Open Mathematics

Research Article

Bing Wang*, Qing-Hua Li, and Zhen-Yu Xiu

A categorical approach to abstract convex spaces and interval spaces

https://doi.org/10.1515/math-2019-0029
Received August 30, 2018; accepted February 20, 2019

Abstract: In this paper, we establish the axiomatic conditions of hull operators and introduce the category of interval spaces. We also investigate their relations with convex spaces from a categorical sense. It is shown that the category **CS** of convex spaces is isomorphic to the category **HS** of hull spaces, and they are all topological over **Set**. Also, it is proved that there is an adjunction between the category **IS** of interval spaces and the category **CS** of convex spaces. In particular, the category **CS(2)** of arity 2 convex spaces can be embedded in **IS** as a reflective subcategory.

Keywords: Convex structure, Hull operator, Interval operator, Adjunction

MSC: 52A01, 54A05, 18D35

1 Introduction

Convexity is an important and basic property in many mathematical areas. However, in some concrete mathematical setting, such as vector spaces, it is not the most suitable setting for studying the basic properties of convex sets. In order to avoid this deficiency, abstract convex structures (convex structures, in short) are defined by three axioms [1], which is a similar way of defining topological structures. Up to now, the convexity theory has become a branch of mathematics dealing with set-theoretic structures satisfying axioms similar to that usual convex sets fulfill. Actually, convex structures appeared in many research areas, such as lattices [2], graphs [3], and topology [4]. Besides, convexity theory is also investigated from the lattice-valued aspect, including L-convex structures [5–13] and M-fuzzifying convex structures [14–19].

Category theory plays an important role in demonstrating the relations between different types of spatial structures. It emerges frequently in general topology and fuzzy topology, especially in crisp and fuzzy convergence theory [20–29]. This motivates us to apply category theory to convex structures since convex structures can be considered as topology-like structures. Actually, like continuous mappings between topological spaces, there is also a special kind of mappings between convex spaces, which is called convexity-preserving mapping. Under a convexity-preserving mapping, convex sets in the range space are inverted to convex sets of the domain. Such mappings arise in various constructions of convexities. For spaces derived from an algebraic structure, convexity-preserving mappings usually agree with the corresponding notions of homomorphisms. In fact, convexity-preserving mappings are exactly the appropriate notions of morphisms

***Corresponding Author: Bing Wang:** School of Mathematical Sciences, Mudanjiang Normal University, Mudanjiang 157011, China, E-mail: wbbit2011@sina.com
Qing-Hua Li: School of Mathematics and Information Sciences, Yantai University, Yantai 264005, China, E-mail: lqh@ytu.edu.cn
Zhen-Yu Xiu: 1. College of Applied Mathematics, Chengdu University of Information Technology, Chengdu 610000, China. 2. College of Mathematics and Econometrics, Hunan University, Hunan 410082, China, E-mail: xiuzhenyu112@sohu.com

in the category of convex spaces. This is just one of our motivations of this paper. That is, we would like to investigate the categorical properties of convex spaces.

A convex structure is completely determined by a special kind of closure operators, which is called algebraic closure operators. Actually, algebraic closure operators can be treated as the hull operators of convex spaces. Except for algebraic closure operators, interval operators, as a generalization of intervals, provide a natural and frequent method of describing or constructing convex structures. There are also close relations between convex structures and interval operators. Inspired by this, we will not only provide a new characterization of convex structures by closure operators and present an axiomatic hull operators, but also focus on the categorical properties of interval spaces and study its relations with convex spaces from a categorical sense.

2 Preliminaries

Throughout this paper, let X denote a nonempty set and 2^X the powerset of X. Let $\{A_j\}_{j\in J} \overset{dir}{\subseteq} 2^X$ denote that $\{A_j\}_{j\in J}$ is a directed subset of 2^X, which means for each $B, C \in \{A_j\}_{j\in J}$, there exists $D \in \{A_j\}_{j\in J}$ such that $B \subseteq D$ and $C \subseteq D$.

Let X, Y be two nonempty sets and $f : X \longrightarrow Y$ be a mapping. Define $f^{\rightarrow} : 2^X \longrightarrow 2^Y$ and $f^{\leftarrow} : 2^Y \longrightarrow 2^X$ as follows:

$$\forall A \in 2^X,\ f^{\rightarrow}(A) = \{f(x) \mid x \in A\};\ \ \forall B \in 2^Y,\ f^{\leftarrow}(B) = \{x \mid f(x) \in B\}.$$

Definition 2.1 ([1]). A convex structure $\mathcal{C}$ on X is a subset of 2^X which satisfies:

(CS1) $\emptyset, X \in \mathcal{C}$;

(CS2) $\{A_i\}_{i\in I} \subseteq \mathcal{C}$ implies $\bigcap_{i\in I} A_i \in \mathcal{C}$;

(CS3) $\{A_j\}_{j\in J} \overset{dir}{\subseteq} \mathcal{C}$ implies $\overset{dir}{\bigcup}_{j\in J} \in \mathcal{C}$.

For a convex structure $\mathcal{C}$ on X, the pair $(X, \mathcal{C})$ is called a convex space.

A mapping $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is called convexity-preserving (CP, in short) provided that $B \in \mathcal{C}_Y$ implies $f^{\leftarrow}(B) \in \mathcal{C}_X$. The category whose objects are convex spaces and whose morphisms are CP mappings will be denoted by **CS**.

Definition 2.2 ([1]). A closure operator on X is a mapping $C : 2^X \longrightarrow 2^X$ which satisfies:

(CL1) $C(\emptyset) = \emptyset$;

(CL2) $A \subseteq C(A)$;

(CL3) $A \subseteq B \Rightarrow C(A) \subseteq C(B)$;

(CL4) $CC(A) = C(A)$.

