A CASE OF COMBINED RADIAL AND AXIAL HEAT FLOW IN COMPOSITE CYLINDERS*

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Introduction. Although several problems of heat flow in composite cylinders have been studied, all the cases considered treat the heat flow in the radial direction only [1, 2, 3]. The case of combined radial and axial heat flow in composite cylinders presents an interesting boundary value problem which has also considerable significance in the theory of vibrations and propagation of electromagnetic waves [4, 5, 6]. In this paper, we consider a case of combined radial and axial heat flow in the unsteady state in finite cylinders composed of two coaxial parts of different materials. The temperature distribution in the cylinder at any instant under the assumed boundary and initial conditions has been obtained by the use of the Laplace transformation. The procedure is illustrated by a numerical calculation in a particular case.

The Problem. Composite cylinder made of two different materials, the inner cylinder $0 \leq r \leq a$ and the outer cylinder $a \leq r \leq b$ having thermal conductivity and diffusivity coefficients ϵ_1 and k_1 and ϵ_2 and k_2 respectively.† Boundary conditions: The flat ends of the cylinder $x = 0$ and $x = l$ kept at zero temperature with the outer surface insulated and perfect thermal contact at $r = a$ between the two coaxial parts. Initially the cylinder is assumed to be heated to constant unit temperature. Required the temperature distribution in the cylinder for any time $t > 0$ (see Fig. 1).
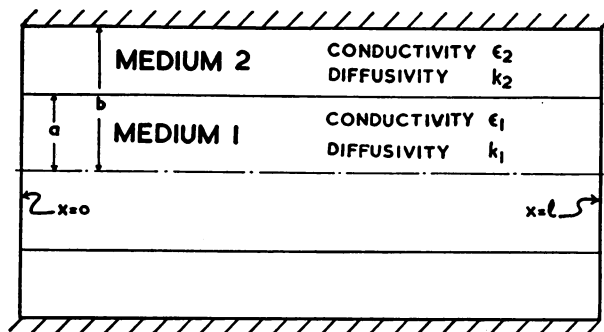

FIG. 1. Composite cylinder, the inner cylinder having conductivity and diffusivity coefficients ϵ_1 and k^1 respectively and the outer ϵ_2 and k_2.

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†Throughout this article the subscripts 1 and 2 refer to the regions $0 \leq r \leq a$ and $a \leq r \leq b$ respectively.

Method of solution. The boundary value problem for the temperatures u_1 and u_2 in the inner and outer cylinders may be stated as follows:

$$\frac{\partial u_1}{\partial t} = k_1\left(\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\frac{\partial u_1}{\partial r} + \frac{\partial^2 u_1}{\partial x^2}\right), \qquad 0 \leq r \leq a, \tag{1}$$

$$\frac{\partial u_2}{\partial t} = k_2\left(\frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r}\frac{\partial u_2}{\partial r} + \frac{\partial^2 u_2}{\partial x^2}\right), \qquad a \leq r \leq b, \tag{2}$$

with the initial conditions

$$u_1(r, x, 0) = u_2(r, x, 0) = 1, \tag{3}$$

and the boundary conditions

$$u_1(r, 0, t) = u_2(r, 0, t) = 0, \tag{4}$$

$$u_1(r, l, t) = u_2(r, l, t) = 0, \tag{5}$$

$$u_1(a, x, t) = u_2(a, x, t), \tag{6}$$

$$\epsilon_1\, \partial u_1(a, x, t)/\partial r = \epsilon_2\, \partial u_2(a, x, t)/\partial r, \tag{7}$$

$$\epsilon_2\, \partial u_2(b, x, t)/\partial r = 0. \tag{8}$$

We require the temperature functions $u_1(r, x, t)$ and $u_2(r, x, t)$ satisfying the equations from (1) to (8) inclusive.

