

A CARDINAL SPLINE APPROACH TO WAVELETS

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ABSTRACT. While it is well known that the m th order B -spline $N_m(x)$ with integer knots generates a multiresolution analysis, $\cdots \subset V_{-1} \subset V_0 \subset \cdots$, with the m th order of approximation, we prove that $\psi(x) := L_{2m}^{(m)}(2x - 1)$, where $L_{2m}(x)$ denotes the $(2m)$ th order fundamental cardinal interpolatory spline, generates the orthogonal complementary wavelet spaces W_k . Note that for $m = 1$, when the B -spline $N_1(x)$ is the characteristic function of the unit interval $[0, 1)$, our basic wavelet $L_2'(2x - 1)$ is simply the well-known Haar wavelet. In proving that $V_{k+1} = V_k \oplus W_k$, we give the exact formulation of $N_m(2x - j)$, $j \in \mathbb{Z}$, in terms of integer translates of $N_m(x)$ and $\psi(x)$. This allows us to derive a wavelet decomposition algorithm without relying on orthogonality nor construction of a dual basis.

1. INTRODUCTION

A very natural and convenient way to introduce wavelets is to follow the notion of multiresolution analysis in [5, 7]. Let ϕ be a function in $L^2 = L^2(\mathbb{R})$ and set $\phi_{k,j}(x) = \phi(2^k x - j)$. For each $k \in \mathbb{Z}$, denote by V_k the L^2 -closure of the algebraic span of $\{\phi_{k,j}; j \in \mathbb{Z}\}$. Then the function ϕ is said to generate a multiresolution analysis (or approximation of L^2) if the following conditions are satisfied:

- (i) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$;
- (ii) $\text{Clos}_{L^2}(\bigcup_{k \in \mathbb{Z}} V_k) = L^2$;
- (iii) $\bigcap_{k \in \mathbb{Z}} V_k = \{0\}$; and
- (iv) for each k , $\{\phi_{k,j}; j \in \mathbb{Z}\}$ is an unconditional basis of V_k .

Suppose that ϕ generates a multiresolution analysis and that there exists some function ψ in L^2 such that the L^2 -closure W_0 of the algebraic span of $\{\psi(\cdot - j); j \in \mathbb{Z}\}$ is the orthogonal complement of V_0 in V_1 . Then ψ is called

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a *basic wavelet* relative to ϕ . We will use the notation $V_1 = V_0 \oplus W_0$, and again set $\psi_{kj}(x) = \psi(2^k x - j)$ and

$$W_k = \text{Clos}_{L^2}(\text{span}\{\psi_{kj}: j \in \mathbb{Z}\}).$$

(Note that in the wavelet literature, a normalization constant $2^{k/2}$ is used in defining ψ_{kj} .) If ψ is a basic wavelet relative to ϕ , then it is clear that the “*wavelet spaces*” W_k , generated by ψ , satisfy the following properties:

- (v) $V_{k+1} = V_k \oplus W_k$, all $k \in \mathbb{Z}$;
- (vi) $W_k \perp W_j$, all $k \neq j$; and
- (vii) $L^2 = \bigoplus_{k \in \mathbb{Z}} W_k$.

As a consequence of (vii), any function in L^2 has a (unique) orthogonal decomposition in terms of functions in W_k , $k \in \mathbb{Z}$. This is called a (complete) *wavelet decomposition*. Of course, in applications, one first approximates an $f \in L^2$ by some $f_k \in V_k$ with a sufficiently large value of k (cf. (i) and (ii)) and then obtains the (unique) wavelet decomposition of f_k , namely:

$$(1.1) \quad f_k = g_{k-1} + \cdots + g_{k-\ell} + f_{k-\ell}$$

for some sufficiently large value of ℓ , so that $f_{k-\ell}$ has arbitrarily small L^2 norm (cf. (iii)). Here, $g_i \in W_i$, $i = k-\ell, \dots, k-1$, and $f_{k-\ell} \in V_{k-\ell}$. Hence, in view of (i), (v), and (vi), the decomposition of f_k in (1.1) is an *orthogonal decomposition* (in the sense that all the components $g_{k-1}, \dots, g_{k-\ell}, f_{k-\ell}$ of f_k are mutually orthogonal).