For a closure operator C on X, the pair (X, C) is called a closure space. It will be called algebraic if it also satisfies that,

(CLA) $C(A) = \bigcup\{C(B) \mid B \text{ is a finite subset of } A\}$.

The pair (X, C) is called an algebraic closure space.

A mapping $f : (X, C_X) \longrightarrow (Y, C_Y)$ between closure spaces is called convexity-preserving (CP, in short) provided that $f^{\rightarrow}(C_X(A)) \subseteq C_Y(f^{\rightarrow}(A))$ for all $A \in 2^X$. The category whose objects are closure spaces and whose morphisms are CP mappings will be denoted by **CLS**, and the full subcategory of algebraic closure spaces by **ACLS**.

For a convex space $(X, \mathcal{C})$, define $C^{\mathcal{C}} : 2^X \longrightarrow 2^X$ by $C^{\mathcal{C}}(A) = \bigcap_{A\subseteq B\in\mathcal{C}} B$. Then $C^{\mathcal{C}}$ is an algebraic closure operator on X. Conversely, for a closure operator (X, C), define $\mathcal{C}^C \subseteq 2^X$ by $\mathcal{C}^C = \{A \in 2^X \mid A = C(A)\}$. Then $\mathcal{C}^C$ is a convex structure on X. Furthermore, they are one-to-one corresponding. In a categorical sense, we have

Theorem 2.3. *The category* **CS** *is isomorphic to the category* **ACLS**.

Definition 2.4 ([29, 30]). A category **C** is called a topological category over **Set** with respect to the usual forgetful functor from **C** to **Set** if it satisfies (TC1), (TC2) and (TC3) or (TC1)′, (TC2) and (TC3).

(TC1) Existence of initial structures: For any set X, any class J, and family $((X_j, \xi_j))_{j\in J}$ of **C**-object and any family $(f_j : X \longrightarrow X_j)_{j\in J}$ of mappings, there exists a unique **C**-structure ξ on X which is initial with respect to the source $(f_j : X \longrightarrow (X_j, \xi_j))_{j\in J}$, this means that for a **C**-object (Y, η), a mapping $g : (Y, \eta) \longrightarrow (X, \xi)$ is a **C**-morphism iff for all $j \in J$, $f_j \circ g : (Y, \eta) \longrightarrow (X_j, \xi_j)$ is a **C**-morphism.

(TC1)′ Existence of final structures: For any set X, any class J, and family $((X_j, \xi_j))_{j\in J}$ of **C**-object and any family $(f_j : X_j \longrightarrow X)_{j\in J}$ of mappings, there exists a unique **C**-structure ξ on X which is final with respect to the sink $(f_j : (X_j, \xi_j) \longrightarrow X))_{j\in J}$, this means that for a **C**-object (Y, η), a mapping $g : (X, \xi) \longrightarrow (Y, \eta)$ is a **C**-morphism iff for all $j \in J$, $g \circ f_j : (X_j, \xi_j) \longrightarrow (Y, \eta)$ is a **C**-morphism.

(TC2) Fibre-smallness: For any set X, the **C**-fibre of X, i.e., the class of all **C**-structures on X, which we denote by $\mathbf{C}(X)$, is a set.

(TC3) Terminal separator property: For any set X with cardinality at most one, there exists exactly one **C**-object with underlying set X (i.e. there exists exactly one **C**-structure on X).

Lemma 2.5 ([29, 30]). *Suppose that $\mathbb{F} : \mathbf{A} \longrightarrow \mathbf{B}$ and $\mathbb{G} : \mathbf{B} \longrightarrow \mathbf{A}$ are concrete functors. Then the following conclusions are equivalent:*

(1) $\{id_Y : \mathbb{F} \circ \mathbb{G}(Y) \longrightarrow Y \mid Y \in \mathbf{B}\}$ *is a natural transformation from the functor* $\mathbb{F} \circ \mathbb{G}$ *to the identity functor* $id_{\mathbf{B}}$ *on* **B**, *and* $\{id_X : X \longrightarrow \mathbb{G} \circ \mathbb{F}(X) \mid X \in \mathbf{A}\}$ *is a natural transformation from the identity functor* $id_{\mathbf{A}}$ *on* **A** *to the functor* $\mathbb{G} \circ \mathbb{F}$.

(2) *For each* $Y \in \mathbf{B}$, $id_Y : \mathbb{F} \circ \mathbb{G}(Y) \longrightarrow Y$ *is a* **B**-*morphism, and for each* $X \in \mathbf{A}$, $id_X : X \longrightarrow \mathbb{G} \circ \mathbb{F}(X)$ *is a* **A**-*morphism.*

In this case, $(\mathbb{F}, \mathbb{G})$ *is called an adjunction between* **A** *and* **B**.

If $(\mathbb{F}, \mathbb{G})$ is an adjunction, then it is easy to verify that $\mathbb{F}$ is a left adjoint of $\mathbb{G}$ or equivalently, $\mathbb{G}$ is a right adjoint of $\mathbb{F}$.

The class of objects of a category **A** is denoted by $|\mathbf{A}|$. For more notions related to category theory we refer to [29] and [30].

3 The categories of convex spaces and hull spaces

In this section, we first study some categorical properties of convex spaces. Then we introduce the axiomatic hull operators and investigate its relations with convex structures.