Let

$$U_1(r, x, s) = \int_0^\infty u_1(r, x, t)e^{-st}\, dt$$

$$U_2(r, x, s) = \int_0^\infty u_2(r, x, t)e^{-st}\, dt$$

be the Laplace transforms of u_1 and u_2. The transforms $U_1(r, x, s)$ and $U_2(r, x, s)$ will then satisfy the equations

$$\frac{\partial^2 U_1}{\partial r^2} + \frac{1}{r}\frac{\partial U_1}{\partial r} + \frac{\partial^2 U_1}{\partial x^2} - \frac{s}{k_1}\, U_1 = -\frac{1}{k_1}, \tag{9}$$

$$\frac{\partial^2 U_2}{\partial r^2} + \frac{1}{r}\frac{\partial U_2}{\partial r} + \frac{\partial^2 U_2}{\partial x^2} - \frac{s}{k_2}\, U_2 = -\frac{1}{k_2}, \tag{10}$$

and the boundary conditions

$$U_1(r, 0, s) = U_2(r, 0, s) = 0, \tag{11}$$

$$U_1(r, l, s) = U_2(r, l, s) = 0, \tag{12}$$

$$U_1(a, x, s) = U_2(a, x, s), \tag{13}$$

$$\epsilon_1\, \partial U_1(a, x, s)/\partial r = \epsilon_2\, \partial U_2(a, x, s)/\partial r, \tag{14}$$

$$\partial U_2(b, x, s)/\partial r = 0. \tag{15}$$

In order that $U_1(r, x, s)$ and $U_2(r, x, s)$ may vanish at $x = 0$ and $x = l$ as required by the boundary conditions (11) and (12), we expand U_1 and U_2 as well as the constants $-1/k_1$ and $-1/k_2$ on the right hand side of equations (9) and (10) in Fourier sine series

$$U_1(r, x, s) = \sum_n V_{1n}(r, s) \sin (n\pi x/l), \qquad n = 1, 3, 5, \cdots ,$$

$$U_2(r, x, s) = \sum_n V_{2n}(r, s) \sin (n\pi x/l), \qquad n = 1, 3, 5, \cdots ,$$

$$-\frac{1}{k_1} = \sum_n b_{1n} \sin (n\pi x/l), \qquad n = 1, 3, 5, \cdots ,$$

$$-\frac{1}{k_2} = \sum_n b_{2n} \sin (n\pi x/l), \qquad n = 1, 3, 5, \cdots .$$

For the sake of brevity, let

$$\alpha_n^2 = s/k_1 + n^2\pi^2/l^2 \tag{16}$$

$$\beta_n^2 = s/k_2 + n^2\pi^2/l^2 \tag{17}$$

The radial functions V_{1n} and V_{2n} now satisfy respectively the equations

$$\frac{\partial^2 V_{1n}}{\partial r^2} + \frac{1}{r}\frac{\partial V_{1n}}{\partial r} - \alpha_n^2\left(V_{1n} + \frac{b_{1n}}{\alpha_n^2}\right) = 0,$$

$$\frac{\partial^2 V_{2n}}{\partial r^2} + \frac{1}{r}\frac{\partial V_{2n}}{\partial r} - \beta_n^2\left(V_{2n} + \frac{b_{2n}}{\beta_n^2}\right) = 0.$$

These have the solutions

$$V_{1n}(r, s) = A_n I_0(\alpha_n r) - \frac{b_{1n}}{\alpha_n^2},$$

$$V_{2n}(r, s) = B_n I_0(\beta_n r) + C_n K_0(\beta_n r) - \frac{b_{2n}}{\beta_n^2},$$

where I_0 and K_0 are respectively the modified Bessel functions of the first and the second kinds of zeroth order. The constants A_n, B_n, etc. are now to be determined from the boundary conditions (13), (14) and (15). Thus we obtain

$$U_1(r, x, s) = \sum_n \left[\epsilon_2\beta_n\left(\frac{b_{1n}}{\alpha_n^2} - \frac{b_{2n}}{\beta_n^2}\right)\frac{\{I_1(\beta_n a)K_1(\beta_n b) - I_1(\beta_n b)K_1(\beta_n a)\}I_0(\alpha_n r)}{\Delta_n(s)}\right.$$
$$\left. - \frac{b_{1n}}{\alpha_n^2}\right] \sin \frac{n\pi x}{l}, \tag{18}$$

$$U_2(r, x, s) = \sum_n \left[\epsilon_1\alpha_n\left(\frac{b_{1n}}{\alpha_n^2} - \frac{b_{2n}}{\beta_n^2}\right)\frac{\{I_0(\beta_n r)K_1(\beta_n b) - I_1(\beta_n b)K_0(\beta_n r)\}I_1(\alpha_n a)}{\Delta_n(s)}\right.$$
$$\left. - \frac{b_{2n}}{\beta_n^2}\right] \sin \frac{n\pi x}{l}, \tag{19}$$

where

$$\Delta_n(s) = \epsilon_2\beta_n I_0(\alpha_n a)[I_1(\beta_n a)K_1(\beta_n b) - I_1(\beta_n b)K_1(\beta_n a)]$$

$$- \epsilon_1\alpha_n I_1(\alpha_n a)[I_0(\beta_n a)K_1(\beta_n b) + I_1(\beta_n b)K_0(\beta_n a)]. \tag{20}$$