A standard method to determine a basic wavelet $\tilde{\psi}$ from ϕ can be summarized very briefly as follows: First, orthonormalize $\{\phi_{0j}\}$, yielding $\{\tilde{\phi}_{0j}\}$; then find the (two-scale) relation of $\tilde{\phi}$ in terms of $\{\tilde{\phi}_{1j}\}$; and finally alternate the signs of the coefficient sequence in this two-scale relation in a clever way to form $\tilde{\psi}$ in terms of $\{\tilde{\psi}_{1j}\}$ (cf. [5, 3]). An extra feature that this basic wavelet $\tilde{\psi}$ has is that $\{\tilde{\psi}_{0j}\}$ is an orthogonal family. For instance, if ϕ is the m th order B -spline N_m , defined by convoluting the characteristic function $\chi_{[0,1]}$ with itself m -times (cf. [8] and [2]), then the procedure outlined above has been demonstrated for $m = 4$ in [5] to yield the orthonormal (cubic spline) wavelets of Battle [1] and Lemarié [4] (also see [7]).

It is noted from these examples that even for small values of m such as $m = 2$ and 4, the price to pay in order to achieve the additional orthogonality property of $\{\tilde{\psi}_{0j}\}$ is quite high. In the first place, it imposes extra computational complexity. Indeed, even for linear and cubic splines, it is extremely difficult to obtain an explicit representation of the basic wavelet $\tilde{\psi}$ (cf. [5, 7]). Of course, for computational purposes, an expression of $\tilde{\psi}$ is not important. However, due to the orthogonality of $\{\tilde{\psi}_{0j}\}$, the rate of decay of the sequences in the formulation for reconstructing $\tilde{\phi}_{1j}$ in terms of $\{\tilde{\phi}_{0j}\}$ and $\{\tilde{\psi}_{0j}\}$, which is essential in the computational algorithms for the wavelet decomposition (1.1) (cf. [3, 6]), seems to be somewhat slow.

In this paper, we take a more direct approach to constructing a basic wavelet ψ from ϕ . While it is essential to generate the same wavelet spaces W_k , so that the properties (v)–(vii) are valid and the same wavelet decomposition (1.1) is attained, we do not impose orthogonality on the family $\{\psi_{0j}\}$. Instead, we look for a ψ so that the sequences for representing ϕ_{1j} in terms of $\{\phi_{0j}\}$ and $\{\psi_{0j}\}$ have sufficiently fast rate of decay. In fact, we believe that this approach will allow us to construct a larger class of compactly supported basic wavelets ψ than the class obtained by Daubechies [3]. Our results in this direction will be presented in a later paper. The objective of this paper is to demonstrate this point of view by considering the B -splines N_m of arbitrary orders.

2. MAIN RESULTS

As usual, let χ_A denote the characteristic function of a set A . Then the m th order B -splines N_m are defined as follows:

$$N_1(x) = \chi_{[0,1)}(x),$$

and for $m = 2, 3, \dots$,

$$N_m(x) = (N_{m-1} * N_1)(x) = \int_0^1 N_{m-1}(x - t) dt.$$

Clearly, the support of N_m is the interval $[0, m]$ and the m th derivative of N_m is given by

$$(2.1) \quad N_m^{(m)}(x) = \sum_{j=0}^m (-1)^j \binom{m}{j} \delta(x - j),$$

where δ denotes, as usual, the Dirac delta distribution with unit mass at the origin (cf. [2]). In this paper, we will consider

$$(2.2) \quad \phi = N_m,$$

which generates a multiresolution analysis (i)–(iv). In fact, its order of approximation is precisely m . (For more details on approximation orders, see [2].)

To describe our basic wavelet ψ relative to ϕ given by (2.2), we need the *fundamental (cardinal) interpolatory spline*:

$$(2.3) \quad L_{2m}(x) = \sum_{j \in \mathbb{Z}} \alpha_j N_{2m}(x + m - j),$$

defined uniquely by the interpolation conditions:

$$(2.4) \quad L_{2m}(k) = \delta_{k,0}, \quad k \in \mathbb{Z},$$

where $\alpha_j = \alpha_j^{(m)}$ and the standard notation for the Kronecker delta is used. That is, by considering the symbols

$$(2.5) \quad A(z) = \sum_{j \in \mathbb{Z}} \alpha_j z^j$$

and

$$(2.6) \quad B(z) = \sum_{j=-m+1}^{m-1} N_{2m}(j+m)z^j,$$

the fundamental interpolation conditions (2.4) can be formulated as

$$(2.7) \quad A(z)B(z) \equiv 1.$$

The importance of this formulation is that the finite Laurent series $B(z)$ can be expressed as

$$(2.8) \quad B(z) = \frac{1}{(2m-1)!} z^{-m+1} \Pi(z),$$

where $\Pi = \Pi_{2m-1}$ is the well-known Euler-Frobenius polynomial of degree $2m-2$ (cf. Schoenberg [8]). We will make use of this important information later.

