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A CARTAN TYPE IDENTITY FOR ISOPARAMETRIC HYPERSURFACES IN SYMMETRIC SPACES

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Abstract. In this paper, we obtain a Cartan type identity for curvature-adapted isoparametric hypersurfaces in symmetric spaces of compact type or non-compact type. This identity is a generalization of Cartan-D'Atri's identity for curvature-adapted (=amenable) isoparametric hypersurfaces in rank one symmetric spaces. Furthermore, by using the Cartan type identity, we show that certain kind of curvature-adapted isoparametric hypersurfaces in a symmetric space of non-compact type are principal orbits of Hermann actions.

1. Introduction. An isoparametric hypersurface in a (general) Riemannian manifold is a connected hypersurface whose sufficiently close parallel hypersurfaces are of constant mean curvature (see [12] for example). In this paper, we assume that all isoparametric hypersurfaces are complete. It is known that all isoparametric hypersurfaces in a symmetric space of compact type are equifocal in the sense of [37] and that, conversely all equifocal hypersurfaces are isoparametric (see [12]). Also, it is known that all isoparametric hypersurfaces in a symmetric space of non-compact type are complex equifocal in the sense of [18] and that, conversely, all curvature-adapted complex equifocal hypersurfaces are isoparametric (see [19, Theorem 15]), where the curvature-adaptedness implies that, for a unit normal vector v, the (normal) Jacobi operator $R(\cdot, v)v$ preserves the tangent space invariantly and commutes with the shape operator A for v, where R is the curvature tensor of the ambient space. It is known that principal orbits of a Hermann action (i.e., the action of a symmetric subgroup of G) of cohomogeneity one on a symmetric space G/K of compact type are curvature-adapted and equifocal (see ([11]). Hence they are isoparametric hypersurfaces. On the other hand, we [20, 23] showed that the principal orbits of a Hermann action (i.e., the action of a (not necessarily compact) symmetric subgroup of G) of cohomogeneity one on a symmetric space G/K of non-compact type are curvature-adapted and complex equifocal, and they have no focal point of non-Euclidean type on the ideal boundary of G/K. Hence they are isoparametric hypersurfaces.

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For an isoparametric hypersurface M in a real space form N of constant curvature c, it is known that the following Cartan's identity holds:

$$\sum_{\lambda \in \mathrm{Spec} A \setminus \{\lambda_0\}} \frac{c + \lambda\lambda_0}{\lambda - \lambda_0} \times m_\lambda = 0 \tag{1.1}$$

for any $\lambda_0 \in \mathrm{Spec} A$, where A is the shape operator of M and $\mathrm{Spec} A$ is the spectrum of A, m_λ is the multiplicity of λ. Here we note that all hypersurfaces in a real space form are curvature-adapted. In general cases, this identity is shown in algebraic method. Also, it is shown in geometrical method in the following three cases:

(i) $c = 0,\ \lambda_0 \neq 0,$

(ii) $c > 0,\ \lambda_0\ :$ any eigenvalue of A_v,

(iii) $c < 0,\ |\lambda_0| > \sqrt{-c}$.

In detail, it is shown by showing the minimality of the focal submanifold for λ_0 and using this fact.

Let $H \curvearrowright G/K$ be a cohomogeneity one action of a compact group H $(\subset G)$ on a rank one symmetric space G/K and M a principal orbit of this action. Since the H-action is of cohomogeneity one, it is hyperpolar. Hence M is an equifocal (hence isoparametric) hypersurface (see [13]). In 1979, D'Atri [8] obtained a Cartan type identity for M in the case where M is amenable (i.e., curvature-adapted). On the other hand, in 1989–1991, Berndt [1, 2] obtained a Cartan type identity (in algebraic method) for curvature-adapted hypersurfaces with constant principal curvature in rank one symmetric spaces other than spheres and hyperbolic spaces. Here we note that, for a curvature-adapted hypersurface in a rank one symmetric space of non-compact type, it has constant principal curvature if and only if it is isoparametric.

In this paper, we obtain the Cartan type identities for curvature-adapted isoparametric hypersurfaces in symmetric spaces and, furthermore, by using the Cartan type identity, we prove that certain kind of curvature-adapted isoparametric hypersurfaces in a symmetric space of non-compact type are principal orbits of Hermann actions. Let M be a hypersurface in a symmetric space $N = G/K$ of compact type or non-compact type and v a unit normal vector field of M. Set $R(v_x) := R(\cdot, v_x)v_x|_{T_xM}$, where R is the curvature tensor of N. For each $r \in \mathbb{R}$, we define a function τ_r over $[0, \infty)$ by

$$\tau_r(s) := \begin{cases} \dfrac{\sqrt{s}}{\tan(r\sqrt{s})} & (s > 0) \\ \dfrac{1}{r} & (s = 0)\,. \end{cases}$$

Also, for each $r \in \mathbb{C}$, we define a complex-valued function $\hat{\tau}_r$ over $(-\infty, 0]$ by

$$\hat{\tau}_r(s) := \begin{cases} \dfrac{\mathbf{i}\sqrt{-s}}{\tan(\mathbf{i}r\sqrt{-s})} & (s < 0) \\ \dfrac{1}{r} & (s = 0)\,, \end{cases}$$

where **i** is the imaginary unit. First we prove the following Cartan type identity for a curvature-adapted isoparametric hypersurface in a simply connected symmetric space of compact type.

THEOREM A. *Let M be a curvature-adapted isoparametric hypersurface in a simply connected symmetric space $N := G/K$ of compact type. For each focal radius r_0 of M, we have*

$$\sum_{(\lambda,\mu)\in S^x_{r_0}} \frac{\mu + \lambda\tau_{r_0}(\mu)}{\lambda - \tau_{r_0}(\mu)} \times m_{\lambda,\mu} = 0\,, \tag{1.2}$$

where $S^x_{r_0} := \{(\lambda,\mu) \in \mathrm{Spec} A_x \times \mathrm{Spec} R(v_x)\,;\ \mathrm{Ker}(A_x - \lambda I) \cap \mathrm{Ker}(R(v_x) - \mu I) \neq \{0\},\ \lambda \neq \tau_{r_0}(\mu)\}$ *and* $m_{\lambda,\mu} := \dim(\mathrm{Ker}(A_x - \lambda I) \cap \mathrm{Ker}(R(v_x) - \mu I))$.

REMARK 1.1. (i) If $\mathrm{Ker}(A_x - \lambda_0 I) \cap \mathrm{Ker}(R(v_x) - \mu_0 I)$ is included by the focal space for the focal radius r_0, then we have $\tau_{r_0}(\mu_0) = \lambda_0$.

(ii) If G/K is a sphere of constant curvature c, then $\mathrm{Spec} R(v_x) = \{c\}$ and $\tau_{r_0}(c)$ is equal to the principal curvature corresponding to r_0. Hence the identity (1.2) coincides with (1.1).

(iii) In the case where G/K is a rank one symmetric space of compact type, the identity (1.2) coincides with the identity obtained by D'Atri [8] (see [8, Theorems 3.7 and 3.9]).

(iv) In the case where G/K is a rank one symmetric space of compact type other than spheres, the identity (1.2) is different from the identity obtained by Berndt [1, 2].

Next, in this paper, we prove the following Cartan type identity for a curvature-adapted isoparametric C^ω-hypersurface in a symmetric space of non-compact type, where C^ω means the real analyticity.

THEOREM B. *Let M be a curvature-adapted isoparametric C^ω-hypersurface in a symmetric space $N := G/K$ of non-compact type. Assume that M has no focal point of non-Euclidean type on the ideal boundary $N(\infty)$ of N. Then M admits a complex focal radius and, for each complex focal radius r_0 of M, we have*

$$\sum_{(\lambda,\mu)\in S^x_{r_0}} \frac{\mu + \lambda\hat{\tau}_{r_0}(\mu)}{\lambda - \hat{\tau}_{r_0}(\mu)} \times m_{\lambda,\mu} = 0\,, \tag{1.3}$$

where $S^x_{r_0} := \{(\lambda,\mu) \in \mathrm{Spec} A_x \times \mathrm{Spec} R(v_x)\,;\ \mathrm{Ker}(A_x - \lambda I) \cap \mathrm{Ker}(R(v_x) - \mu I) \neq \{0\},\ \lambda \neq \hat{\tau}_{r_0}(\mu)\}$ *and* $m_{\lambda,\mu} := \dim(\mathrm{Ker}(A_x - \lambda I) \cap \mathrm{Ker}(R(v_x) - \mu I))$.

REMARK 1.2. (i) The notion of a complex focal radius was introduced in [18]. This quantity indicates the position of a focal point of the complexification $M^{\mathbb{C}}$ ($\subset G^{\mathbb{C}}/K^{\mathbb{C}}$) of a submanifold M in a symmetric space G/K of non-compact type (see [19]).

(ii) If $\mathrm{Ker}(A_x - \lambda_0 I) \cap \mathrm{Ker}(R(v_x) - \mu_0 I)$ is included by the focal space for the complex focal radius r_0, then we have $\hat{\tau}_{r_0}(\mu_0) = \lambda_0$.

(iii) If G/K is a hyperbolic space of constant curvature c, then $\mathrm{Spec} R(v_x) = \{c\}$ and $\hat{\tau}_{r_0}(c)$ is equal to the principal curvature corresponding to r_0. Hence the identity (1.3) coincides with (1.1).

(iv) In the case where G/K is a rank one symmetric space of non-compact type and r_0 is a real focal radius, the identity (1.3) coincides with the identity obtained by D'Atri [8] (see [8, Theorems 3.7 and 3.9]).

(v) In the case where G/K is a rank one symmetric space of non-compact type other than hyperbolic spaces, the identity (1.3) is different from the identity obtained by J. Berndt [1, 2].

(vi) For a curvature-adapted and isoparametric hypersurface M in G/K, the following conditions (a)–(c) are equivalent:

(a) M has no focal point of non-Euclidean type on $N(\infty)$,

(b) M is proper complex equifocal in the sense of [20],

(c) $\mathrm{Ker}(A_x \pm \sqrt{-\mu} I) \cap \mathrm{Ker}(R(v_x) - \mu I) = \{0\}$ holds for each $\mu \in \mathrm{Spec} R(v_x) \setminus \{0\}$.

(vii) Principal orbits of a Hermann type action of cohomogeneity one on G/K are curvature-adapted isoparametric C^ω-hypersurface having no focal point of non-Euclidean type on $N(\infty)$ (see [20, Theorem B] and the above (iii)).