The temperature distribution functions $u_1(r, x, t)$ and $u_2(r, x, t)$ may be now obtained from $U_1(r, x, s)$ and $U_2(r, x, s)$ by the inversion integrals [7]

$$u_1(r, x, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} U_1(r, x, s)e^{st} \, ds \tag{21}$$

$$u_2(r, x, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} U_2(r, x, s)e^{st} \, ds \tag{22}$$

The integrals in (21) and (22) may be expressed as $2\pi i$ times the sum of the residues of the corresponding integrands at their poles. In evaluating the residue of $U_1(r, x, s)$ exp (st) it will be noted that the first term in $U_1(r, x, s)$ has got the factor $\alpha_n^2\beta_n\Delta_n(s)$ in the denominator. It will be seen further that $\beta_n = 0$ does not give rise to a pole since the expression remains finite (on account of the singularity of K_1 at the origin) as $\beta_n \to 0$ so that the only poles are those due to $\alpha_n^2 = 0$ and $\Delta_n = 0$. However, the residue of the first term at $\alpha_n = 0$ cancels with that due to the second term and hence the only significant poles are those of $\Delta_n = 0$. Similar remarks apply in evaluating the residue of $U_2(r, x, s)$ exp (st). One obtains therefore

$$u_1(r, x, t) = \sum_n \frac{4\pi\epsilon_2}{l^2} \frac{k_1 - k_2}{k_1 k_2} n \sin \frac{n\pi x}{l}$$

$$\cdot \sum_j \frac{\{I_1(\beta_{nj}a)K_1(\beta_{nj}b) - I_1(\beta_{nj}b)K_1(\beta_{nj}a)\}I_0(\alpha_{nj}r)}{\alpha_{nj}^2\beta_{nj}\Delta_n'(\lambda_{nj})} e^{\lambda_{nj}t}, \tag{23}$$

$$u_2(r, x, t) = \sum_n \frac{4\pi\epsilon_1}{l^2} \frac{k_1 - k_2}{k_1 k_2} n \sin \frac{n\pi x}{l}$$

$$\cdot \sum_j \frac{\{I_0(\beta_{nj}r)K_1(\beta_{nj}b) - I_1(\beta_{nj}b)K_0(\beta_{nj}r)\}I_1(\alpha_{nj}a)}{\alpha_{nj}\beta_{nj}^2\Delta_n'(\lambda_{nj})} e^{\lambda_{nj}t}, \tag{24}$$

where λ_{nj} are the zeroes of $\Delta_n(s) = 0$, $(n = 1, 3, 5, \cdots)$ and α_{nj} and β_{nj} are the corresponding values defined by the relations (16) and (17) when s has the values λ_{nj}.

The zeroes of Δ_n may be obtained from equation (20) by solving the equation

$$\frac{\epsilon_1 a_n}{\epsilon_2\beta_n} \frac{I_1(\alpha_n a)}{I_0(\beta_n a)} = \frac{I_1(\beta_n a)K_1(\beta_n b) - I_1(\beta_n b)K_1(\beta_n a)}{I_0(\beta_n a)K_1(\beta_n b) + I_1(\beta_n b)K_0(\beta_n a)} \tag{25}$$

graphically. The equation (25) may be transformed into a form more suitable for numerical work by the substitutions

$$\alpha_n a = ix, \qquad \beta_n a = iy, \qquad \beta_n b = i\rho y, \qquad (\rho = b/a),$$

x and y being related on account of (16) and (17) by the equation

$$y^2 = \sigma^2(x^2 + n^2\pi^2 a^2\delta^2/l^2), \tag{26}$$

where $\sigma^2 = k_1/k_2$ and $\delta^2 = (k_1 - k_2)/k_1$ are dimensionless constants. Introducing another dimensionless constant $\sigma'^2 = \epsilon_1/\epsilon_2$ and transforming the I and K functions into the corresponding J and Y functions* by the relations

$$I_\nu(iz) = i^\nu J_\nu(z),$$

$$K_\nu(iz) = i^{-\nu+1}[-J_\nu(z) + iY_\nu(z)]\pi/2,$$

we obtain in place of (25)

$$\sigma'^2 x \frac{J_1(x)}{J_0(x)} = y \frac{J_1(\rho y)\,Y_1(y) - J_1(y)\,Y_1(\rho y)}{J_1(\rho y)\,Y_0(y) - J_0(y)\,Y_1(\rho y)}. \tag{27}$$

Equation (27) has real roots and may be solved by plotting the right and left hand sides as functions of x. (Note that y on the right hand side is not the corresponding ordinate, but determined by (26)).