Our basic wavelet $\psi = \psi_m$ relative to $\phi = N_m$ is given by

$$(2.9) \quad \psi(x) = L_{2m}^{(m)}(2x-1).$$

Clearly, for $m=1$, when $\phi = \chi_{[0,1]}$, the corresponding basic wavelet $\psi = \psi_1$ is simply the *Haar wavelet*. In general, we must verify the properties (v)–(vii).

In the first place, W_0 is orthogonal to V_0 , so that

$$(2.10) \quad W_k \perp V_k, \quad \text{all } k \in \mathbb{Z},$$

is trivial. Indeed, for all ℓ_1 and $\ell_2 \in \mathbb{Z}$, it follows from (2.1) and (2.4) that, with $j = \ell_2 - \ell_1$,

$$\begin{aligned} \int_{-\infty}^{\infty} \psi(x-\ell_1) \bar{\phi}(x-\ell_2) dx &= \int_{-\infty}^{\infty} \psi(x) \phi(x-j) dx \\ &= \int_{-\infty}^{\infty} L_{2m}^{(m)}(2x-1) N_m(x-j) dx \\ &= \frac{(-1)^m}{2^m} \int_{-\infty}^{\infty} L_{2m}(2x-1) N_m^{(m)}(x-j) dx \\ &= \sum_{\nu=0}^m \frac{1}{2^m} (-1)^{m-\nu} \binom{m}{\nu} \int_{-\infty}^{\infty} L_{2m}(2x-1) \delta(x-j-\nu) dx \\ &= \sum_{\nu=0}^m \frac{1}{2^m} (-1)^{m-\nu} \binom{m}{\nu} L_{2m}(2j+2\nu-1) = 0. \end{aligned}$$

On the other hand, while the inclusion $V_0 \subset V_1$ is a simple consequence of the two-scale identity:

$$(2.11) \quad \phi(x) = \sum_{j=0}^m 2^{-m+1} \binom{m}{j} \phi(2x-j),$$

which can be described in terms of Fourier transforms by the identity:

$$(2.12) \quad \hat{\phi}(\omega) = \frac{1}{2} P(z) \hat{\phi}\left(\frac{\omega}{2}\right),$$

where

$$(2.13) \quad z := e^{-i\frac{\omega}{2}}$$

and

$$(2.14) \quad P(z) = \sum_{j=0}^m 2^{-m+1} \binom{m}{j} z^j = 2^{-m+1} (1+z)^m;$$

the inclusion $W_0 \subset V_1$, so that

$$(2.15) \quad W_k \subset V_{k+1}, \quad \text{all } k \in \mathbb{Z},$$

is a consequence of the following lemma.

Lemma 1. *Let z be as in (2.13). Then*

$$(2.16) \quad \hat{\psi}(\omega) = \frac{1}{2} Q(z) \hat{\phi}\left(\frac{\omega}{2}\right),$$

where

$$(2.17) \quad Q(z) := \frac{(1-z)^m}{z^{m-1}} A(z).$$

Here, $A(z)$ is the symbol of $\{\alpha_j\}$ defined in (2.5). As a trivial consequence of (2.10) and (2.15), we have the required orthogonality property (vi), namely:

$$W_k \perp W_j, \quad \text{all } k \neq j \text{ in } \mathbb{Z}.$$

So, to conclude that ψ is a basic wavelet relative to ϕ , it is sufficient to verify

$$(2.18) \quad V_1 = V_0 + W_0.$$

Indeed, if (2.18) holds, then (v) and (vii) follow from (2.10) and (i)–(iii). However, in order to give a computational algorithm for achieving the wavelet decomposition (1.1), we must give the explicit rule that describes the decomposition (2.18). That is, we need to determine the ℓ^2 sequences $\{a_n\}$ and $\{b_n\}$ that express each $\phi_{1\ell}$, $\ell \in \mathbb{Z}$, in terms of $\{\phi_{0n}\}$ and $\{\psi_{0n}\}$ as follows:

$$(2.19) \quad \phi(2x - \ell) = \sum_n a_{\ell-2n} \phi(x - n) + \sum_n b_{\ell-2n} \psi(x - n).$$