The proof of Theorem B is performed by showing **the minimality of the focal submanifold** $F := \{\exp^\perp((\mathrm{Re}\, r_0)v_x + (\mathrm{Im}\, r_0)Jv_x)\ ;\ x \in M^{\mathbb{C}}\}$ of the complexification $M^{\mathbb{C}}$ of M (see Figure 1), where $\exp^\perp$ is the normal exponential map of the submanifold $M^{\mathbb{C}}$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$, J is the complex structure of $G^{\mathbb{C}}/K^{\mathbb{C}}$ and v is a unit normal vector field of M (in G/K). Here we note that $\exp^\perp((\mathrm{Re}\, r_0)v_x + (\mathrm{Im}\, r_0)Jv_x)$ is equal to the point $\gamma^{\mathbb{C}}_{v_x}(r_0)$ of the complexified geodesic $\gamma^{\mathbb{C}}_{v_x}$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$. In the case where G/K is of rank greater than one and M is not homogeneous, the proof of the minimality of F is performed by showing **the minimality of the lift** $\widetilde{F} := (\pi \circ \phi)^{-1}(F)$ **of** F **to the path space** $H^0([0,1], \mathfrak{g}^{\mathbb{C}})$, where ϕ is the parallel transport map for $G^{\mathbb{C}}$ (which is an anti-Kaehlerian submersion of $H^0([0,1], \mathfrak{g}^{\mathbb{C}})$ onto $G^{\mathbb{C}}$) and π is the natural projection of $G^{\mathbb{C}}$ onto $G^{\mathbb{C}}/K^{\mathbb{C}}$ (which also is an anti-Kaehlerian submersion). Here we note that the minimality of F is trivial in the case where M is homogeneous. By using Theorem B, we prove the following fact for the number of distinct principal curvatures

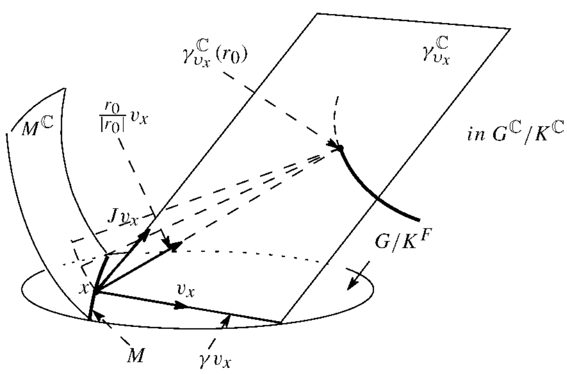


FIGURE 1.

of a curvature-adapted isoparametric C^ω-hypersurfaces in a symmetric sapce of non-compact type.

By using Theorem B, we prove the following main result.

THEOREM C. *Let M be a curvature-adapted isoparametric C^ω-hypersurface in a symmetric space N of non-compact type. Assume that M has no focal point of non-Euclidean type on $N(\infty)$. Then M is a principal orbit of a Hermann action.*

REMARK 1.3. In this theorem, are indispensable both the condition of the curvature-adaptedness and the condition for the non-existenceness of non-Euclidean type focal point on the ideal boundary. In fact, we have the following examples. Let G/K be an irreducible symmetric space of non-compact type such that the (restricted) root system of G/K is non-reduced. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ ($\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{k} = \mathrm{Lie}\, K$) be the Cartan decomposition associated with a symmetric pair (G, K) and $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$. Also, let $\triangle_+$ be the positive root system of G/K with respect to $\mathfrak{a}$ and Π the simple root system of $\triangle_+$, where we fix a lexicographic ordering of the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$. Set $\mathfrak{n} := \sum_{\lambda\in\triangle_+} \mathfrak{g}_\lambda$ and $N := \exp \mathfrak{n}$, where $\mathfrak{g}_\lambda$ is the root space for λ and exp is the exponential map of G. If G/K is of rank one, then any orbit of the N-action on G/K is a full irreducible curvature-adapted isoparametric C^ω-hypersurface but it has a focal point of non-Euclidean type on $N(\infty)$ (see [25]). On the other hand, it is a principal orbit of no Hermann action. Thus, in this theorem, is indispensable the condition for the non-existenceness of a focal point of non-Euclidean type on the ideal boundary. Let H_λ be the element of $\mathfrak{a}$ defined by $\langle H_\lambda, \bullet\rangle = \lambda(\bullet)$. Assume that the (restricted) root system of G/K is of type (BC_n). Take an element λ of Π such that 2λ belongs to $\triangle_+$, and one-dimensional subspaces l of $\mathbb{R}H_\lambda + \mathfrak{g}_\lambda$. Set $S := \exp((\mathfrak{a} + \mathfrak{n}) \ominus l)$, where exp is the exponential map of G and $(\mathfrak{a} + \mathfrak{n}) \ominus l$ is the orthogonal complement of l in $\mathfrak{a} + \mathfrak{n}$. Then S is a subgroup of $AN := \exp(\mathfrak{a} + \mathfrak{n})$ and any orbit of the S-action on G/K is a full irreducible isoparametric C^ω-hypersurface but it is not curvature-adapted (see [25]). Furthermore, we can find an orbit having no focal point of non-Euclidean type on $N(\infty)$ among orbits of the S-action. On the other hand, it is a principal orbit of no Hermann action. Thus the condition of the curvature-adaptedness is indispensable in this theorem.

In Section 2, we recall basic notions. In Section 3, we prove Theorem A. In Section 4, we define the mean curvature of a proper anti-Kaehlerian Fredholm submanifold and prepare a lemma to prove Theorem B. In Section 5, we prove Theorems B and C.

2. Basic notions. In this section, we recall basic notions which are used in the proof of Theorems A and B. First we recall the notion of an equifocal hypersurface in a symmetric space. Let M be a complete (oriented embedded) hypersurface in a symmetric space $N = G/K$ and fix a global unit normal vector field v of M. Let γ_{v_x} be the normal geodesic of M with $\gamma'_{v_x}(0) = v_x$, where $x \in M$ and $\gamma'_{v_x}(0)$ is the velocity vector of γ_{v_x} at 0. If $\gamma_{v_x}(s_0)$ is a focal point of M along γ_{v_x}, then s_0 is called a *focal radius of M at x*. Denote by $\mathcal{FR}_{M,x}$ the set of all focal radii of M at x. If M is compact and if $\mathcal{FR}_{M,x}$ is independent of the choice of x, then it is called an *equifocal hypersurface*. This notion is the hypersurface version of an equifocal submanifold defined in [37].

Next we recall the notion of a complex equifocal hypersurface in a symmetric space of non-compact type. Let M be a complete (oriented embedded) hypersurface in a symmetric space $N = G/K$ of non-compact type and fix a global unit normal vector field v of M. Let $\mathfrak{g}$ be the Lie algebra of G and θ be the Cartan involution of G with $\mathrm{Fix}\,\theta = K$, where $\mathrm{Fix}\,\theta$ is the fixed point group of θ. Denote by the same symbol θ the involution of $\mathfrak{g}$ induced from θ. Set $\mathfrak{p} := \mathrm{Ker}(\theta + \mathrm{id})$. The subspace $\mathfrak{p}$ is identified with the tangent space $T_{eK}N$ of N at eK, where e is the identity element of G. Let M be a complete (oriented embedded) hypersurface in N. Fix a global unit normal vector field v of M. Denote by A the shape operator of M (for v). Take $X \in T_xM$ $(x = gK)$. The M-Jacobi field Y along γ_x with $Y(0) = X$ (hence $Y'(0) = -A_xX$) is given by

$$Y(s) = (P_{\gamma_x|_{[0,s]}} \circ (D^{co}_{sv_x} - sD^{si}_{sv_x} \circ A_x))(X)\,,$$

where $P_{\gamma_x|_{[0,s]}}$ is the parallel translation along $\gamma_x|_{[0,s]}$, $D^{co}_{sv_x}$ (resp. $D^{si}_{sv_x}$) is given by

$$D^{co}_{sv_x} = g_* \circ \cos(\mathbf{i}\mathrm{ad}(sg_*^{-1}v_x)) \circ g_*^{-1}$$
$$\left(\text{resp. } D^{si}_{sv_x} = g_* \circ \frac{\sin(\mathbf{i}\mathrm{ad}(sg_*^{-1}v_x))}{\mathbf{i}\mathrm{ad}(sg_*^{-1}v_x)} \circ g_*^{-1}\right).$$

Here ad is the adjoint representation of the Lie algebra $\mathfrak{g}$ of G. All focal radii of M at x are catched as real numbers s_0 with $\mathrm{Ker}(D^{co}_{s_0v_x} - s_0D^{si}_{s_0v_x} \circ A_x) \neq \{0\}$. So, we [18] defined the notion of a *complex focal radius of M at x* as a complex number z_0 with $\mathrm{Ker}(D^{co}_{z_0v_x} - z_0D^{si}_{z_0v_x} \circ A^{\mathbb{C}}_x) \neq \{0\}$, where $D^{co}_{z_0v_x}$ (resp. $D^{si}_{z_0v_x}$) is a $\mathbb{C}$-linear transformation of $(T_xN)^{\mathbb{C}}$ defined by

$$D^{co}_{z_0v_x} = g^{\mathbb{C}}_* \circ \cos(\mathbf{i}\mathrm{ad}^{\mathbb{C}}(z_0g_*^{-1}v_x)) \circ (g^{\mathbb{C}}_*)^{-1}$$
$$\left(\text{resp. } D^{si}_{sv_x} = g^{\mathbb{C}}_* \circ \frac{\sin(\mathbf{i}\mathrm{ad}^{\mathbb{C}}(z_0g_*^{-1}v_x))}{\mathbf{i}\mathrm{ad}^{\mathbb{C}}(z_0g_*^{-1}v_x)} \circ (g^{\mathbb{C}}_*)^{-1}\right),$$

where $g^{\mathbb{C}}_*$ (resp. $\mathrm{ad}^{\mathbb{C}}$) is the complexification of g_* (resp. ad). Also, we call $\mathrm{Ker}(D^{co}_{z_0v_x} - z_0D^{si}_{z_0v_x} \circ A^{\mathbb{C}}_x)$ the *foccal space* of the complex focal radius z_0 and its complex dimension the *multiplicity* of the complex focal radius z_0, In [19], it was shown that, in the case where M is of class C^ω, complex focal radii of M at x indicate the positions of focal points of the extrinsic complexification $M^{\mathbb{C}} (\hookrightarrow G^{\mathbb{C}}/K^{\mathbb{C}})$ of M along the complexified geodesic $\gamma^{\mathbb{C}}_{v_x}$, where $G^{\mathbb{C}}/K^{\mathbb{C}}$ is the anti-Kaehlerian symmetric space associated with G/K. See [19] (also [26]) about the detail of the definition of the extrinsic complexification. Denote by $\mathcal{CFR}_x$ the set of all complex focal radii of M at x. If $\mathcal{CFR}_x$ is independent of the choice of x, then M is called a *complex equifocal hypersurface*. Here we note that we should call such a hypersurface an equi-complex focal hypersurface but, for simplicity, we call it a complex equifocal hypersurface. This notion is the hypersurface version of a complex equifocal submanifold defined in [18].

