi: (x, t, \xi, \tau) \xrightarrow{\chi} (y, s, \eta, \sigma) \in R^{2n}$, *defined in a conic neighborhood* $\mathfrak{U}$ *of* $(0, 0, \xi^0, \tau^0)$, *homogeneous of degree one in* (ξ, τ), *and such that for* $(x, t, \xi, \tau) \in \mathfrak{U}$,

(i) $(x, t) \in \Gamma \Rightarrow s = 0$,

(ii) $p(x, t; \xi, \tau) = p(0, 0; \eta, \sigma) \overset{\text{def}}{=} p_0(\eta, \sigma)$.

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Proof. For χ to be canonical means that the Poisson brackets of the image points satisfy

$$\{y_i, y_j\} = \{y_i, s\} = \{y_i, \sigma\} = \{s, \eta_i\} = \{\eta_i, \sigma\} = 0,$$

$$\{s, \sigma\} = 1, \qquad \{y_i, \eta_j\} = \delta_{ij},$$

where

$$\{u, v\} = \sum_l \left(\frac{\partial u}{\partial x_l}\frac{\partial v}{\partial \xi_l} - \frac{\partial u}{\partial \xi_l}\frac{\partial v}{\partial x_l}\right) + \left(\frac{\partial u}{\partial t}\frac{\partial v}{\partial \tau} - \frac{\partial u}{\partial \tau}\frac{\partial v}{\partial t}\right)$$

$$\overset{\text{def}}{=} H_v u \overset{\text{def}}{=} \left[h_v^{(1)} \cdot \left(\nabla_x, \frac{\partial}{\partial t}\right) - h_v^{(2)} \cdot \left(\nabla_\xi, \frac{\partial}{\partial \tau}\right)\right] u, \qquad h_v = (h_v^{(1)}, h_v^{(2)}).$$

Our construction of χ is a modification of that given by Duistermaat and Hörmander [1] in free space. The functions η_i will be constructed successively on $\Gamma \times R^n$ by assigning each on an initial manifold transverse to the linear span of those h_{η_i} which are already known, and such that $h_{\eta_i}^{(1)}$ is tangential to Γ. This last fact will enable us to construct s such that $s = 0$ on Γ. Once η is constructed, σ is determined by (ii). To extend η and σ off of Γ, we shall use the equation $h_p \eta = 0$ together with (ii); a simple application of the chain rule shows that this construction implies the canonical relations $\{\eta_i, \sigma\} = 0$. Finally, we shall use the initial condition $(y, s)(0, 0, \eta, \sigma) = (y, s)$ together with the equations $h_{\eta_i}(y, s) = h_\sigma(y, s) = 0$ to determine (y, s).

We now construct χ. Let N_1 be a neighborhood of $(0, 0)$ in Γ, and C_1 a conic neighborhood of (ξ^0, τ^0) in R^n such that for $(x, t) \in N_1$ and $0 \neq (\xi, \tau) \in C_1$,

$$\langle n, \nabla_{\xi,\tau}\rangle p \neq 0, \tag{1}$$

where n is the normal to Γ at (x, t). On N_1, let $v_1(x, t)$ be a nonsingular tangential vector field such that $\langle(\xi^0, \tau^0), v_1(0, 0)\rangle = \xi_1^0$, and define

$$\eta_1(x, t, \xi, \tau) = \langle(\xi, \tau), v_1(x, t)\rangle, \qquad (x, t, \xi, \tau) \in N_1 \times C_1. \tag{2}$$

Because of (1), $h_p^{(1)}$ is not tangential to N_1, and hence we can extend the definition of η_1 off of $N_1 \times C_1$ by using (2) as an initial condition for

$$\{\eta_1, p\} \equiv H_p \eta_1 = 0. \tag{3}$$

We define recursively triples $\{N_i, v_i, \eta_i\}$, $i = 2, \cdots, n-1$ as follows.

Let $N_i \ni (0, 0)$ be a smooth $(n-1)$-dimensional surface in N_{i-1}, transverse to the span of the vectors $v_1, \cdots, v_{i-1}$ and not orthogonal to (ξ^0, τ^0) unless $\xi_i^0 = \xi_{i+1}^0 = \cdots = \xi_{n-1}^0 = 0$. Let v_i be a nonsingular vector field in N_i such that $\langle(\xi^0, \tau^0), v_i(0, 0)\rangle = \xi_i^0$. Define

$$\eta_i = \langle(\xi, \tau), v_i(x, t)\rangle, \qquad (x, t, \xi, \tau) \in N_i \times C_1, \tag{4}$$