The main theorem in this paper is to determine these two sequences $\{a_n\}$ and $\{b_n\}$. To facilitate the statement of this theorem, it is more convenient to introduce two Laurent series:

$$(2.20) \quad \begin{aligned} G(z) &:= \frac{1}{(2m-1)! 2^m} \cdot \frac{(1+z)^m \Pi(z)}{z^{2m-1}} A(z^2) \\ &= \frac{1}{(2m-1)! 2^m} \cdot \frac{(1+z)^m \Pi(z)}{z^{2m-1}} \sum_{j \in \mathbb{Z}} \alpha_j z^{2j} \end{aligned}$$

and

$$(2.21) \quad \begin{aligned} H(z) &:= \frac{-1}{[(2m-1)!]^2 2^{2m-1}} \cdot \frac{(1-z)^m \Pi(z) \Pi(-z)}{z^{2m-1}} A(z^2) \\ &= \frac{-1}{[(2m-1)!]^2 2^{2m-1}} \cdot \frac{(1-z)^m \Pi(z) \Pi(-z)}{z^{2m-1}} \sum_{j \in \mathbb{Z}} \alpha_j z^{2j}. \end{aligned}$$

Recall that $\Pi(z)$ is the Euler-Frobenius polynomial of degree $2m - 2$, with coefficients:

$$(2m - 1)! N_{2m}(j), \quad j = 1, \dots, 2m - 1,$$

(cf. (2.8) and (2.6)). We have the following result.

Theorem 1. *Let $G(z)$ and $H(z)$ be defined as in (2.20) and (2.21). Then the sequences $\{a_n\}$ and $\{b_n\}$ that give the wavelet decomposition (2.19) are the coefficients of the following Laurent expansions of $G(z)$ and $H(z)$:*

$$(2.22) \quad G(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n}$$

and

$$(2.23) \quad H(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n}.$$

We remark that since $\Pi(z)$ is only a polynomial, it follows from (2.20)–(2.23) that the rate of exponential decay of the sequences $\{a_n\}$ and $\{b_n\}$ does not exceed that of $\{\alpha_{[n/2]}\}$. To give this rate of decay, let the roots of $\Pi = \Pi_{2m-1}$ be denoted by r_j , $j = 1, \dots, 2m - 2$, labeled in decreasing order; that is,

$$r_{2m-2} < r_{2m-3} < \dots < r_m < -1 < r_{m-1} < \dots < r_1 < 0,$$

where $r_1 r_{2m-2} = \dots = r_{m-1} r_m = 1$ (cf. Schoenberg [8, pp. 37–38]). Then we have

$$(2.24) \quad \alpha_n = O(|r_m|^{-|n|}), \quad n \rightarrow \pm\infty.$$

Hence, in view of the formulations (2.20)–(2.23), we have

$$(2.25) \quad a_n, b_n = O(|r_m|^{-|n|/2}), \quad n \rightarrow \pm\infty.$$

3. PROOFS OF RESULTS

First, we remark that Lemma 1 can be easily verified by using Fourier transforms and the identity

$$N_{2m}^{(m)}(x) = \sum_{j=0}^m (-1)^j \binom{m}{j} N_m(x - j),$$

(cf. [2]). To establish the theorem, we need the following result.

Lemma 2. *Let $\Pi = \Pi_{2m-1}$ be the Euler-Frobenius polynomials. Then*

$$(3.1) \quad \Pi(z)(1+z)^{2m} - \Pi(-z)(1-z)^{2m} = 2^{2m} z \Pi(z^2)$$

for all z .