Next we recall the notion of an anti-Kaehlerian equifocal hypersurface in an anti-Kaehlerian symmetric space. Let J be a parallel complex structure on an even dimensional pseudo-Riemannian manifold $(M, \langle\ ,\ \rangle)$ of half index. If $\langle JX, JY\rangle = -\langle X, Y\rangle$ holds for every $X,\ Y \in TM$, then $(M, \langle\ ,\ \rangle, J)$ is called an *anti-Kaehlerian manifold*. Let $N = G/K$ be a symmetric space of non-compact type and $G^{\mathbb{C}}/K^{\mathbb{C}}$ the anti-Kaehlerian symmetric space associated with G/K. See [19] about the anti-Kaehlerian structure of $G^{\mathbb{C}}/K^{\mathbb{C}}$. Let f be an isometric immersion of an anti-Kaehlerian manifold $(M, \langle\ ,\ \rangle, J)$ into $G^{\mathbb{C}}/K^{\mathbb{C}}$. If $\widetilde{J} \circ f_* = f_* \circ J$, then M is called an *anti-Kaehlerian submanifold* immersed by f. Let A be the shape tensor of M. We have $A_{\widetilde{J}v}X = A_v(JX) = J(A_vX)$, where $X \in TM$ and $v \in T^{\perp}M$. If $A_vX = aX + bJX$ $(a, b \in \mathbb{R})$, then X is called a *J-eigenvector for $a + b$***i**. Let $\{e_i\}_{i=1}^n$ be an orthonormal system of T_xM such that $\{e_i\}_{i=1}^n \cup \{Je_i\}_{i=1}^n$ is an orthonormal base of T_xM. We call such an orthonormal system $\{e_i\}_{i=1}^n$ a *J-orthonormal base* of T_xM. If there exists a J-orthonormal base consisting of J-eigenvectors of A_v, then we say that A_v *is diagonalizable with respect to a J-orthonormal base*. Then we set $\mathrm{Tr}_J A_v := \sum_{i=1}^n \lambda_i$ as $A_ve_i = (\mathrm{Re}\,\lambda_i)e_i + (\mathrm{Im}\,\lambda_i)Je_i$ $(i = 1, \ldots, n)$. We call this quantity the J-trace of A_v. If, for each unit normal vector $v \in M$, the shape operator A_v is diagonalizable with respect to a J-orthonormal tangent base, if the normal Jacobi operator $R(v)$ preserves the tangent space T_xM (x : the base point of v) invariantly and if A_v and $R(v)$ commute, then we call M a *curvature-adapted anti-Kaehlerian submanifold*, where R is the curvature tensor of $G^{\mathbb{C}}/K^{\mathbb{C}}$. Assume that M is an anti-Kaehlerian hypersurface (i.e., $\mathrm{codim}\, M = 2$) and that it is orientable. Denote by $\exp^{\perp}$ the normal exponential map of M. Fix a global parallel orthonormal normal base $\{v, Jv\}$ of M. If $\exp^{\perp}(av_x + bJv_x)$ is a focal point of (M, x), then we call the complex number $a + b$**i** a *complex focal radius along the geodesic* γ_{v_x}. Assume that the number (which may be 0 and ∞) of distinct complex focal radii along the geodesic γ_{v_x} is independent of the choice of $x \in M$. Furthermore assume that the number is not equal to 0. Let $\{r_{i,x}\ ;\ i = 1, 2, \ldots\}$ be the set of all complex focal radii along γ_{v_x}, where $|r_{i,x}| < |r_{i+1,x}|$ or "$|r_{i,x}| = |r_{i+1,x}|$ & $\mathrm{Re}\, r_{i,x} > \mathrm{Re}\, r_{i+1,x}$" or "$|r_{i,x}| = |r_{i+1,x}|$ & $\mathrm{Re}\, r_{i,x} = \mathrm{Re}\, r_{i+1,x}$ & $\mathrm{Im}\, r_{i,x} = -\mathrm{Im}\, r_{i+1,x} < 0$". Let r_i $(i = 1, 2, \ldots)$ be complex-valued functions on M defined by assigning $r_{i,x}$ to each $x \in M$. We call this function r_i the *i-th complex focal radius function for* v. If the number of distinct complex focal radii along γ_{v_x} is independent of the choice of $x \in M$, complex focal radius functions for v are constant on M and they have constant multiplicity, then M is called an *anti-Kaehlerian equifocal hypersurface*. We ([19]) showed the following fact.

FACT 3. *Let M be a complete (embedded) C^{ω}-hypersurface in G/K. Then M is complex equifocal if and only if $M^{\mathbb{C}}$ is anti-Kaehler equifocal.*

Next we recall the notion of an anti-Kaehlerian isoparametric hypersurface in an infinite dimensional anti-Kaehlerian space. Let f be an isometric immersion of an anti-Kaehlerian Hilbert manifold $(M, \langle\ ,\ \rangle, J)$ into an infinite dimensional anti-Kaehlerian space $(V, \langle\ ,\ \rangle, \widetilde{J})$. See [19, Section 5] about the definitions of an anti-Kaehlerian Hilbert manifold and an infinite dimensional anti-Kaehlerian space. If $\widetilde{J} \circ f_* = f_* \circ J$ holds, then we call M an *anti-Kaehlerian Hilbert submanifold in* $(V, \langle\ ,\ \rangle, \widetilde{J})$ *immersed by* f. If M is of finite codimension and there exists an orthogonal time-space decomposition $V = V_- \oplus V_+$ such that $\widetilde{J}V_\pm = V_\mp$, $(V, \langle\ ,\ \rangle_{V_\pm})$ is a Hilbert space, the distance topology associated with $\langle\ ,\ \rangle_{V_\pm}$ coincides with the original topology of V and, for each $v \in T^\perp M$, the shape operator A_v is a compact operator with respect to $f^*\langle\ ,\ \rangle_{V_\pm}$, then we call M an *anti-Kaehlerian Fredholm submanifold* (rather than *anti-Kaehlerian Fredholm Hilbert submanifold*). Let $(M, \langle\ ,\ \rangle, J)$ be an orientable anti-Kaehlerian Fredholm hypersurface in an anti-Kaehlerian space $(V, \langle\ ,\ \rangle, \widetilde{J})$ and A be the shape tensor of $(M, \langle\ ,\ \rangle, J)$. Fix a global unit normal vector field v of M. If there exists $X(\neq 0) \in T_xM$ with $A_{v_x}X = aX + bJX$, then we call the complex number $a + b\mathbf{i}$ a *J-eigenvalue of A_{v_x}* (or a *complex principal curvature of M at x*) and call X a *J-eigenvector of A_{v_x} for $a + b\mathbf{i}$*. Here we note that this relation is rewritten as $A^{\mathbb{C}}_{v_x}X^{(1,0)} = (a + b\mathbf{i})X^{(1,0)}$, where $X^{(1,0)} := \frac{1}{2}(X - \mathbf{i}JX)$. Also, we call the space of all J-eigenvectors of A_{v_x} for $a + b\sqrt{-1}$ a *J-eigenspace of A_{v_x} for $a + b\mathbf{i}$*. We call the set of all J-eigenvalues of A_{v_x} the *J-spectrum of A_{v_x}* and denote it by $\mathrm{Spec}_J A_{v_x}$. $\mathrm{Spec}_J A_{v_x} \setminus \{0\}$ is described as follows:

$$\mathrm{Spec}_J A_{v_x} \setminus \{0\} = \{\lambda_i\ ;\ i = 1, 2, \dots\}$$

$$\left(\begin{array}{c} |\lambda_i| > |\lambda_{i+1}| \text{ or } ``|\lambda_i| = |\lambda_{i+1}| \ \&\ \mathrm{Re}\,\lambda_i > \mathrm{Re}\,\lambda_{i+1}" \\ \text{or } ``|\lambda_i| = |\lambda_{i+1}| \ \&\ \mathrm{Re}\,\lambda_i = \mathrm{Re}\,\lambda_{i+1} \ \&\ \mathrm{Im}\,\lambda_i = -\mathrm{Im}\,\lambda_{i+1} > 0" \end{array}\right).$$

Also, the J-eigenspace for each J-eigenvalue of A_{v_x} other than 0 is of finite dimension. We call the J-eigenvalue λ_i the *i-th complex principal curvature of M at x*. Assume that the number (which may be ∞) of distinct complex principal curvatures of M is constant over M. Then we can define functions $\widetilde{\lambda}_i$ $(i = 1, 2, \dots)$ on M by assigning the i-th complex principal curvature of M at x to each $x \in M$. We call this function $\widetilde{\lambda}_i$ the *i-th complex principal curvature function* of M. If the number of distinct complex principal curvatures of M is constant over M, each complex principal curvature function is constant over M and it has constant multiplicity, then we call M an *anti-Kaehler isoparametric hypersurface*. Let $\{e_i\}_{i=1}^\infty$ be an orthonormal system of $(T_xM, \langle\ ,\ \rangle_x)$. If $\{e_i\}_{i=1}^\infty \cup \{Je_i\}_{i=1}^\infty$ is an orthonormal base of T_xM, then we call $\{e_i\}_{i=1}^\infty$ a *J-orthonormal base*. If there exists a J-orthonormal base consisting of J-eigenvectors of A_{v_x}, then A_{v_x} is said to *be diagonalized with respect to the J-orthonormal base*. If M is anti-Kaehlerian isoparametric and, for each $x \in M$, the shape operator A_{v_x} is diagonalized with respect to a J-orthonormal base, then we call M a *proper anti-Kaehlerian isoparametric hypersurface*.

In [18], we defined the notion of the parallel transport map for a semi-simple Lie group G as a pseudo-Riemannian submersion of a pseudo-Hilbert space $H^0([0,1], \mathfrak{g})$ onto G. See [18] in detail. Also, in [19], we defined the notion of the parallel transport map for the complexification $G^{\mathbb{C}}$ of a semi-simple Lie group G as an anti-Kaehlerian submersion of an infinite dimensional anti-Kaehlerian space $H^0([0,1], \mathfrak{g}^{\mathbb{C}})$ onto $G^{\mathbb{C}}$. See [19] in detail. Let G/K be a symmetric space of non-compact type and $\phi : H^0([0,1], \mathfrak{g}^{\mathbb{C}}) \to G^{\mathbb{C}}$ the parallel transport map for $G^{\mathbb{C}}$ and $\pi : G^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}$ the natural projection. We [19] showed the following fact.