Definition 3.1. For a nonempty, let $F_{\mathcal{C}}(X)$ denote the fibre

$$\{(X, \mathcal{C}) \mid \mathcal{C} \text{ is a convex structure on } X.\}$$

of X. For convex spaces $(X, \mathcal{C}_1)$ and $(X, \mathcal{C}_2)$, we say $(X, \mathcal{C}_1)$ is finer than $(X, \mathcal{C}_2)$, or $(X, \mathcal{C}_2)$ is coarser than $(X, \mathcal{C}_1)$, denoted by $(X, \mathcal{C}_1) \leqslant_{\mathcal{C}} (X, \mathcal{C}_2)$, if the identity mapping $id_X : (X, \mathcal{C}_1) \longrightarrow (X, \mathcal{C}_2)$ is CP. We also write $\mathcal{C}_1 \leqslant_{\mathcal{C}} \mathcal{C}_2$.

Example 3.2. Let X be a nonempty set.

(1) Define $\mathcal{C}_\star$ by $\mathcal{C}_\star = 2^X$. Then $\mathcal{C}_\star$ is the finest convex structure on X, which is called the discrete convex structure on X.

(2) Define $\mathcal{C}^\star$ by $\mathcal{C}^\star = \{\emptyset, X\}$. Then $\mathcal{C}^\star$ is the coarsest convex structure on X, which is called the indiscrete convex structure on X.

Theorem 3.3. *The category* **CS** *is topological over* **Set**.

Proof. We first prove the existence of final structures. Let $((X_\lambda, \mathcal{C}_\lambda))_{\lambda\in\Lambda}$ be a family of convex spaces and let X be a nonempty set. Let further $((f_\lambda : (X_\lambda, \mathcal{C}_\lambda) \longrightarrow X))_{\lambda\in\Lambda}$ be a sink. Define $\mathcal{C} \subseteq 2^X$ by

$$\mathcal{C} = \{A \in 2^X \mid \forall\lambda \in \Lambda,\ f_\lambda^{\leftarrow}(A) \in \mathcal{C}_\lambda\}.$$

Since $f_\lambda^{\leftarrow}$ preserves arbitrary meets and directed joins, we can verify that $\mathcal{C}$ is a convex structure on X.

Let further $(Y, \mathcal{C}_Y)$ be a convex space and $g : X \longrightarrow Y$ be a mapping. Assume that $g \circ f_\lambda$ is CP for all $\lambda \in \Lambda$. We have for all $B \in \mathcal{C}_Y$,

$$\forall\lambda \in \Lambda,\ f_\lambda^{\leftarrow}(g^{\leftarrow}(B)) = (g \circ f_\lambda)^{\leftarrow}(B) \in \mathcal{C}_\lambda.$$

By definition of $\mathcal{C}$, we obtain $g^{\leftarrow}(B) \in \mathcal{C}$. This implies that $g : (X, \mathcal{C}) \longrightarrow (Y, \mathcal{C}_Y)$ is CP, as desired.

Secondly, the class of all convex structures on a fixed set X is a subset of $2^{(2^X)}$, which means that the **CS** fibre of X is a set.

Finally, for a one point set $X = \{x\}$, there exists only one convex structure $\mathcal{C} = \{\emptyset, \{x\}\}$ on X. Hence, **CS** satisfies the terminal separator property. Therefore, **CS** is a topological category in the sense of [29]. That is, a well-fibred topological category in the terminology of [30]. □

Corollary 3.4. *$(F_{\mathcal{C}}(X), \leqslant_{\mathcal{C}})$ is a complete lattice.*

Next we will introduce the axiomatic hull operators and study its relations with convex structures.

Definition 3.5. A hull operator on X is a mapping $co : 2^X \longrightarrow 2^X$ which satisfies:

(H1) $co(\emptyset) = \emptyset$, $co(X) = X$;

(H2) $A \subseteq co(A)$;

(H3) $A \subseteq B \Rightarrow co(A) \subseteq co(B)$;

(H4) $co(co(A)) = co(A)$.

(H5) $co(\overset{dir}{\bigcup}_{j\in J} A_j) = \bigcup_{j\in J} co(A_j)$.

For a hull operator co on X, the pair (X, co) is called a hull space. Actually, a hull operator on X is a closure operator on X which satisfies (H5).

Definition 3.6. A mapping $f : (X, co_X) \longrightarrow (Y, co_Y)$ is called convexity-preserving (CP, in short) provided that $f^{\rightarrow}(co_X(A)) \subseteq co_Y(f^{\rightarrow}(A))$ for all $A \in 2^X$.

The category whose objects are hull spaces and whose morphisms are CP mappings will be denoted by **HS**.

Proposition 3.7. *Let $(X, \mathcal{C})$ be a convex space and define $co^{\mathcal{C}} : 2^X \longrightarrow 2^X$ by*

$$\forall A \in 2^X,\ co^{\mathcal{C}}(A) = \bigcap_{A\subseteq B\in\mathcal{C}} B.$$

Then $co^{\mathcal{C}}$ is a hull operator on X.

Proof. By Theorem 2.3, $co^{\mathcal{C}}$ is an algebraic closure operator on X. Thus $co^{\mathcal{C}}$ satisfies (H1)–(H4). It suffices to verify (H5). For $\{A_j\}_{j\in J} \overset{dir}{\subseteq} 2^X$, take any $x \notin \bigcup_{j\in J} co^{\mathcal{C}}(A_j) = \bigcup_{j\in J}\bigcap_{A_j\subseteq B\in\mathcal{C}} B$. Then there exists $B_j \in 2^X$ such that $A_j \subseteq B \in \mathcal{C}$ and $x \notin B_j$ for each $j \in J$. Let $C_j = co^{\mathcal{C}}(A_j)$. By (C2) and (H3), we know $A_j \subseteq C_j \in \mathcal{C}$ and $\{C_j\}_{j\in J}$ is directed. Put $B = \bigcup_{j\in J} C_j$. By (C3), we obtain $\bigcup_{j\in J} A_j \subseteq B \in \mathcal{C}$. Further, since $C_j \subseteq B_j$, it follows that $x \notin C_j$ for each $j \in J$. This implies that $x \notin B$. As a consequence, we obtain $B \in 2^X$ such that $\bigcup_{j\in J} A_j \subseteq B \in \mathcal{C}$ and $x \notin B$. This means that $x \notin co^{\mathcal{C}}(\overset{dir}{\bigcup}_{j\in J} A_j)$. By the arbitrariness of x, we have $co^{\mathcal{C}}(\overset{dir}{\bigcup}_{j\in J} A_j) \subseteq \bigcup_{j\in J} co^{\mathcal{C}}(A_j)$. The inverse inequality holds obviously. Therefore, $co^{\mathcal{C}}$ is a hull operator. □