Let

$$x = \xi_{nj} \qquad j = 1, 2, 3, \cdots ; \qquad n = 1, 3, 5, \cdots$$

be the roots of equation (27), the double subscript indicating that ξ_{nj} is the jth root of $\Delta_n = 0$. Let the corresponding values of $\alpha_{nj}a$, etc. be

$$\alpha_{nj}a = i\xi_{nj}\,, \qquad \beta_{nj}a = i\sigma\eta_{nj}\,, \qquad \beta_{nj}b = i\rho\sigma\eta_{nj}\,,$$

where

$$\eta_{nj}^2 = \xi_{nj}^2 + (n\pi a\delta/l)^2 \text{ by equation (26). Then}$$

$$\lambda_{nj} = -(\xi_{nj}^2/a^2 + n^2\pi^2/l^2)k_1\;.$$

$u_1(r, x, t)$ and $u_2(r, x, t)$ can be now expressed in terms of ξ_{nj} and η_{nj} as follows:

$$u_1(r, x, t) = \frac{8\pi a^3\epsilon_2}{l^2}\frac{k_1 - k_2}{k_1\sigma}\sum_n n \sin\frac{n\pi x}{l}\sum_j \frac{F_{11}(\rho\sigma\eta_{nj}\,,\,\sigma\eta_{nj})J_0(\xi_{nj}r/a)}{\xi_{nj}^2\eta_{nj}D_n(\lambda_{nj})}e^{\lambda_{nj}t}, \tag{28}$$

$$u_2(r, x, t) = \frac{8\pi a^3\epsilon_1}{l^2}\frac{k_1 - k_2}{k_1\sigma^2}\sum_n n \sin\frac{n\pi x}{l}\sum_j \frac{F_{10}(\rho\sigma\eta_{nj}\,,\,\sigma\eta_{nj}r/a)J_1(\xi_{nj})}{\xi_{nj}\eta_{nj}^2 D_n(\lambda_{nj})}e^{\lambda_{nj}t}, \tag{29}$$

where

$$D_n(\lambda_{nj}) = \frac{\epsilon_2 a}{\sigma}\left(\frac{\eta_{nj}}{\xi_{nj}} - \sigma'^2\frac{\xi_{nj}}{\eta_{nj}}\right)J_1(\xi_{nj})F_{11}(\rho\sigma\eta_{nj}\,,\,\sigma\eta_{nj})$$

$$+ \frac{\epsilon_1 b}{\sigma}\frac{\xi_{nj}}{\eta_{nj}}J_1(\xi_{nj})F_{00}(\rho\sigma\eta_{nj}\,,\,\sigma\eta_{nj})$$

$$+ \epsilon_2 b J_0(\xi_{nj})F_{10}(\sigma\eta_{nj}\,,\,\rho\sigma\eta_{nj}) \tag{30}$$

$$+ \frac{\epsilon_1 a}{\sigma'^2}\left(1 - \frac{\sigma'^2}{\sigma^2}\right)J_0(\xi_{nj})F_{01}(\sigma\eta_{nj}\,,\,\rho\sigma\eta_{nj}),$$

$$F_{\mu\nu} = J_\mu(x)Y_\nu(y) - J_\nu(y)Y_\mu(x). \tag{31}$$

*Ref. [3], Appendix III.

Verification of the solution.* As the series solution established by the Laplace transform method is purely formal, it is necessary to show that it satisfies all the conditions of the boundary value problem and is unique. It is obvious that the series solutions (28) and (29) for $u_1(r, x, t)$ and $u_2(r, x, t)$ respectively satisfy the boundary conditions (4) and (5). It is also seen that the boundary conditions (7) and (8) are satisfied by direct substitution of the expressions for u_1 and u_2 , and (6) is satisfied on account of the relation $\Delta_n(\lambda_{ni}) = 0$. It only remains, therefore, to verify that the initial condition, viz., $u_1 = u_2 = 1$ for $t = 0$. This is done most conveniently with the contour integral form of the solutions (21) and (22). Consider u_1 for example.