FACT. 4. *Let M be a complete anti-Kaehlerian hypersurface in an anti-Kaehlerian symmetric space $G^{\mathbb{C}}/K^{\mathbb{C}}$. Then M is anti-Kaehlerian equifocal if and only if each component of $(\pi\circ\phi)^{-1}(M)$ is anti-Kaehlerian isoparametric.*

Next we recall the notion of a focal point of non-Euclidean type on the ideal boundary $N(\infty)$ of a hypersurface M in a Hadamard manifold N which was introduced in [23] for a submanifold of general codimension. Assume that M is orientable. Let v be a unit normal vector field of M and $\gamma_{v_x}:[0,\infty)\to N$ the normal geodesic of M of direction v_x. If there exists an M-Jacobi field Y along γ_{v_x} satisfying $\lim_{t\to\infty}||Y(t)||/t=0$, then we call $\gamma_{v_x}(\infty)$ $(\in N(\infty))$ a *focal point* of M *on the ideal boundary* $N(\infty)$ *along* γ_{v_x}, where $\gamma_{v_x}(\infty)$ is the asymptotic class of γ_{v_x}. Also, if there exists an M-Jacobi field Y along γ_{v_x} satisfying $\lim_{t\to\infty}||Y(t)||/t=0$ and $\mathrm{Sec}(v_x,Y(0))\neq 0$, then we call $\gamma_{v_x}(\infty)$ a *focal point of non-Euclidean type of M on $N(\infty)$ along* γ_{v_x}, where $\mathrm{Sec}(v_x,Y(0))$ is the sectional curvature for the 2-plane spanned by v_x and $Y(0)$. If, for any point x of M, $\gamma_{v_x}(\infty)$ and $\gamma_{-v_x}(\infty)$ are not a focal point of non-Euclidean type of M on $N(\infty)$, then we say that M *has no focal point of non-Euclidean type on the ideal boundary* $N(\infty)$. According to [19, Theorem 1] and [23, Theorem A], we have the following fact.

FACT 5. *Let M be a curvature-adapted and isoparametric C^ω-hypersurface in a symmetric space $N:=G/K$ of non-compact type. Then the following conditions* (i) *and* (ii) *are equivalent:*

(i) *M has no focal point of non-Euclidean type on the ideal boundary $N(\infty)$.*
(ii) *Each component of $(\pi\circ\phi)^{-1}(M^{\mathbb{C}})$ is proper anti-Kaehlerian isoparametric.*

3. Proof of Theorem A. In this section, we shall prove Theorem A. Let M be a curvature-adapted isoparametric hypersurface in a simply connected symmetric space G/K of compact type, v a unit normal vector field of M and $C(\subset T^\perp_xM)$ the Coxeter domain (i.e., the fundamental domain (containing 0) of the Coxeter group of M at x). The boundary ∂C of C consists of two points and it is described as $\partial C=\{r_1v_x,r_2v_x\}$ $(r_2<0<r_1)$. We may assume that $|r_1|\leq|r_2|$ by replacing v with $-v$ if necessary. Note that the set $\mathcal{FR}_M$ of all focal radii of M is equal to $\{kr_1+(1-k)r_2\,;\,k\in\mathbb{Z}\}$. Set $F_i:=\{\gamma_{v_x}(r_i)\,;\,x\in M\}$ $(i=1,2)$, which are all of focal submanifolds of M. The hypersurface M is the r_i-tube over F_i $(i=1,2)$. Let π be the natural projection of G onto G/K and ϕ the parallel transport map for G. Let $\widetilde{M}$ be a component of $(\pi\circ\phi)^{-1}(M)$, which is an isoparametric hypersurface in $H^0([0,1],\mathfrak{g})$. The set $\mathcal{PC}_{\widetilde{M}}$ of all principal curvatures other than zero of $\widetilde{M}$ is equal to $\left\{\frac{1}{kr_1+(1-k)r_2}\,;\,k\in\mathbb{Z}\right\}$. Set $\lambda_{2k-1}:=\frac{1}{kr_1+(1-k)r_2}$ $(k=1,2,\dots)$ and $\lambda_{2k}:=\frac{1}{-(k-1)r_1+kr_2}$ $(k=1,2,\dots)$. Then we have $|\lambda_{i+1}|<|\lambda_i|$ or $\lambda_i=-\lambda_{i+1}>0$ for any $i\in\mathbb{N}$. Denote by m_i the multiplicity of λ_i. Denote by A (resp. $\widetilde{A}$) the shape operator of M for v (resp. $\widetilde{M}$ for v^L), where v^L is the horizontal lift of v to $\widetilde{M}$ with respect to $\pi\circ\phi$. Fix $r_0\in\mathcal{FR}_M$. The focal map $f_{r_0}:M\to G/K$ is defined by $f_{r_0}(x):=\gamma_{v_x}(r_0)$ $(x\in M)$. Let $F:=f_{r_0}(M)$, which is either F_1 or F_2. Denote by A^F the shape tensor of F and ψ_t the geodesic flow of G/K.

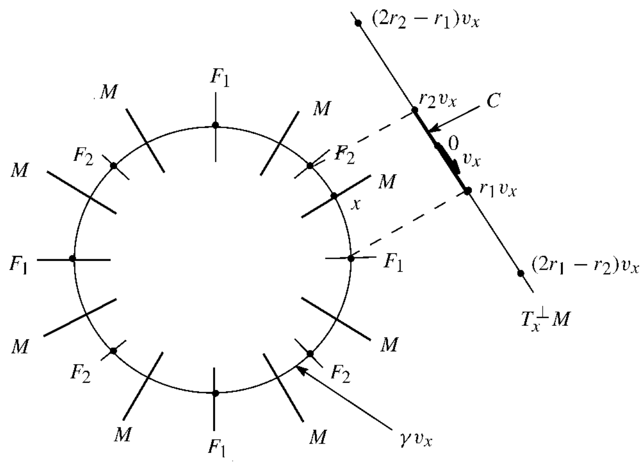


FIGURE 2.

PROOF OF THEOREM A. Define a set S_x by

$$S_x := \{(\lambda, \mu) \in \operatorname{Spec} A_x \times \operatorname{Spec} R(v_x)\,;\ \operatorname{Ker}(A_x - \lambda I) \cap \operatorname{Ker}(R(v_x) - \mu I) \neq \{0\}\}\,.$$

Since M is curvature adapted, we have

$$T_x M = \bigoplus_{(\lambda,\mu)\in S_x} (\operatorname{Ker}(A_x - \lambda I) \cap \operatorname{Ker}(R(v_x) - \mu I))\,.$$

Define a distribution D on M by $D_x := \bigoplus_{(\lambda,\mu)\in S^x_{r_0}} (\operatorname{Ker}(A_x - \lambda I) \cap \operatorname{Ker}(R(v_x) - \mu I))$ and $D^\perp$ the orthogonal complementary distribution of D in TM. Let $X \in \operatorname{Ker}(A_x - \lambda I) \cap \operatorname{Ker}(R(v_x) - \mu I)$ ($(\lambda, \mu) \in S^x_{r_0}$) and Y be the Jacobi field along $\gamma_{r_0 v_x}$ with $Y(0) = X$ and $Y'(0) = -A_{r_0 v_x} X$ $(= -r_0 \lambda X)$. This Jacobi field Y is described as

$$Y(s) = \left(\cos(s r_0 \sqrt{\mu}) - \frac{\lambda \sin(s r_0 \sqrt{\mu})}{\sqrt{\mu}}\right) P_{\gamma_{r_0 v}|_{[0,s]}}(X)\,.$$

Since $Y(1) = f_{r_0 *} X$, we have

$$f_{r_0 *} X = \left(\cos(r_0 \sqrt{\mu}) - \frac{\lambda \sin(r_0 \sqrt{\mu})}{\sqrt{\mu}}\right) P_{\gamma_{r_0 v_x}}(X)\,, \tag{3.1}$$

which is not equal to 0 because $(\lambda, \mu) \in S^x_{r_0}$. From this relation, we have $T_{f_{r_0}(x)} F = P_{\gamma_{r_0 v_x}}(D)$. On the other hand, we have

$$\begin{aligned} \widetilde{\nabla}_{f_{r_0 *} X} \psi_{r_0}(v_x) &= \frac{1}{r_0} Y'(1) \\ &= -\left(\sqrt{\mu} \sin(r_0 \sqrt{\mu}) + \lambda \cos(r_0 \sqrt{\mu})\right) P_{\gamma_{r_0 v_x}}(X)\,. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we have

$$A^F_{\psi_{r_0}(v_x)} f_{r_0 *} X = -\frac{\mu + \lambda \tau_{r_0}(\mu)}{\lambda - \tau_{r_0}(\mu)} f_{r_0 *} X\,.$$

Hence we can derive the following relation:

$$\mathrm{Tr}\, A^F_{\psi_{r_0}(v_x)} = -\sum_{(\lambda,\mu)\in S^x_{r_0}} \frac{\mu + \lambda\tau_{r_0}(\mu)}{\lambda - \tau_{r_0}(\mu)} \times m_{\lambda,\mu}\,, \tag{3.3}$$

where $S^x_{r_0}$ and $m_{\lambda,\mu}$ are as in the statement of Theorem A. On the other hand, it is not difficult to show the existence of a transnormal function on G/K having M and F as a regular level and a singular level, respectively. Hence, according to [28, Theorem 1.3], F is austere and hence minimal. Therefore, we obtain the desired identity from (3.3). □

4. The mean curvature of a proper anti-Kaehlerian Fredholm submanifold. In this section, we define the notion of a proper anti-Kaehlerian Fredholm submanifold and its mean curvature vector. Let M be an anti-Kaehlerian Fredholm submanifold in an infinite dimensional anti-Kaehlerian space V and A be the shape tensor of M. Denote by the same symbol J the complex structures of M and V. If A_v is diagonalized with respect to a J-orthonormal base for each unit normal vector v of M, then we call M a *proper anti-Kaehlerian Fredholm submanifold*. Assume that M is such a submanifold. Let v be a unit normal vector of M. If the series $\sum_{i=1}^{\infty} m_i\lambda_i$ exists, then we call it the *J-trace of A_v* and denote it by $\mathrm{Tr}_J A_v$, where $\{\lambda_i\,;\, i = 1, 2, \dots\} = \mathrm{Spec}_J A_v \setminus \{0\}$ (λ_i's are ordered as stated in Section 2) and $m_i = \frac{1}{2}\mathrm{dimKer}(A_v - \lambda_i I)$ $(i = 1, 2, \dots)$, where $\lambda_i I$ means $(\mathrm{Re}\,\lambda_i)I + (\mathrm{Im}\,\lambda_i)J$. Note that, if $\sharp(\mathrm{Spec}_J A_v)$ is finite, then we promise $\lambda_i = 0$ and $m_i = 0$ $(i > \sharp(\mathrm{Spec}_J A_v \setminus \{0\}))$, where $\sharp(\cdot)$ is the cardinal number of $(\cdot)$. Define a normal vector field H of M by $\langle H_x, v\rangle = \mathrm{Tr}_J A_v$ $(x \in M,\ v \in T^{\perp}_x M)$. We call H the *mean curvature vector of M*.

Let G/K be a symmetric space of non-compact type and $\phi : H^0([0,1], \mathfrak{g}^{\mathbb{C}}) \to G^{\mathbb{C}}$ be the parallel transport map for the complexification $G^{\mathbb{C}}$ of G and π be the natural projection of $G^{\mathbb{C}}$ onto the anti-Kaehlerian symmetric space $G^{\mathbb{C}}/K^{\mathbb{C}}$. We have the following fact, which will be used in the proof of Theorem B in the next section.