Proposition 3.8. *If $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is a CP mapping, then so is $f : (X, co^{\mathcal{C}_X}) \longrightarrow (Y, co^{\mathcal{C}_Y})$.*

Proof. Since $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is a CP mapping, it follows that

$$\forall B \in 2^Y,\ B \in \mathcal{C}_Y \text{ implies } f^{\leftarrow}(B) \in \mathcal{C}_X.$$

Then for each $A \in 2^X$, we have

$$f^{\leftarrow}(co^{\mathcal{C}_Y}(f^{\rightarrow}(A))) = \bigcap_{f^{\rightarrow}(A)\subseteq B\in\mathcal{C}_Y} f^{\leftarrow}(B) \supseteq \bigcap_{A\subseteq f^{\leftarrow}(B)\in\mathcal{C}_X} f^{\leftarrow}(B) \supseteq \bigcap_{A\subseteq C\in\mathcal{C}_X} C = co^{\mathcal{C}_X}(A).$$

This shows $f^{\rightarrow}(co^{\mathcal{C}_X}(A)) \subseteq co^{\mathcal{C}_Y}(f^{\rightarrow}(A))$, as desired. □

By Propositions 3.7 and 3.8, we obtain a functor $\mathbb{F} : \mathbf{CS} \longrightarrow \mathbf{HS}$ as follows:

$$\mathbb{F} : \begin{cases} \mathbf{CS} & \longrightarrow & \mathbf{HS} \\ (X, \mathcal{C}) & \longmapsto & (X, co^{\mathcal{C}}) \\ f & \longmapsto & f. \end{cases}$$

Proposition 3.9. *Let (X, co) be a hull space and define $\mathcal{C}^{co} = \{A \in 2^X \mid A = co(A)\}$. Then $\mathcal{C}^{co}$ is a convex structure on X.*

Proof. (C1) is obvious. We need only verify (C2) and (C3).

(C2) Take any $\{A_i\}_{i\in I} \subseteq \mathcal{C}^{co}$. Then for each $i \in I$, $co(A_i) = A_i$. By (H3), $co(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} co(A_i)$. In order to show the inverse inequality, take any $x \notin co(\bigcap_{i\in I} A_i) \supseteq \bigcap_{i\in I} A_i$. Then there exists $i_0 \in I$ such that $x \notin A_{i_0} = co(A_{i_0})$. This implies that $x \notin \bigcap_{i\in I} co(A_i)$. By the arbitrariness of x, we obtain $\bigcap_{i\in I} co(A_i) \subseteq co(\bigcap_{i\in I} A_i)$. Hence, it follows that $\bigcap_{i\in I} co(A_i) = co(\bigcap_{i\in I} A_i)$. This means that $\bigcap_{i\in I} A_i \in \mathcal{C}^{co}$.

(C3) Take any $\{A_j\}_{j\in J} \overset{dir}{\subseteq} \mathcal{C}^{co}$. Then $A_j = co(A_j)$ for each $j \in J$. By (H5), it follows that

$$co(\overset{dir}{\bigcup}_{j\in J} A_j) = \overset{dir}{\bigcup}_{j\in J} co(A_j) = \overset{dir}{\bigcup}_{j\in J} A_j.$$

This means that $\overset{dir}{\bigcup}_{j\in J} A_j \in \mathcal{C}^{co}$. □

Proposition 3.10. *If $f : (X, co_X) \longrightarrow (Y, co_Y)$ is a CP mapping, then so is $f : (X, \mathcal{C}^{co_X}) \longrightarrow (Y, \mathcal{C}^{co_Y})$.*

Proof. Since $f : (X, co_X) \longrightarrow (Y, co_Y)$ is a CP mapping, we have

$$\forall A \in 2^X,\ co_X(A) \subseteq f^{\leftarrow}(co_Y(f^{\rightarrow}(A))).$$

Then for each $B \in \mathcal{C}^{co_Y}$, it follows that

$$co_X(f^{\leftarrow}(B)) \subseteq f^{\leftarrow}(co_Y(f^{\rightarrow}(f^{\leftarrow}(B)))) \subseteq f^{\leftarrow}(co_Y(B)) = f^{\leftarrow}(B).$$

This implies that $co_X(f^{\leftarrow}(B)) = f^{\leftarrow}(B)$. Hence, $f^{\leftarrow}(B) \in \mathcal{C}^{co_X}$. □

By Propositions 3.9 and 3.10, we obtain a functor $\mathbb{G} : \mathbf{HS} \longrightarrow \mathbf{CS}$ as follows:

$$\mathbb{G} : \begin{cases} \mathbf{HS} & \longrightarrow & \mathbf{CS} \\ (X, co) & \longmapsto & (X, \mathcal{C}^{co}) \\ f & \longmapsto & f. \end{cases}$$

Theorem 3.11. *The category* **CS** *is isomorphic to the category* **HS**.