To verify this lemma, we recall the identity

$$(3.2) \quad \frac{\Pi(z)}{(1-z)^{2m}} = \sum_{j=0}^{\infty} (j+1)^{2m-1} z^j$$

(cf. [8]). Hence, it follows that

$$\begin{aligned} \frac{\Pi(z)}{(1-z)^{2m}} - \frac{\Pi(-z)}{(1+z)^{2m}} &= \sum_{j=0}^{\infty} (j+1)^{2m-1} (z^j - (-1)^j z^j) \\ &= 2^{2m} z \sum_{k=0}^{\infty} (k+1)^{2m-1} z^{2k} = 2^{2m} z \frac{\Pi(z^2)}{(1-z^2)^{2m}} \end{aligned}$$

which is equivalent to (3.1). \square

To establish Theorem 1, we first note that (2.19) is equivalent to

$$\frac{1}{2} \hat{\phi} \left(\frac{\omega}{2} \right) e^{-i\ell\omega/2} = \sum_{n \in \mathbb{Z}} a_{\ell-2n} e^{-in\omega} \hat{\phi}(\omega) + \sum_{n \in \mathbb{Z}} b_{\ell-2n} e^{-in\omega} \hat{\psi}(\omega)$$

which, in view of the two-scale identities (2.12) and (2.16), is in turn equivalent to

$$(3.3) \quad P(z) \sum_{n \in \mathbb{Z}} a_{\ell-2n} z^{2n-\ell} + Q(z) \sum_{n \in \mathbb{Z}} b_{\ell-2n} z^{2n-\ell} = 1,$$

where P and Q are given in (2.14) and (2.17), and the notation (2.13) is used. By a change of indices, it is clear that (3.3), and hence (2.19), is equivalent to the set of two identities:

$$(3.4) \quad \begin{cases} P(z) \frac{G(z) + G(-z)}{2} + Q(z) \frac{H(z) + H(-z)}{2} = 1, \\ P(z) \frac{G(z) - G(-z)}{2} + Q(z) \frac{H(z) - H(-z)}{2} = 1. \end{cases}$$

Now, by applying Lemma 2, and recalling (2.7), (2.8), (2.20), and (2.21), it is straightforward to arrive at (3.4). \square

4. DECOMPOSITION AND RECONSTRUCTION ALGORITHMS

The sequences $\{a_n\}$ and $\{b_n\}$ which have at least the exponential decay rate (2.25) can now be used to yield the wavelet decomposition (1.1). Indeed, writing

$$(4.1) \quad \begin{cases} f_j := \sum_m c_m^j \phi_{jm}, & j = k - \ell, \dots, k, \\ g_j := \sum_m d_m^j \psi_{jm}, & j = k - \ell, \dots, k - 1, \end{cases}$$

we note that, as a consequence of the decomposition (2.19), the decomposition

$$(4.2) \quad f_j = f_{j-1} + g_{j-1}, \quad j = k - \ell, \dots, k,$$

is equivalent to the recursive relations:

$$(4.3) \quad \begin{cases} c_\ell^{j-1} = \sum_n a_{n-2\ell} c_n^j, & j = k - \ell + 1, \dots, k, \\ d_\ell^{j-1} = \sum_n b_{n-2\ell} c_n^j, & j = k - \ell + 1, \dots, k. \end{cases}$$

Hence, (4.3) can be used to arrive at the wavelet decomposition (1.1), namely,

$$(4.4) \quad f_k = g_{k-1} + \cdots + g_{k-\ell} + f_{k-\ell},$$

where $f_{k-\ell} \in V_{k-\ell}$ and $g_j \in W_j$, $j = k - \ell, \dots, k - 1$, are formulated in (4.1).

Similarly, the two-scale relations (2.11) and (2.16) can be used to reconstruct f_k from $f_{k-\ell}$ and $g_{k-\ell}, \dots, g_{k-1}$. (Of course, in applications, these wavelets $g_{k-1}, \dots, g_{k-\ell}$ have probably been processed and are necessarily different from those in (4.4).) For notational convenience, let us write

$$(4.5) \quad P(z) = \sum_{j=0}^m p_j z^j$$

(that is, $p_j = 2^{-m+1} \binom{m}{j}$) and

$$(4.6) \quad Q(z) = \sum_{j \in \mathbb{Z}} q_j z^j.$$

Then, in view of (2.11) and (2.16), the decomposition (4.2) is equivalent to

$$(4.7) \quad c_\ell^j = \sum_n p_{\ell-2n} c_n^{j-1} + \sum_n q_{\ell-2n} d_n^{j-1}, \quad j = k - \ell + 1, \dots, k.$$

Note that $\{p_n\}$ is a finite sequence, and in view of (2.17) and (2.24), we have

$$q_n = O(|r_m|^{-|n|}), \quad n \rightarrow \pm\infty.$$

For more details on algorithms of this type and their applications, the reader is referred to Mallat [6].

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