LEMMA 4.1. *Let M be a curvature-adapted anti-Kaehlerian submanifold in $G^{\mathbb{C}}/K^{\mathbb{C}}$ and A (resp. $\widetilde{A}$) be the shape tensor of M (resp. $(\pi\circ\phi)^{-1}(M)$). Assume that, for each unit normal vector v of M and each J-eigenvalue μ of $R(v)$, $\mathrm{Ker}(A_v - \sqrt{-\mu}I) \cap \mathrm{Ker}(R(v) - \mu I) = \{0\}$ holds. Then the following statements* (i) *and* (ii) *hold:*

(i) *$(\pi\circ\phi)^{-1}(M)$ is a proper anti-Kaehlerian Fredholm submanifold.*

(ii) *For each unit normal vector v of M, $\mathrm{Tr}_J \widetilde{A}_{v^L} = \mathrm{Tr}_J A_v$ holds, where v^L is the horizontal lift of v to $(\pi\circ\phi)^{-1}(M)$ and $\mathrm{Tr}_J A_v$ is the J-trace of A_v.*

PROOF. We can show the statement (i) in terms of [19, Lemmas 9, 12 and 13]. By imitating the proof of [18, Theorem C], we can show the statement (ii), where we also use the above lemmas in [19]. □

5. Proofs of Theorems B and C. In this section, we first prove Theorem B. Let M be a curvature-adapted isoparametric C^{ω}-hypersurface in a symmetric space G/K of non-compact type. Assume that M admits no focal point of non-Euclidean type on the ideal boundary of G/K. Denote by A the shape tensor of M and R the curvature tensor of G/K.

Let v be a unit normal vector field of M, which is uniquely extended to a unit normal vector field of the extrinsic complexification $M^{\mathbb{C}}(\subset G^{\mathbb{C}}/K^{\mathbb{C}})$ of M. Since M is a curvature-adapted isoparametric hypersurface admitting no focal point of non-Euclidean type on the ideal boundary $N(\infty)$, it admits a complex focal radius. Let r_0 be one of complex focal radii of M. The focal map $f_{r_0} : M^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}$ for r_0 is defined by $f_{r_0}(x) := \exp^{\perp}(r_0 v_x)(= \gamma^{\mathbb{C}}_{v_x}(r_0))$ $(x \in M^{\mathbb{C}})$, where $r_0 v_x$ means $(\mathrm{Re} r_0)v_x + (\mathrm{Im} r_0)Jv_x$ (J : the complex structure of $G^{\mathbb{C}}/K^{\mathbb{C}}$). Let $F := f_{r_0}(M^{\mathbb{C}})$, which is an anti-Kaehlerian submanifold in $G^{\mathbb{C}}/K^{\mathbb{C}}$ (see Figure 1). Without loss of generality, we may assume $o := eK \in M$. Denote by $\widehat{A}$ and A^F the shape tensor of $M^{\mathbb{C}}$ and F, respectively. Let ψ_t be the geodesic flow of $G^{\mathbb{C}}/K^{\mathbb{C}}$. Then we have the following fact.

LEMMA 5.1. *For any* $x \in M\ (\subset M^{\mathbb{C}})$, *the following relation holds:*

$$\mathrm{Tr}_J A^F_{\psi_{|r_0|}\left(\frac{r_0}{|r_0|}v_x\right)} = -\frac{r_0}{|r_0|}\sum_{(\lambda,\mu)\in S^x_{r_0}} \frac{\mu + \lambda\hat{\tau}_{r_0}(\mu)}{\lambda - \hat{\tau}_{r_0}(\mu)} \times m_{\lambda,\mu}\,,$$

where $S^x_{r_0}$ *and* $m_{\lambda,\mu}$ *are as in the statement of Theorem B.*

PROOF. Let $S_x := \{(\lambda,\mu) \in \mathrm{Spec}A_{v_x} \times \mathrm{Spec}R(v_x)\,;\ \mathrm{Ker}(A_{v_x} - \lambda I) \cap \mathrm{Ker}(R(v_x) - \mu I) \neq \{0\}\}$. Since M is curvature adapted, we have $T_xM = \bigoplus_{(\lambda,\mu)\in S_x}(\mathrm{Ker}(A_x - \lambda I) \cap \mathrm{Ker}\,(R(v_x) - \mu I))$. Set $D_x := \bigoplus_{(\lambda,\mu)\in S^x_{r_0}}(\mathrm{Ker}(A_x - \lambda I) \cap \mathrm{Ker}(R(v_x) - \mu I))$ and $D^{\perp}_x$ the orthogonal complement of D_x in T_xM. The tangent space $T_x(M^{\mathbb{C}})$ is identified with the complexification $(T_xM)^{\mathbb{C}}$. Under this identification, the shape operator $\widehat{A}_{v_x}$ is identified with the complexification $A^{\mathbb{C}}_x$ of A_x. Let $X \in \mathrm{Ker}(A_x - \lambda I)^{\mathbb{C}} \cap \mathrm{Ker}(R(v_x) - \mu I)^{\mathbb{C}}$ $((\lambda,\mu) \in S^x_{r_0})$ and Y be the Jacobi field along $\gamma_{r_0 v_x}$ with $Y(0) = X$ and $Y'(0) = -\hat{A}_{r_0 v_x}X\ (= -r_0\lambda X = -\lambda\,((\mathrm{Re}r_0)X + (\mathrm{Im}r_0)JX))$, where $\gamma_{r_0 v_x}$ is the geodesic in $G^{\mathbb{C}}/K^{\mathbb{C}}$ with $\dot{\gamma}_{r_0 v_x}(0) = r_0 v_x (= (\mathrm{Re}r_0)v_x + (\mathrm{Im}r_0)Jv_x)$. This Jacobi field Y is described as

$$Y(s) = \left(\cos(\mathbf{i}sr_0\sqrt{-\mu}) - \frac{\lambda\sin(\mathbf{i}sr_0\sqrt{-\mu})}{\mathbf{i}\sqrt{-\mu}}\right)P_{\gamma_{r_0 v_x}|_{[0,s]}}(X)\,.$$

Since $Y(1) = f_{r_0 *}X$, we have

$$f_{r_0 *}X = \left(\cos(\mathbf{i}r_0\sqrt{-\mu}) - \frac{\lambda\sin(\mathbf{i}r_0\sqrt{-\mu})}{\mathbf{i}\sqrt{-\mu}}\right)P_{\gamma_{r_0 v_x}}(X) \tag{5.1}$$

which is not equal to 0 because $(\lambda,\mu) \in S^x_{r_0}$. This relation implies that $T_{f_{r_0}(x)}F = P_{\gamma_{r_0 v_x}}(D^{\mathbb{C}}_x)$. On the other hand, we have

$$\begin{aligned}\widetilde{\nabla}_{f_{r_0 *}X}\psi_{|r_0|}\left(\frac{r_0}{|r_0|}v_x\right) &= \frac{1}{|r_0|}Y'(1)\\ &= -\frac{r_0}{|r_0|}\left(\mathbf{i}\sqrt{-\mu}\sin(\mathbf{i}r_0\sqrt{-\mu}) + \lambda\cos(\mathbf{i}r_0\sqrt{-\mu})\right)P_{\gamma_{r_0 v_x}}(X)\,.\end{aligned} \tag{5.2}$$

From (5.1) and (5.2), we have

$$A^F_{\psi_{|r_0|}\left(\frac{r_0}{|r_0|}v_x\right)}f_{r_0 *}X = \frac{-\frac{r_0}{|r_0|}\left(\mu + \lambda\hat{\tau}_{r_0}(\mu)\right)}{\lambda - \hat{\tau}_{r_0}(\mu)}f_{r_0 *}X\,. \tag{5.3}$$

The desired relation follows from this relation. □

Set $\kappa(\lambda,\mu) := \frac{-\frac{r_0}{|r_0|}(\mu+\lambda\hat{\tau}_{r_0}(\mu))}{\lambda-\hat{\tau}_{r_0}(\mu)}$ $((\lambda,\mu)\in S^x_{r_0})$. Next we prepare the following lemma.

LEMMA 5.2. *Let* $(\lambda_1,\mu_1)\in S^x_{r_0}$. *Then we have*

(i) $(\exp_{G^{\mathbb{C}}} r_0 v_x)_*^{-1}\psi_{|r_0|}\Big(\frac{r_0}{|r_0|}v_x\Big) = \frac{r_0}{|r_0|}v_x\,,$

where $\exp_{G^{\mathbb{C}}}$ *is the exponential map of* $G^{\mathbb{C}}$,

(ii) $$(\exp_{G^{\mathbb{C}}} r_0 v_x)_*^{-1}\Big(\mathrm{Ker}\Big(A^F_{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_x)} - \kappa(\lambda_1,\mu_1)I\Big)\Big) = \bigoplus_{(\lambda,\mu)\in S^x_{r_0}(\lambda_1,\mu_1)}\Big(\mathrm{Ker}(A_{v_x}-\lambda I)^{\mathbb{C}}\cap\mathrm{Ker}(R(v_x)-\mu I)^{\mathbb{C}}\Big),$$

where $S^x_{r_0}(\lambda_1,\mu_1) = \{(\lambda,\mu)\in S^x_{r_0}\,;\,\kappa(\lambda,\mu)=\kappa(\lambda_1,\mu_1)\}$,

(iii) *if* $\lambda_1 \neq \pm\sqrt{-\mu_1}$, *then* $\kappa(\lambda_1,\mu_1)\neq \pm\frac{r_0}{|r_0|}\sqrt{-\mu_1}$.