Proof. It suffices to show that $\mathbb{G}\circ\mathbb{F} = \mathbb{I}_{\mathbf{CS}}$ and $\mathbb{F}\circ\mathbb{G} = \mathbb{I}_{\mathbf{HS}}$. That is, for each $(X, \mathcal{C}) \in |\mathbf{CS}|$ and each $(X, co) \in |\mathbf{HS}|$, $co^{\mathcal{C}^{co}} = co$ and $\mathcal{C}^{co^{\mathcal{C}}} = \mathcal{C}$.

For each $A \in 2^X$, we have

$$co^{\mathcal{C}^{co}}(A) = \bigcap_{A \subseteq B \in \mathcal{C}^{co}} B = \bigcap_{A \subseteq B = co(B)} B = co(A)$$

and

$$A \in \mathcal{C}^{co^{\mathcal{C}}} \Longleftrightarrow A = co^{\mathcal{C}}(A) = \bigcap_{A \subseteq B \in \mathcal{C}} B \Longleftrightarrow A \in \mathcal{C}.$$

This completes the proof. □

4 The category of interval spaces

In this section, we will introduce the category of interval spaces with interval spaces as objects and with interval-preserving mappings as morphisms. Then we will study its categorical properties.

Definition 4.1 ([1]). An interval operator on X is a mapping $\mathcal{I} : X \times X \longrightarrow 2^X$ which satisfies:
(I1) $x, y \in \mathcal{I}(x, y)$;
(I2) $\mathcal{I}(x, y) = \mathcal{I}(y, x)$.
For an interval operator $\mathcal{I}$ on X, the pair $(X, \mathcal{I})$ is called an interval space.

Definition 4.2 ([1]). A mapping $f : (X, \mathcal{I}_X) \longrightarrow (Y, \mathcal{I}_Y)$ is called interval-preserving (IP in short) provided that

$$\forall x, y \in X,\ f^{\rightarrow}(\mathcal{I}_X(x, y)) \subseteq \mathcal{I}_Y(f(x), f(y)).$$

It is easy to check that all interval spaces and IP mappings form a category, denoted by **IS**.

Definition 4.3. For a nonempty set X, let $F_{\mathcal{I}}(X)$ denote the fibre

$$\{(X, \mathcal{I}) \mid \mathcal{I} \text{ is an interval operator on } X\}$$

of X. For interval spaces $(X, \mathcal{I}_1)$ and $(X, \mathcal{I}_2)$, we say $(X, \mathcal{I}_1)$ is finer than $(X, \mathcal{I}_2)$, or $(X, \mathcal{I}_2)$ is coarser than $(X, \mathcal{I}_1)$, denoted by $(X, \mathcal{I}_1) \leqslant_{\mathcal{I}} (X, \mathcal{I}_2)$, if the identity mapping $id_X : (X, \mathcal{I}_1) \longrightarrow (X, \mathcal{I}_2)$ is IP. We also write $\mathcal{I}_1 \leqslant_{\mathcal{I}} \mathcal{I}_2$.

Example 4.4. Let X be a nonempty set.
(1) Define $\mathcal{I}_\star : X \times X \longrightarrow 2^X$ by $\mathcal{I}_\star(x, y) = \{x, y\}$ for each $x, y \in X$. Then $\mathcal{I}_\star$ is the finest interval operator on X, which is called the discrete interval operator on X.
(2) Define $\mathcal{I}^\star : X \times X \longrightarrow 2^X$ by $\mathcal{I}^\star(x, y) = X$. Then $\mathcal{I}^\star$ is the coarsest interval operator on X, which is called the indiscrete interval operator on X.
(3) Suppose that $\mathbb{R}$ is the set of real numbers. Define $\mathcal{I}_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ by

$$\forall a, b \in \mathbb{R},\ \mathcal{I}_{\mathbb{R}}(a, b) = [min\{a, b\}, max\{a, b\}].$$

Then $\mathcal{I}_{\mathbb{R}}$ is an interval operator on $\mathbb{R}$.
(4) Suppose that d is a metric on X. Define $\mathcal{I}_d : X \times X \longrightarrow 2^X$ by

$$\forall x, y \in X,\ \mathcal{I}_d(x, y) = \{z \in X \mid d(x, y) = d(x, z) + d(z, y)\}.$$

Then $\mathcal{I}_d$ is an interval operator on X.

Proposition 4.5. *Let $(X, \mathcal{I}_X)$, $(Y, \mathcal{I}_Y)$ and $(Z, \mathcal{I}_Z)$ be interval spaces. If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are IP, then $g \circ f : (X, \mathcal{I}_X) \longrightarrow (Z, \mathcal{I}_Z)$ is IP.*

Proof. The proof is easy and omitted. □

In the category **IS**, important constructions like the formulations of products and subspaces, are always possible.

Theorem 4.6. *The category* **IS** *is topological over* **Set**.

Proof. We first prove the existence of initial structures. Let $((X_\lambda, \mathcal{I}_\lambda))_{\lambda\in\Lambda}$ be a family of interval spaces and let X be a nonempty set. Let further $((f_\lambda : X \longrightarrow (X_\lambda, \mathcal{I}_\lambda)))_{\lambda\in\Lambda}$ be a source. Define $\mathcal{I} : X \times X \longrightarrow 2^X$ by

$$\forall x, y \in X,\ \mathcal{I}(x, y) = \bigcap_{\lambda\in\Lambda} f_\lambda^{\leftarrow}(\mathcal{I}_\lambda(f_\lambda(x), f_\lambda(y))).$$

Then it is easy to verify that $\mathcal{I}$ is an interval operator on X.