For $t = 0$ we have from (18) and (21)

$$u_1(r, x, 0) = \sum_n \frac{4}{n\pi} \sin \frac{n\pi x}{l} + \sum_n \frac{4}{n\pi} \sin \frac{n\pi x}{l} \cdot \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} A_n I_0(\alpha_n r) \, ds \qquad (32)$$

where

$$A_n = \frac{\epsilon_2(k_1 - k_2)}{k_1 k_2} \left(\frac{n\pi}{l}\right)^2 \frac{I_1(\beta_n a) K_1(\beta_n b) - I_1(\beta_n b) K_1(\beta_n a)}{\beta_n \alpha_n^2 \Delta_n(s)} . \qquad (33)$$

since

$$\sum_n \frac{4}{n\pi} \sin \frac{n\pi x}{l} = 1 \qquad \text{we may write} \qquad u_1 = 1 + v_1$$

where

$$v_1 = \sum_n \frac{4}{n\pi} \sin \frac{n\pi x}{l} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} A_n I_0(\alpha_n r) \, ds. \qquad (34)$$

It thus suffices to show that $v_1 \equiv 0$. The path of integration is a straight line parallel to the imaginary axis such that all the poles of the integrand lie to the left of this line. As the poles λ_{ni} are all negative we can choose the path with any $\gamma > 0$. We shall choose γ large and positive. Since $\alpha_n^2 = s/k_1 + n^2\pi^2/l^2$ and $\beta_n^2 = s/k_2 + n^2\pi^2/l^2$ it is clear that $|\alpha_n^2|$ and $|\beta_n^2|$ will be large both for large ξ and large η. Further, if $k_1 > k_2$ (say) we have on the path of integration $|\alpha_n| < |\beta_n|$.

Replacing now the modified Bessel functions in equation (33) for A_n by their asymptotic expansions for large argument and retaining only the dominant terms in the numerator and denominator we find that the integral in (34) becomes, apart from a constant factor

$$n^2 \left(\frac{a}{r}\right)^{1/2} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{-\alpha_n(a - r)}}{\epsilon_2 \beta_n + \epsilon_1 \alpha_n} \frac{ds}{(\alpha_n \beta_n)^{3/2}} .$$

It may be shown that the absolute value of this expression is less than

$$2(\epsilon_1 + \epsilon_2)\left(\frac{a}{r}\right)^{1/2} n^2 \exp\left\{-\left(\frac{\gamma}{2k_1} + \frac{n^2\pi^2}{2l^2}\right)(a - r)\right\} \int_0^\infty \frac{d\eta}{(P^2 + Q^2\eta^2)^{1/2} |\alpha_n|^{3/2} |\beta_n|^{1/2}}$$

Thus the absolute values of the terms of the series in (34) are majorized by

$$c_1 \left(\frac{a}{r}\right)^{1/2} \exp\left\{-\frac{r}{2k_1}(a - r)\right\} \sum_n K(n) \exp\left\{-\frac{n^2\pi^2}{2l^2}(a - r)\right\}, \qquad (35)$$

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*Ref. [3], Appendix I.

where $K(n)$ is $0(nu)$ with a fixed finite μ. The expression (35) shows that v_1 can be made arbitrarily small by making γ large. Hence it follows that $v_1 \equiv 0$. Similar reasoning shows that u_2 also satisfies the initial condition.

The proof of the uniqueness of the solution is well known and need not be repeated here.

A numerical example. To illustrate the numerical procedure, the temperature distribution in a composite cylinder having the following parameters is calculated