PROOF. The relation of (i) is trivial. Let $(\lambda,\mu)\in S^x_{r_0}(\lambda_1,\mu_1)$. The restriction $f_{r_0*}|_{\mathrm{Ker}(A_{v_x}-\lambda I)^{\mathbb{C}}\cap\mathrm{Ker}(R(v_x)-\mu I)^{\mathbb{C}}}$ of f_{r_0*} is equal to $P_{\gamma_{r_0 v_x}}|_{\mathrm{Ker}(A_{v_x}-\lambda I)^{\mathbb{C}}\cap\mathrm{Ker}(R(v_x)-\mu I)^{\mathbb{C}}}$ up to constant multiple by (5.1). Also, we have $P_{\gamma_{r_0 v_x}} = (\exp_{G^{\mathbb{C}}} r_0 v_x)_*$. These facts together with (5.3) deduce

$$\begin{aligned}(\exp_{G^{\mathbb{C}}} r_0 v_x)_* &\Big(\mathrm{Ker}(A_{v_x}-\lambda I)^{\mathbb{C}}\cap\mathrm{Ker}(R(v_x)-\mu I)^{\mathbb{C}}\Big)\\ &= f_{r_0*}\Big(\mathrm{Ker}(A_{v_x}-\lambda I)^{\mathbb{C}}\cap\mathrm{Ker}(R(v_x)-\mu I)^{\mathbb{C}}\Big)\\ &\subset \mathrm{Ker}\Big(A^F_{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_x)} - \kappa(\lambda_1,\mu_1)I\Big)\,.\end{aligned}$$

From this fact, the relation of (ii) follows. Now we shall show the statement (iii). Let $r_0 = a_0 + b_0\sqrt{-1}$ $(a_0, b_0\in\mathbb{R})$. Suppose that $\kappa(\lambda_1,\mu_1) = \pm\frac{r_0}{|r_0|}\sqrt{-\mu_1}$. By squaring both sides of this relation, we have

$$\big(\hat{\tau}_{r_0}(\mu_1)^2+\mu_1\big)\big(\lambda_1^2+\mu_1\big) = 0\,.$$

Hence we have $\lambda_1 = \pm\sqrt{-\mu_1}$. Thus the statement (iii) is shown. □

Denote by $\hat{R}$ the curvature tensor of $G^{\mathbb{C}}/K^{\mathbb{C}}$. By using these lemmas, we prove Theorem B. According to Lemma 5.1, we have only to show $\mathrm{Tr}_J A^F_{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_x)} = 0$ $(x\in M)$. In the case where M is homogeneous, we can show this relation by imitating the process of the proof of [15, Corollary 1.1].

SIMPLE PROOF OF THEOREM B IN RANK ONE CASE. We have only to show $\mathrm{Tr}_J A^F_{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_x)} = 0$. Assume that G/K is of rank one. Define a complex linear function $\Phi : T^{\perp}_{f_{r_0}(x)}F \to \mathbb{C}$ by $\Phi(w) = \mathrm{Tr}_J A^F_w$ $(w\in T^{\perp}_{f_{r_0}(x)}F)$. Since M is curvature-adapted, we have $T_xM = \bigoplus_{(\lambda,\mu)\in S_x}\big(\mathrm{Ker}(A_{v_x}-\lambda I)\cap\mathrm{Ker}(R(v_x)-\mu I)\big)$. Set

$$\hat{S}^y_{r_0} := \{(\lambda,\mu)\in(\mathrm{Spec}_J\hat{A}_{v_y})\times(\mathrm{Spec}_J\hat{R}(v_y))\,;\,\mathrm{Ker}(\hat{A}_{v_y}-\lambda I)\cap\mathrm{Ker}(\hat{R}(v_y)-\mu I)\neq\{0\}\ \&\ \lambda\neq\hat{f}_{r_0}(\mu)\}$$

($y \in M^{\mathbb{C}}$). Define a distribution $\hat{D}$ on $M^{\mathbb{C}}$ by

$$\hat{D}_y := \bigoplus_{(\lambda,\mu)\in\hat{S}^y_{r_0}} \left(\mathrm{Ker}(\hat{A}_{v_y} - \lambda I) \cap \mathrm{Ker}(\hat{R}(v_y) - \mu I)\right) \quad (y \in M^{\mathbb{C}})$$

and $\hat{D}^{\perp}$ the orthogonal complementary distribution of $\hat{D}$ in $T(M^{\mathbb{C}})$. Also, define a distribution D on M by $D_x := \bigoplus_{(\lambda,\mu)\in\hat{S}^x_{r_0}} (\mathrm{Ker}(A_x - \lambda I) \cap \mathrm{Ker}(R(v_x) - \mu I))$ ($x \in M$) and $D^{\perp}$ the orthogonal complementary distribution of D in TM. Under the identification of $T_x(M^{\mathbb{C}})$ with $(T_xM)^{\mathbb{C}}$, $\hat{D}_x$ is identified with the complexification $(D_x)^{\mathbb{C}}$ of D_x. The focal map f_{r_0} is a submersoin of $M^{\mathbb{C}}$ onto F and the fibres of f_{r_0} are integral manifolds of $\hat{D}^{\perp}$. Let L be the integral manifold of $\hat{D}^{\perp}$ through x and set $L_{\mathbb{R}} := L \cap M$. It is shown that L is the extrinsic complexification of $L_{\mathbb{R}}$. Set $Q := \{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_x)\,;\, x \in L\}$ and $Q_{\mathbb{R}} := \{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_x)\,;\, x \in L_{\mathbb{R}}\}$. It is shown that Q is the extrinsic complexification of $Q_{\mathbb{R}}$ and that Q is a complex hypersurface without geodesic point in $T^{\perp}_{f_{r_0}(x)}F$, that is, it is not contained in any complex affine hyperplane of $T^{\perp}_{f_{r_0}(x)}F$. According to Lemma 5.1, we have

$$\Phi\left(\psi_{|r_0|}\left(\frac{r_0}{|r_0|}v_y\right)\right) = -\frac{r_0}{|r_0|} \sum_{(\lambda,\mu)\in S^y_{r_0}} \frac{\mu + \lambda\hat{\tau}_{r_0}(\mu)}{\lambda - \hat{\tau}_{r_0}(\mu)} \times m_{\lambda,\mu}\,.$$

Let $(\widetilde{\lambda}, \widetilde{\mu})$ be a pair of continuous functions on $L_{\mathbb{R}}$ such that $(\widetilde{\lambda}(y), \widetilde{\mu}(y)) \in S^y_{r_0}$ for any $y \in L$. Since G/K is of rank one, $\widetilde{\mu}$ is constant on $L_{\mathbb{R}}$. The complex focal radius having $\mathrm{Ker}(A_y - \widetilde{\lambda}(y)\,I) \cap \mathrm{Ker}(R(v_y) - \widetilde{\mu}(y)\,I)$ as a part of the focal space is the complex number z_0 satisfying $\mathrm{Ker}(D^{co}_{z_0v_y} - z_0 D^{si}_{z_0v_y} \circ A^{\mathbb{C}}_y)|_{\mathrm{Ker}(A_y-\widetilde{\lambda}(y)\,I)\cap\mathrm{Ker}(R(v_y)-\widetilde{\mu}(y)\,I)} \neq \{0\}$, that is, it is equal to $(1/\sqrt{\widetilde{\mu}(y)})\arctan(\sqrt{\widetilde{\mu}(y)}/\widetilde{\lambda}(y))$, which is independent of the choice of $y \in L_{\mathbb{R}}$ by the isoparametricness (hence complex equifocality) of M. Hence $\widetilde{\lambda}$ is constant on $L_{\mathbb{R}}$. Therefore Φ is constant along $Q_{\mathbb{R}}$. Since Φ is of class C^{ω} and $Q_{\mathbb{R}}$ is a half-dimensional totally real submanifold in Q, Φ is constant along Q. Furthermore, this fact together with the linearity of Φ imply $\Phi \equiv 0$. In particular, we have $\mathrm{Tr}\, A^F_{\psi_{r_0}(v_x)} = 0$. □

PROOF OF THEOREM B (GENERAL CASE). According to Lemma 5.1, we have only to show $\mathrm{Tr}_J A^F_{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_{x_0})} = 0$ ($x_0 \in M$). We shall show this relation by investigating the focal submanifold of $(\pi\circ\phi)^{-1}(M^{\mathbb{C}})$ corresponding to r_0, where ϕ (: $H^0([0,1],\mathfrak{g}^{\mathbb{C}}) \to G^{\mathbb{C}}$) is the parallel transport map for $G^{\mathbb{C}}$ and π is the natural projection of $G^{\mathbb{C}}$ onto $G^{\mathbb{C}}/K^{\mathbb{C}}$. Let $\widetilde{M^{\mathbb{C}}}$ be the complete extension of $(\pi\circ\phi)^{-1}(M^{\mathbb{C}})$. Let v^L be the horizontal lift of v to $\widetilde{M}^{\mathbb{C}}$. Since $\pi\circ\phi$ is an anti-Kaehlerian submersion, the complex focal radii of $M^{\mathbb{C}}$ (hence M) are those of $\widetilde{M}^{\mathbb{C}}$. Let r_0 be a complex focal radius of M (hence $\widetilde{M}^{\mathbb{C}}$). The focal map $\widetilde{f}_{r_0}$ for r_0 is defined by $\widetilde{f}_{r_0}(x) = x + r_0 v^L_x$ ($x \in \widetilde{M}^{\mathbb{C}}$). Set $\widetilde{F} := \widetilde{f}_{r_0}(\widetilde{M}^{\mathbb{C}})$. Denote by $\widetilde{A}$ (resp. $A^{\widetilde{F}}$) the shape tensor of $\widetilde{M}^{\mathbb{C}}$ (resp. $\widetilde{F}$). Let $\mathrm{Spec}_J \widetilde{A}_{v^L_{\hat{0}}} \setminus \{0\} = \{\lambda_i\,;\, i = 1,2,\dots\}$ ("$|\lambda_i| > |\lambda_{i+1}|$" or "$|\lambda_i| = |\lambda_{i+1}|$ & $\mathrm{Re}\lambda_i > \mathrm{Re}\lambda_{i+1}$" or "$|\lambda_i| = |\lambda_{i+1}|$ & $\mathrm{Re}\lambda_i = \mathrm{Re}\lambda_{i+1}$ & $\mathrm{Im}\lambda_i = -\mathrm{Im}\lambda_{i+1} > 0$"). The set of all complex focal radii of $M^{\mathbb{C}}$ (hence M) is equal to $\{1/\lambda_i\,;\, i = 1,2,\dots\}$. We have $r_0 = 1/\lambda_{i_0}$ for some i_0. Define a distribution $\widetilde{D}_i$ ($i = 0,1,2,\dots$) on $\widetilde{M}^{\mathbb{C}}$ by