Let further $(Y, \mathcal{I}_Y)$ be an interval space and $g : Y \longrightarrow X$ be a mapping. Assume that $f_\lambda \circ g$ is IP for all $\lambda \in \Lambda$. We then have for each $y_1, y_2 \in Y$ and for each $\lambda \in \Lambda$,

$$(f_\lambda \circ g)^{\rightarrow}(\mathcal{I}_Y(y_1, y_2)) \subseteq \mathcal{I}_\lambda(f_\lambda(g(y_1)), f_\lambda(g(y_2))).$$

From this we obtain

$$g^{\rightarrow}(\mathcal{I}_Y(y_1, y_2)) \subseteq \bigcap_{\lambda\in\Lambda} f_\lambda^{\leftarrow}(\mathcal{I}_\lambda(f_\lambda(g(y_1)), f_\lambda(g(y_2)))) = \mathcal{I}(g(y_1), g(y_2)).$$

This means that $g : (Y, \mathcal{I}_Y) \longrightarrow (X, \mathcal{I})$ is IP, as desired.

Secondly, the class of all interval operators on a fixed set X is a subset of $2^{((2^X)^{X\times X})}$, which means that the **IS** fibre of X is a set.

Finally, for a one point set $X = \{x\}$, there exists only one interval operator $\mathcal{I}$ on X, which is defined by $\mathcal{I}(x, x) = X$. Hence, **IS** satisfies the terminal separator property. Therefore, **IS** is a topological category over **Set**. □

Corollary 4.7. $(F_{\mathcal{I}}(X), \leqslant_{\mathcal{I}})$ *is a complete lattice.*

Example 4.8 (Product Spaces)**.** Let $\{(X_\lambda, \mathcal{I}_\lambda)\}_{\lambda\in\Lambda}$ be a family of interval spaces. The interval operator $\Pi - \mathcal{I}$ on $\prod_{\lambda\in\Lambda} X_\lambda$ which is initial with respect to the projections $(p_\lambda)_{\lambda\in\Lambda}$ is called the product interval operator and the pair $(\prod_{\lambda\in\Lambda} X_\lambda, \Pi - \mathcal{I})$ is called the product space. By definition, we have for $x, y \in \prod_{\lambda\in\Lambda} X_\lambda$,

$$\Pi - \mathcal{I}(x, y) = \bigcap_{\lambda\in\Lambda} p_\lambda^{\leftarrow}(\mathcal{I}_\lambda(p_\lambda(x), p_\lambda(y))) = \prod_{\lambda\in\Lambda} \mathcal{I}_\lambda(p_\lambda(x), p_\lambda(y)).$$

Example 4.9 (Subspaces)**.** Let $(X, \mathcal{I}_X)$ be an interval space and let $Y \subseteq X$. The interval operator $\mathcal{I}_Y$ on Y which is initial with respect to the inclusion mapping $id_Y : Y \longrightarrow X$ is called the sub-interval operator and the pair $(Y, \mathcal{I}_Y)$ is called the subspace of $(X, \mathcal{I}_X)$. By definition, we have for $x, y \in Y$,

$$\mathcal{I}_Y(x, y) = \mathcal{I}_X(x, y) \cap Y.$$

5 Relations between IS and CS

In this section, we will focus on the relations between **IS** and **CS**. In particular, we will propose a full subcategory of **CS**, consisted of arity 2 convex spaces and study its relations with interval spaces.

Definition 5.1. A convex space $(X, \mathcal{C})$ is called arity 2 if it satisfies

(AR2) $\forall A \in 2^X, \forall x, y \in A, co^{\mathcal{C}}(\{x, y\}) \subseteq A$ implies $A \in \mathcal{C}$.

Let **CS(2)** denote the full subcategory of **CS**, consisted of arity 2 convex spaces.

Next we will study the relations between **CS** (**CS(2)**) and **IS**.

Proposition 5.2. *Let* $(X, \mathcal{C})$ *be a convex space and define* $\mathcal{I}^{\mathcal{C}} : X \times X \longrightarrow 2^X$ *by*

$$\forall x, y \in X,\ \mathcal{I}^{\mathcal{C}}(x, y) = co^{\mathcal{C}}(\{x, y\}) = \bigcap_{x,y \in A \in \mathcal{C}} A.$$

Then $\mathcal{I}^{\mathcal{C}}$ *is an interval operator on* X.

Proof. The verifications of (I1) and (I2) are straightforward and omitted. □

Proposition 5.3. *If* $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ *is a CP mapping, then* $f : (X, \mathcal{I}^{\mathcal{C}_X}) \longrightarrow (Y, \mathcal{I}^{\mathcal{C}_Y})$ *is a IP mapping.*

Proof. Since $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is a CP mapping, it follows that $f^{\leftarrow}(B) \in \mathcal{C}_X$ for each $B \in \mathcal{C}_Y$. Then for each $x, y \in X$, we have

$$\begin{aligned} f^{\leftarrow}(\mathcal{I}^{\mathcal{C}_Y}(f(x), f(y))) &= \bigcap_{f(x), f(y) \in B \in \mathcal{C}_Y} f^{\leftarrow}(B) \\ &\supseteq \bigcap_{x,y \in f^{\leftarrow}(B) \in \mathcal{C}_X} f^{\leftarrow}(B) \\ &\supseteq \bigcap_{x,y \in A \in \mathcal{C}_X} A = \mathcal{I}^{\mathcal{C}_X}(x, y). \end{aligned}$$

This implies that $f^{\rightarrow}(\mathcal{I}^{\mathcal{C}_X}(x, y)) \subseteq \mathcal{I}^{\mathcal{C}_Y}(f(x), f(y))$, as desired. □

By Propositions 5.2 and 5.3, we obtain a functor $\mathbb{H}$ as follows:

$$\mathbb{H} : \begin{cases} \mathbf{CS} & \longrightarrow & \mathbf{IS} \\ (X, \mathcal{C}) & \longmapsto & (X, \mathcal{I}^{\mathcal{C}}) \\ f & \longmapsto & f. \end{cases}$$

Proposition 5.4. *Let* $(X, \mathcal{I})$ *be an interval space and define* $\mathcal{C}^{\mathcal{I}}$ *as follows:*

$$\mathcal{C}^{\mathcal{I}} = \{A \in 2^X \mid \forall x, y \in A,\ \mathcal{I}(x, y) \subseteq A\}.$$

Then $(X, \mathcal{C}^{\mathcal{I}})$ *is an arity 2 convex space.*

Proof. (C1) is obvious. We need only verify (C2), (C3) and (AR2).