and extend η_i by the equations

$$\{\eta_i, \eta_j\} = 0, \qquad j < i, \tag{5}$$

and

$$\{\eta_i, p\} = 0. \tag{6}$$

The consistency of the construction of η using (5) and (6) follows from the identity $[H_u, H_v] = H_{\{u,v\}}$. For example, the equations $\{\eta_j, \eta_k\} = 0$ are satisfied by construction along a submanifold M_k, $j < k$, and $\{\eta_j, p\} = \{\eta_k, p\} = 0$ along integral curves of H_p through M_k. On these integral curves, then,

$$\{p, \{\eta_j, \eta_k\}\} = H_{\{\eta_j,\eta_k\}}p = [H_{\eta_j}, H_{\eta_k}]p = H_{\eta_j}\{p, \eta_k\} - H_{\eta_k}\{p, \eta_j\} = 0,$$

so that $\{\eta_j, \eta_k\} = 0$ along these curves.

Remark. If Γ is the hyperplane $t = 0$, it suffices to set $\eta = \xi$ on $N_1 \times C_1$ and use (6) to extend the definition of η_i.

The condition $\partial p(0, 0; \xi^0, \tau^0)/\partial\tau \neq 0$, together with the above construction, ensures that there is a conic neighborhood $\mathfrak{U}$ of $(0, 0; \xi^0, \tau^0)$ in which σ is uniquely defined by (ii) if we set $\sigma(0, 0; \xi^0, \tau^0) = \tau^0$. Locally, σ is defined as a function of $Z \equiv (\eta, p)$, from which we conclude that

$$\{\sigma, \eta_j\} = H_{\eta_j}\sigma = \sum_k \frac{\partial\sigma}{\partial\eta_k}\{\eta_j, \eta_k\} + \frac{\partial\sigma}{\partial p}\{\eta_j, p\} = 0.$$

According to [1], we can now determine (y, s) in $\mathfrak{U}$ by assigning (y, s) on an n-dimensional manifold transverse to the span of h_{η_j}, $j = 1, \cdots, n-1$, and h_σ, provided that these vectors together with the radial vector $(0, 0; \xi, \tau)$ are linearly independent. Such a manifold is the subspace $x = 0$, $t = 0$. To see this, we need only observe that

$$h_{\eta_j}(0, 0, \xi, \tau) = (e_j, 0, 0, \partial\eta_j/\partial t),$$

where e_j is a standard unit basis vector in R^{n-1}, and that the nth component of $h_\sigma(0, 0, \xi, \tau)$ is (cf. (ii))

$$\frac{\partial\sigma}{\partial\tau}(0, 0, \xi, \tau) = \frac{\partial p_0}{\partial\sigma}\left\{\frac{\partial p}{\partial\tau} - \sum\frac{\partial p_0}{\partial\eta_j}\frac{\partial\eta_j}{\partial\tau}\right\} = \frac{\partial p_0}{\partial\sigma}\frac{\partial p}{\partial\tau}(0, 0, \xi, \tau) \neq 0.$$

We assign initial conditions

$$(y, s)(0, 0, \xi, \tau) = (0, 0). \tag{7}$$

Using (7) together with equations $H_{\eta_i}(y, s) = (e_i 0)$, $H_\sigma(y, s) = (0, 1)$, serves to define (y, s) in $\mathfrak{U}$.

There remains to show that (i) holds. But s is invariant on the integral curves of each H_{η_j}, and if $(x, t) \in N_j$, then $h^{(1)}_{\eta_j} = v_j$ is tangent to Γ. Given $(x', t', \xi', \tau') \in \mathfrak{U}$ with $(x', t') \in N_1$, we follow successively the integral curves of H_{η_i}, $i = 1, \cdots, n-1$, through $(x^i, t^i, \xi^i, \tau^i)$ till (x, t) hits N_{i+1} at (x^{i+1}, t^{i+1}) and $(\xi, \tau) = (\xi^{i+1}, \tau^{i+1})$, with N_n defined as the point $(0, 0)$. We conclude that for some $P = (0, 0, \xi^n, \tau^n) \in \mathfrak{U}$, $s(x, t, \xi, \tau) = s(P) = 0$ by (7). Theorem 1 is proved.

As a corollary of the proof, we note that if Γ is the hyperplane $t = 0$, then χ can be extended to a conical neighborhood of any cone $C = (0, 0, V\setminus\{0\})$, where V is a closed simply connected cone, and where $\partial p/\partial\tau \neq 0$ on $C\setminus\{0\}$. Since $\chi(0, 0, \xi, \tau) = (0, 0, \xi, \tau)$, condition (ii), together with a simple homotopy argument, allows us to drop the assumption that V be simply connected.

Remark. If p has the parity of m, then χ extends by homogeneity to a two-sided conic set.

Some applications and extensions will be reported on elsewhere.

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