$(\widetilde{D}_0)_u := \mathrm{Ker}\widetilde{A}_{\widetilde{v}^L_u}$ and $(\widetilde{D}_i)_u := \mathrm{Ker}(\widetilde{A}_{\widetilde{v}^L_u} - \lambda_i I)$ $(i = 1, 2, \dots)$, where $u \in \widetilde{M}^{\mathbb{C}}$. Since M is a curvature-adapted isoparametric submanifold admitting no focal point of non-Euclidean type on $N(\infty)$, $\widetilde{M}^{\mathbb{C}}$ is proper anti-Kaehlerian isoparametric by Fact 5. Therefore, we have $T\widetilde{M}^{\mathbb{C}} = \overline{\widetilde{D}_0 \oplus \left(\bigoplus_i \widetilde{D}_i\right)}$ and $\mathrm{Spec}_J \widetilde{A}_{\widetilde{v}^L_u}$ is independent of the choice of $u \in \widetilde{M}^{\mathbb{C}}$. Take $u_0 \in \widetilde{M}^{\mathbb{C}}$ with $(\pi \circ \phi)(u_0) = x_0$. Let $X_i \in (\widetilde{D}_i)_{u_0}$ $(i \neq i_0)$ and $X_0 \in (\widetilde{D}_0)_{u_0}$. Then we have $\widetilde{f}_{r_0*}X_i = (1 - r_0\lambda_i)X_i$ and $\widetilde{f}_{r_0*}X_0 = X_0$. Hence we have $T_{\widetilde{f}_{r_0}(u_0)}\widetilde{F} = (\widetilde{D}_0)_{u_0} \oplus \left(\bigoplus_{i \neq i_0}(\widetilde{D}_i)_{u_0}\right)$ and $\mathrm{Ker}(\widetilde{f}_{r_0})_{*u_0} = (\widetilde{D}_{i_0})_{u_0}$, which implies that $\widetilde{D}_{i_0}$ is integrable. On the other hand, we have $A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})}\widetilde{f}_{r_0*}X_i = (\lambda_i r_0)/|r_0| X_i$ and $A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})}\widetilde{f}_{r_0*}X_0 = 0$, where $\widetilde{\psi}$ is the geodesic flow of $H^0([0,1], \mathfrak{g}^{\mathbb{C}})$. Therefore, we obtain $A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})}\widetilde{f}_{r_0*}X_i = \frac{\lambda_i|\lambda_{i_0}|}{\lambda_{i_0}-\lambda_i}\widetilde{f}_{r_0*}X_i$. Hence we have $\mathrm{Tr}_J A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})} = \sum_{i \neq i_0}\frac{\lambda_i|\lambda_{i_0}|}{\lambda_{i_0}-\lambda_i} \times m_i$, where $m_i := \frac{1}{2}\dim \widetilde{D}_i$. According to Theorem 2 of [19], each leaf of $\widetilde{D}_{i_0}$ is a complex sphere. Let L be the leaf of $\widetilde{D}_{i_0}$ through u_0 and u_0^* be the anti-podal point of u_0 in the complex sphere L. Similarly we can show $\mathrm{Tr}_J A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}(\widetilde{v}^L)_{u_0^*})} = \sum_{i \neq i_0}\frac{\lambda_i|\lambda_{i_0}|}{\lambda_{i_0}-\lambda_i} \times m_i$. Thus we have $\mathrm{Tr}_J A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})} = \mathrm{Tr}_J A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}(\widetilde{v}^L)_{u_0^*})}$. On the other hand, it follows from $\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}(\widetilde{v}^L)_{u_0^*}) = -\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})$ that $\mathrm{Tr}_J A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})} = -\mathrm{Tr}_J A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}(\widetilde{v}^L)_{u_0^*})}$. Hence we obtain

$$\mathrm{Tr}_J A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})} = 0\,. \tag{5.4}$$

It follows from (i) and (ii) of Lemma 5.2 that $F := f_{r_0}(M^{\mathbb{C}})$ is a curvature adapted anti-Kaehlerian submanifold. Also, it follows from (iv) of Remark 1.2, (5.3), (i) and (iii) of Lemma 5.2 that, for each unit normal vector w of F and each $\mu \in \mathrm{Spec}_J R(w) \setminus \{0\}$, $\mathrm{Ker}(A^F_w \pm \sqrt{-\mu}I) \cap \mathrm{Ker}(R(w) - \mu I) = \{0\}$ holds. Therefore, it follows from Lemma 4.1 that $\widetilde{F}$ is a proper anti-Kaehlerian Fredholm submanifold and, for each unit normal vector w of F, we have $\mathrm{Tr}_J A^{\widetilde{F}}_{w^L} = \mathrm{Tr}_J A^F_w$. It is clear that $\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})$ is the horizontal lift of $\psi_{|r_0|}(\frac{r_0}{|r_0|}v_{x_0})$ to $\widetilde{f}_{r_0}(u_0)$. Hence we have

$$\mathrm{Tr}_J A^F_{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_{x_0})} = \mathrm{Tr}_J A^{\widetilde{F}}_{\widetilde{\psi}_{|r_0|}(\frac{r_0}{|r_0|}v^L_{u_0})}\,. \tag{5.5}$$

From (5.4) and (5.5), we have $\mathrm{Tr}_J A^F_{\psi_{|r_0|}(\frac{r_0}{|r_0|}v_{x_0})} = 0$. This completes the proof. □

Now we prepare the following lemma to prove Theorem C.

LEMMA 5.3. *Let M be a curvature-adapted isoparametric C^ω-hypersurface in a symmetric space $N := G/K$ of non-compact type. Assume that M has no focal point of non-Euclidean type on $N(\infty)$. Then, for any complex focal radius r of M, we have*

$$\mathrm{Spec}\left(A_x|_{\mathrm{Ker}\,R(v_x)}\right) \subset \left\{\frac{1}{\mathrm{Re}\,r},\ 0\right\}$$

and

$$\mathrm{Spec}\left(A_x|_{\mathrm{Ker}(R(v_x)-\mu I)}\right)\subset\left\{\frac{\sqrt{-\mu}}{\tanh(\sqrt{-\mu}\mathrm{Re}\,r)},\ \sqrt{-\mu}\tanh(\sqrt{-\mu}\mathrm{Re}\,r)\right\}$$

for $\mu\in\mathrm{Spec}R(v_x)\setminus\{0\}$, *where* x *is an arbitrary point of* M.

PROOF. For simplicity, we set $D_\mu := \mathrm{Ker}(R(v_x)-\mu\,\mathrm{id})$ for each $\mu\in\mathrm{Spec}\,R(v_x)$. Let r_0 be the complex focal radius of M with $\mathrm{Re}r_0=\max_r\mathrm{Re}r$, where r runs over the set of all complex focal radii of M. Let $(\lambda,\mu)\in S^x_{r_0}\setminus\{(0,0)\}$ and r a complex focal radius including $\mathrm{Ker}(A_v-\lambda I)\cap D_\mu$ as the focal space, that is, $\lambda=\hat\tau_r(\mu)$ (see (ii) of Remark 1.2). Set $c_{\lambda,\mu}:=-\frac{\mu+\lambda\hat\tau_{r_0}(\mu)}{\lambda-\hat\tau_{r_0}(\mu)}$. We shall show $\mathrm{Re}\,c_{\lambda,\mu}\le 0$. The argument divides into the following three cases:

$$\text{(i) } \mu=0 \qquad \text{(ii) } 0<\sqrt{-\mu}<|\lambda| \qquad \text{(iii) } |\lambda|<\sqrt{-\mu}\,.$$

First we consider the case (i). Then we have $c_{\lambda,\mu}=\frac{\lambda}{1-\lambda r_0}$. Also, we can show $\lambda=1/r$. Hence we have

$$c_{\lambda,\mu}=\frac{1}{r-r_0}\,. \tag{5.6}$$

Furthermore, we have $\mathrm{Re}\,c_{\lambda,\mu}\le 0$ from the choice of r_0. Next we consider the case (ii). Since $\lambda=\hat\tau_r(\mu)$ and λ is a real number with $|\lambda|>\sqrt{-\mu}$, we can show $\lambda=\hat\tau_{\mathrm{Re}\,r}(\mu)(=\frac{\sqrt{-\mu}}{\tanh(\sqrt{-\mu}\mathrm{Re}\,r)})$ and $r\equiv\mathrm{Re}\,r\ (\mathrm{mod}\,(\pi\mathbf{i})/\sqrt{-\mu})$. Hence we have $c_{\lambda,\mu}=\hat\tau_{(r_0-\mathrm{Re}\,r)}(\mu)$, where we note that $\mathrm{Re}r\not\equiv r_0\ (\mathrm{mod}\,(\pi\mathbf{i})/\sqrt{-\mu})$ because $(\lambda,\mu)\in S^x_{r_0}$. Therefore, we obtain

$$\mathrm{Re}\,c_{\lambda,\mu}=\frac{\sqrt{-\mu}\left(1+\tan^2(\sqrt{-\mu}\mathrm{Im}r_0)\right)\tanh(\sqrt{-\mu}(\mathrm{Re}r-\mathrm{Re}r_0))}{\tanh^2(\sqrt{-\mu}(\mathrm{Re}r-\mathrm{Re}r_0))+\tan^2(\sqrt{-\mu}\mathrm{Im}r_0)}\le 0 \tag{5.7}$$

because $\mathrm{Re}r\le\mathrm{Re}r_0$. Next we consider the case (iii). Since $\lambda=\hat\tau_r(\mu)$ and λ is a real number with $|\lambda|<\sqrt{-\mu}$, we can show $\lambda=\hat\tau_{(\mathrm{Re}\,r+\frac{\pi\mathbf{i}}{2\sqrt{-\mu}})}(\mu)(=\sqrt{-\mu}\tanh(\sqrt{-\mu}\mathrm{Re}\,r))$ and $r\equiv\mathrm{Re}\,r+\frac{\pi\mathbf{i}}{2\sqrt{-\mu}}\ (\mathrm{mod}\,\frac{\pi\mathbf{i}}{\sqrt{-\mu}})$. Hence we have $c_{\lambda,\mu}=\hat\tau_{(r_0-\mathrm{Re}r+\frac{\pi\mathbf{i}}{2\sqrt{-\mu}})}(\mu)$. Therefore, we obtain

$$\mathrm{Re}c_{\lambda,\mu}=\frac{\sqrt{-\mu}\left(1+\tan^2(\sqrt{-\mu}\mathrm{Im}r_0)\right)\tanh(\sqrt{-\mu}(\mathrm{Re}r-\mathrm{Re}r_0))}{1+\tanh^2(\sqrt{-\mu}(\mathrm{Re}r-\mathrm{Re}r_0))\tan^2(\sqrt{-\mu}\mathrm{Im}r_0)}\le 0\,. \tag{5.8}$$

Thus $\mathrm{Re}c_{\lambda,\mu}\le 0$ is shown in general. Hence, from the identity in Theorem B, $\mathrm{Re}c_{\lambda,\mu}=0$ ($(\lambda,\mu)\in S^x_{r_0}$) follows, where we note that $c_{0,0}=0$. In case of (i), it follows from (5.6) that $\mathrm{Re}\left(\frac{1}{r-r_0}\right)=0$. Hence we have $\mathrm{Re}\,r=\mathrm{Re}\,r_0(<\infty)$ or $r=\infty$. If $\mathrm{Re}\,r=\mathrm{Re}\,r_0(<\infty)$, then we have $\lambda=1/r=1/\mathrm{Re}\,r_0=\hat\tau_{\mathrm{Re}\,r_0}(0)$ (which does not happen if r_0 is real because $(\lambda,0)\in S^x_{r_0}$). Also, if $r=\infty$, then we have $\lambda=0$. Thus we have

$$\mathrm{Spec}(A_x|_{D_0})\subset\left\{\frac{1}{\mathrm{Re}\,r_0},\ 0\right\}. \tag{5.9}$$

In case of (ii), it follows from (5.7) that $\mathrm{Re}r=\mathrm{Re}r_0$. Hence we have $\lambda=\hat\tau_{\mathrm{Re}\,r_0}(\mu)$ (which does not happen if $r_0\equiv\mathrm{Re}\,r_0\ (\mathrm{mod}\,(\pi\mathbf{i})/\sqrt{-\mu})$ because $(\lambda,\mu)\in S^x_{r_0}$). In case of (iii), it follows from (5.8) that $\mathrm{Re} r = \mathrm{Re} r_0$. Hence we have $\lambda = \hat{\tau}_{(\mathrm{Re}\, r_0 + \frac{\pi \mathbf{i}}{2\sqrt{-\mu}})}(\mu)$ (which does not happen if $r_0 \equiv \mathrm{Re}\, r_0 + \frac{\pi \mathbf{i}}{2\sqrt{-\mu}} \pmod{(\pi \mathbf{i})/\sqrt{-\mu}}$ because $(\lambda, \mu) \in S^x_{r_0}$). Hence we have

$$\mathrm{Spec}(A_x|_{D_\mu}) \subset \left\{ \frac{\sqrt{-\mu}}{\tanh(\sqrt{-\mu}\mathrm{Re} r_0)},\ \sqrt{-\mu} \tanh(\sqrt{-\mu}\mathrm{Re} r_0) \right\}. \tag{5.10}$$

This complete the proof. $\square$

Next we prove Theorem C in terms of this Lemma and its proof.