$$k_1 = 1, \qquad k_2 = 0.1, \qquad \epsilon_1 = 0.4, \qquad \epsilon_2 = 0.04$$

$$a = 1, \qquad b = 1.5, \qquad l = 10; \qquad u_1(r, x, 0) = u_2(r, x, 0) = 1$$

Equation (27) leading to the roots ξ_{nj} in this particular case is plotted in Fig. 2.
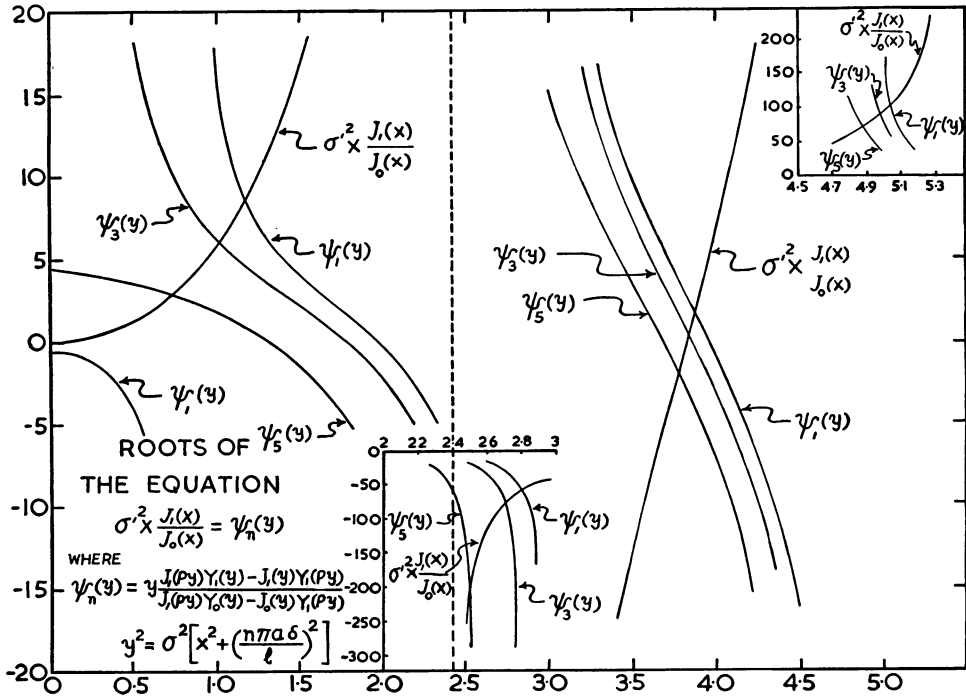

FIG. 2. Graphical solution of the first few roots of the eq. 27 for $n = 1, 3, 5$.

The first few roots are given in the table below.

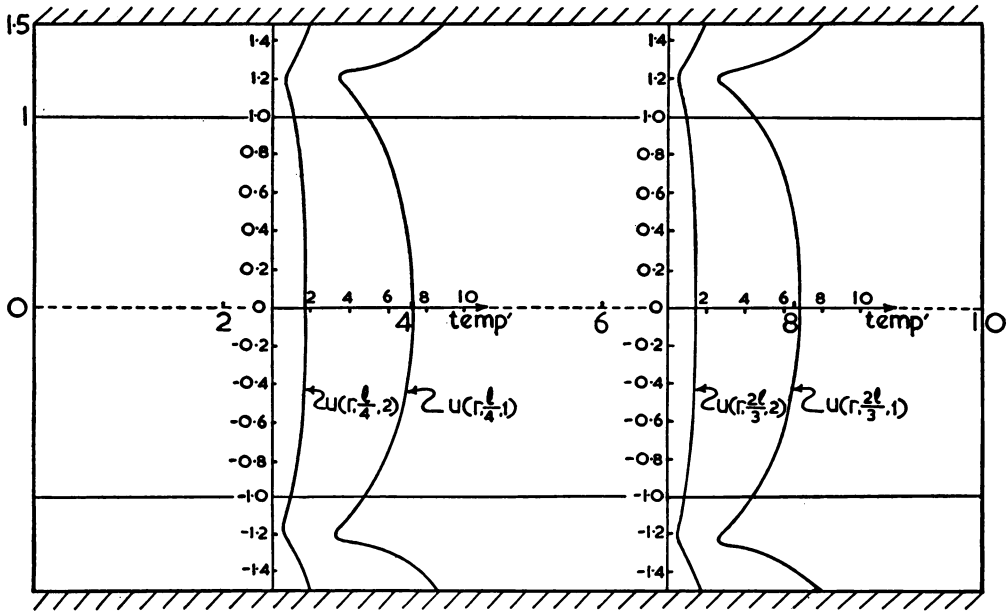| n \ j	1	2	3	4
1	1.18	2.83	3.88	5.04
3	1.03	2.74	3.85	4.98
5	0.75	2.52	3.78	4.88

FIG. 3. Distribution of temperature in the composite cylinder along the radius at $x = l/4$ and $x = 2l/3$ for $t = 1$ and $t = 2$ secs.

The distribution of temperature along the radius at $x = 2l/3$ and $l/4$ for $t = 1$ sec. and 2 secs. is plotted in Fig. 3.

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