(C2) Take any $\{A_i\}_{i \in I} \subseteq \mathcal{C}^{\mathcal{I}}$. Then for each $i \in I$ and for each $x, y \in A_i$, $\mathcal{I}(x, y) \subseteq A_i$. This implies that

$$\begin{aligned} x, y \in \bigcap_{i \in I} A_i &\Longleftrightarrow \forall i \in I,\ x, y \in A_i \\ &\Longleftrightarrow \forall i \in I,\ \mathcal{I}(x, y) \subseteq A_i \\ &\Longleftrightarrow \mathcal{I}(x, y) \subseteq \bigcap_{i \in I} A_i. \end{aligned}$$

Hence, $\bigcap_{i \in I} A_i \in \mathcal{C}^{\mathcal{I}}$.

(C3) Take any $\{A_j\}_{j \in J} \overset{dir}{\subseteq} \mathcal{C}^{\mathcal{I}}$. Then for each $x, y \in \overset{dir}{\bigcup}_{j \in J} A_j$, there exist $j_1, j_2 \in J$ such that $x \in A_{j_1}$ and $y \in A_{j_2}$. Since $\{A_j\}_{j \in J}$ is directed, there exists $j_3 \in J$ such that $A_{j_1} \subseteq A_{j_3}$ and $A_{j_2} \subseteq A_{j_3}$. This implies that $x, y \in A_{j_3} \in \mathcal{C}^{\mathcal{I}}$. Then it follows that $\mathcal{I}(x, y) \subseteq A_{j_3} \subseteq \overset{dir}{\bigcup}_{j \in J} A_j$. This means $\overset{dir}{\bigcup}_{j \in J} A_j \in \mathcal{C}^{\mathcal{I}}$.

(AR2) Take any $A \in 2^X$ such that

$$\forall x, y \in A,\ co^{\mathcal{C}^{\mathcal{I}}}(\{x, y\}) \subseteq A.$$

In order to show $A \in \mathcal{C}^{\mathcal{I}}$, take any $x, y \in A$. It follows that

$$co^{\mathcal{C}^{\mathcal{I}}}(\{x, y\}) = \bigcap_{x,y \in B \in \mathcal{C}^{\mathcal{I}}} B \supseteq \mathcal{I}(x, y).$$

Then we have $\mathcal{I}(x, y) \subseteq co^{\mathcal{C}^{\mathcal{I}}}(\{x, y\}) \subseteq A$ for each $x, y \in A$. This implies that $A \in \mathcal{C}^{\mathcal{I}}$, as desired. □

Proposition 5.5. *If $f : (X, \mathcal{I}_X) \longrightarrow (Y, \mathcal{I}_Y)$ is a IP mapping, then $f : (X, \mathcal{C}^{\mathcal{I}_X}) \longrightarrow (Y, \mathcal{C}^{\mathcal{I}_Y})$ is a CP mapping.*

Proof. Since $f : (X, \mathcal{I}_X) \longrightarrow (Y, \mathcal{I}_Y)$ is a IP mapping, it follows that

$$\forall x, y \in X,\ f^{\rightarrow}(\mathcal{I}_X(x, y)) \subseteq \mathcal{I}_Y(f(x), f(y)).$$

Then for each $B \in \mathcal{C}^{\mathcal{I}_Y}$, take any $x, y \in f^{\leftarrow}(B)$. It follows that $f(x), f(y) \in B \in \mathcal{C}^{\mathcal{I}_Y}$. This means that $\mathcal{I}_Y(f(x), f(y)) \subseteq B$. Further we have

$$\mathcal{I}_X(x, y) \subseteq f^{\leftarrow}(\mathcal{I}_Y(f(x), f(y))) \subseteq f^{\leftarrow}(B).$$

This implies that $f^{\leftarrow}(B) \in \mathcal{C}^{\mathcal{I}_X}$, as desired. □

By Propositions 5.4 and 5.5, we obtain a functor $\mathbb{K} : \mathbf{IS} \longrightarrow \mathbf{CS}$ as follows:

$$\mathbb{K} : \begin{cases} \mathbf{IS} & \longrightarrow & \mathbf{CS} \\ (X, \mathcal{I}) & \longmapsto & (X, \mathcal{C}^{\mathcal{I}}) \\ f & \longmapsto & f. \end{cases}$$

Theorem 5.6. *$(\mathbb{K}, \mathbb{H})$ is an adjunction between* **IS** *and* **CS**.

Proof. Since $\mathbb{K}$ and $\mathbb{H}$ are both concrete functors, we need only verify that $\mathbb{K} \circ \mathbb{H} \leqslant_{\mathcal{C}} \mathbb{I}_{\mathbf{CS}}$ and $\mathbb{H} \circ \mathbb{K} \geqslant_{\mathcal{I}} \mathbb{I}_{\mathbf{IS}}$. That is to say, for each $(X, \mathcal{C}) \in |\mathbf{CS}|$ and $(X, \mathcal{I}) \in |\mathbf{IS}|$, $\mathcal{C}^{\mathcal{I}^{\mathcal{C}}} \leqslant_{\mathcal{C}} \mathcal{C}$ and $\mathcal{I} \leqslant_{\mathcal{I}} \mathcal{I}^{\mathcal{C}^{\mathcal{I}}}$.