PROOF OF THEOREM C. According to the proof of Lemma 5.3, the real parts of complex focal radii of M coincide with one another. Denote by s_0 this real part. Then, according to Lemma 5.3, we have

$$\mathrm{Spec}(A_x|_{D_0}) \subset \left\{ \frac{1}{s_0}, 0 \right\}$$

and

$$\mathrm{Spec}(A_x|_{D_\mu}) \subset \left\{ \frac{\sqrt{-\mu}}{\tanh(\sqrt{-\mu} s_0)},\ \sqrt{-\mu} \tanh(\sqrt{-\mu} s_0) \right\} \quad (\mu \in \mathrm{Spec}\, R(v_x) \setminus \{0\}).$$

Set $D_0^V := \mathrm{Ker}\left(A_x|_{D_0} - \frac{1}{s_0} \mathrm{id} \right)$, $D_0^H := \mathrm{Ker} A_x|_{D_0}$,

$$D_\mu^V := \mathrm{Ker}\left(A_x|_{D_\beta} - \frac{\sqrt{-\mu}}{\tanh(\sqrt{-\mu} s_0)} \mathrm{id} \right)$$

and

$$D_\mu^H := \mathrm{Ker}\left(A_x|_{D_\beta} - \sqrt{-\mu} \tanh(\sqrt{-\mu} s_0)\, \mathrm{id} \right).$$

According to (ii) of Remark 1.2, if $D_0^V \oplus \left(\bigoplus_{\mu \in \mathrm{Spec}\, R(v_x) \setminus \{0\}} D_\mu^V \right) \neq \{0\}$, then s_0 is a (real) focal radius of M whose focal space is equal to $D_0^V \oplus \left(\bigoplus_{\mu \in \mathrm{Spec}\, R(v_x) \setminus \{0\}} D_\mu^V \right) \neq \{0\}$. Let η_{sv} ($s \in \mathbb{R}$) be the end-point map for sv. Set $M_s := \eta_{sv}(M)$. Set $F := M_{s_0}$. If s_0 is a (real) focal radius of M, then F is the only focal submanifold of M, and if s_0 is not a (real) focal radius of M, then F is a parallel submanifold of M. Without loss of generality, we may assume that $eK \in F$. Define a unit normal vector field v^s of M_s ($0 \leq s < s_0$) by $v^s_{\eta_{sv}(x)} = \gamma'_{v_x}(s)$ ($x \in M$). Denote by A^s ($0 \leq s < s_0$) the shape operator of M_s (for v^s) and A^F the shape tensor of F. Set $(D_0^V)^s := (\eta_{sv})_*(D_0^V)$ ($0 \leq s < s_0$) and $(D_\mu^V)^s := (\eta_{sv})_*(D_\mu^V)$ ($0 \leq s < s_0$, $\mu \in \mathrm{Spec}\, R(v_x) \setminus \{0\}$). Also, set $(D_0^H)^s := (\eta_{sv})_*(D_0^H)$ ($s \in \mathbb{R}$) and $(D_\mu^H)^s := (\eta_{sv})_*(D_\mu^H)$ ($s \in \mathbb{R}$, $\mu \in \mathrm{Spec}\, R(v_x) \setminus \{0\}$). Easily we have

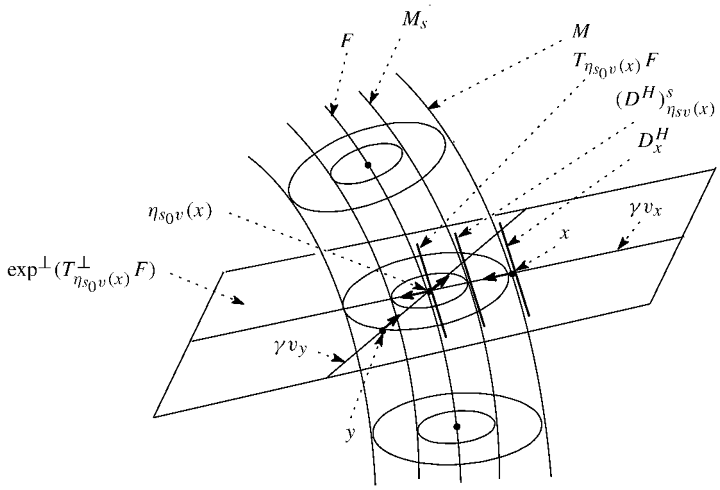
$$T_{\eta_{s_0 v}(x)} F = (D_0^H)^{s_0}_{\eta_{s_0 v}(x)} \oplus \left(\bigoplus_{\mu \in \mathrm{Spec}\, R(v_x) \setminus \{0\}} (D_\mu^H)^{s_0}_{\eta_{s_0 v}(x)} \right). \tag{5.11}$$

Also, we can show

$$A^s_{\eta_{sv}(x)}|_{(D_0^H)^s_{\eta_{sv}(x)}} = 0 \quad (0 \leq s < s_0)$$

and

$$A^s_{\eta_{sv}(x)}|_{(D_\beta^H)^s_{\eta_{sv}(x)}} = \mu \tanh(\sqrt{-\mu}(s_0 - s))\, \mathrm{id} \quad (0 \leq s < s_0).$$



$$D^H_x := (D^H_0)_x \oplus \Big(\bigoplus_{\beta \in \triangle_+|_{\mathbb{R} v_x}} (D^H_\beta)_x \Big)$$

$$(D^H)^s_{\eta_{sv}(x)} := (D^H_0)^s_{\eta_{sv}(x)} \oplus \Big(\bigoplus_{\beta \in \triangle_+|_{\mathbb{R} v_x}} (D^H_\beta)^s_{\eta_{sv}(x)} \Big)$$

FIGURE 3.

Hence we have

$$A^F_{\psi_{s_0}(v_x)}|_{(D^H_0)^{s_0}_{\eta_{s_0 v}(x)}} = 0$$

and

$$A^F_{\psi_{s_0}(v_x)}|_{(D^H_\beta)^{s_0}_{\eta_{s_0 v}(x)}} = \left(\lim_{s \to s_0 - 0} \sqrt{-\mu} \tanh(\sqrt{-\mu}(s_0 - s)) \right) \mathrm{id} = 0\,,$$

where ψ is the geodesic flow of G/K. From these relations and (5.11), we obtain $A^F_{\psi_{s_0}(v_x)} = 0$. Since this relation holds for any $x \in M$, F is totally geodesic. Denote by $\exp^\perp$ the normal exponential map for F. Since the real parts of complex focal radii of M coincide with one another, the normal umbrella $\exp^\perp(T^\perp_x F)$'s $(x \in F)$ do not intersect with one another. From this fact, an involutive diffeomorphism $\tau : G/K \to G/K$ having F as the fixed point set is well-defined by $\tau(\exp^\perp(w)) := \exp^\perp(-w)$ $(w \in T^\perp F)$. For each $s \in \mathbb{R} \setminus \{s_0\}$, the restriction $\tau|_{M_s}$ of τ to M_s coincides with the end-point map $\eta_{2(s_0 - s)v^s}$ for $2(s_0 - s)v^s$. Since F is totally geodesic, we see that $\eta_{2(s_0-s)v^s}$ (hence $\tau|_{M_s}$) is an isometry of M_s. From this fact, it follows that τ is an isometry of G/K. Hence F is reflective. Furthermore, by imitating the proof of [16, Proposition 1.12], we can show that F is an orbit of a Hermann action on G/K as follows. Take $\mathrm{Exp}\, Z_0 \in F$, where Exp is the exponential map of G/K at o. Set $\mathfrak{m} := \mathrm{Ad}(\exp(-Z_0))((\exp\, Z_0)_*^{-1}(T_{\mathrm{Exp}\, Z_0} F))$, where Ad is the adjoint operator of G. Define a subalgebra $\mathfrak{k}'$ of $\mathfrak{g}$ by $\mathfrak{k}' := \{X \in \mathfrak{k}\,;\ \mathrm{ad}(X)\mathfrak{m} = \mathfrak{m}\}$ and set $\mathfrak{h} := \mathfrak{k}' + \mathfrak{m}$, which is a subalgebra of $\mathfrak{g}$. Set $H := I(\exp\, Z_0)(\exp(\mathfrak{h}))$, where $I(\exp\, Z_0)$ is the inner automorphism of G by exp Z_0. Easily we can show that $T_{\mathrm{Exp}\, Z_0}(H\mathrm{Exp}\, Z_0) = T_{\mathrm{Exp}\, Z_0}F$ and hence $H\mathrm{Exp}\, Z_0 = F$. Define an involution $\hat{\tau}$ of G by $\hat{\tau}(g) := \tau \circ g \circ \tau^{-1}$ $(g \in G)$. It is easy to show that $(\mathrm{Fix}\,\hat{\tau})_0 \subset H \subset \mathrm{Fix}\,\hat{\tau}$. Thus $H \curvearrowright G/K$ is a Hermann action. Let $H^{\mathbb{C}}$ be the complexification of H and $M^{\mathbb{C}}(\subset G^{\mathbb{C}}/K^{\mathbb{C}})$ be the complete complexification of M. See [22] about the definition of the complete complexification of M. Since both $H^{\mathbb{C}} \cdot o$ and $M^{\mathbb{C}}$ are anti-Kaehler equifocal submanifolds having $F^{\mathbb{C}}$ as a focal submanifold, they are equal to one of the partial tubes over $F^{\mathbb{C}}$ stated in Section 5 in [22]. Thus they coincides with each other. Furthermore, from this fact, we can derive $H \cdot o = M$. This completes the proof. □

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