On one hand, take any $x, y \in X$. Then

$$\mathcal{I}(x, y) \subseteq \bigcap_{x,y \in A \in \mathcal{C}^{\mathcal{I}}} A = co^{\mathcal{C}^{\mathcal{I}}}(\{x, y\}) = \mathcal{I}^{\mathcal{C}^{\mathcal{I}}}(x, y).$$

On the other hand, take any $A \in 2^X$. Then

$$\begin{aligned} A \in \mathcal{C} &\Longrightarrow \forall x, y \in A,\ co^{\mathcal{C}}(\{x, y\}) \subseteq A \\ &\Longleftrightarrow \forall x, y \in A,\ \mathcal{I}^{\mathcal{C}}(x, y) \subseteq A \\ &\Longleftrightarrow A \in \mathcal{C}^{\mathcal{I}^{\mathcal{C}}}. \end{aligned}$$

This means that $\mathcal{C} \subseteq \mathcal{C}^{\mathcal{I}^{\mathcal{C}}}$, that is, $\mathcal{C}^{\mathcal{I}^{\mathcal{C}}} \leqslant_{\mathcal{C}} \mathcal{C}$, as desired. □

By Propositions 5.2 and 5.4, we know $\mathbb{K}^{\star} \triangleq \mathbb{K} : \mathbf{IS} \longrightarrow \mathbf{CS(2)}$ and $\mathbb{H}^{\star} \triangleq \mathbb{H}|_{\mathbf{CS(2)}} : \mathbf{CS(2)} \longrightarrow \mathbf{IS}$ are still functors. Moreover, we have the following result.

Theorem 5.7. *$(\mathbb{K}^{\star}, \mathbb{H}^{\star})$ is an adjunction between* **IS** *and* **CS(2)**. *Moreover, $\mathbb{K}^{\star}$ is a left inverse of $\mathbb{H}^{\star}$.*

Proof. By Theorem 5.6, it suffices to show that $\mathcal{C}^{\mathcal{I}^{\mathcal{C}}} = \mathcal{C}$ for each arity 2 convex space $(X, \mathcal{C})$. Take any $A \in 2^X$. Then

$$\begin{aligned} A \in \mathcal{C} &\Longleftrightarrow \forall x, y \in A,\ co^{\mathcal{C}}(\{x, y\}) \subseteq A \quad ((X, \mathcal{C}) \text{ is arity 2}) \\ &\Longleftrightarrow \forall x, y \in A, \mathcal{I}^{\mathcal{C}}(x, y) \subseteq A \\ &\Longleftrightarrow A \in \mathcal{C}^{\mathcal{I}^{\mathcal{C}}}. \end{aligned}$$

This means $\mathcal{C}^{\mathcal{I}^{\mathcal{C}}} = \mathcal{C}$. □

Corollary 5.8. *The category* **CS(2)** *can be embedded in the category* **IS** *as a reflective subcategory.*

6 Conclusions

In this paper we provided a categorical approach to abstract convex theory. On one hand, we introduced the axiomatic conditions of hull operators and showed that the resulting category is isomorphic to the category of convex spaces. On the other hand, we investigated the relations between convex spaces and interval spaces. We showed that there is an adjunction between the category of interval spaces and the category of convex spaces. Furthermore, the category of arity 2 convex spaces can be embedded in the category of interval spaces as a reflective subcategory. As is shown in this paper, category theory is an effective tool to deal with convex structures and interval operators. This also implies that category theory will be significant in the research on the theory of convex structures. In the future, we will consider applying category theory to fuzzy convex structures and establishing the relations between fuzzy convex structures and some other related structures.

Acknowledgement: This work is supported by the the Filling Project of Education Department of Heilongjiang Province (NO. 1352MSYYB008), the Scientific Research Project of Mudanjiang Normal University (NO. GP2017001), the Project of Shandong Province Higher Educational Science and Technology Program (NO. J18KA245).

References

[1] Van De Vel M., Theory of Convex Structures, 1993, North-Holland, Amsterdam
[2] Van De Vel M., Binary convexities and distributive lattices, Proc. London Math. Soc., 1984, 48, 3, 1–33
[3] Soltan V.P., d-convexity in graphs, Soviet Math. Dokl., 1983, 28, 419–421
[4] Van Mill J., Supercompactness and Wallman Spaces, 1997, Math. Centre Tracts 85, Amsterdam
[5] Jin Q., Li L., On the embedding of L-convex spaces in stratified L-convex spaces, SpringerPlus, 2016, 5(1), 1610
[6] Maruyama Y., Lattice-valued fuzzy convex geometry, RIMS Kokyuroku, 2009, 164, 22–37
[7] Pang B., Shi F.-G., Subcategories of the category of L-convex spaces, Fuzzy Sets Syst., 2017, 313, 61–74
[8] Pang B., Shi F.-G., Strong inclusion orders between L-subsets and its applications in L-convex spaces, Quaest. Math., 2018, 41, 8, 1021–1043
[9] Pang B., Shi F.-G., Fuzzy counterparts of hull operators and interval operators in the framework of L-convex spaces, Fuzzy Sets Syst. (2018), https:// doi.org/10.1016/j.fss.2018.05.012
[10] Pang B., Xiu Z.-Y., An axiomatic approach to bases and subbases in L-convex spaces and their applications, Fuzzy Sets Syst. (2018), https://doi.org/10.1016/j.fss.2018.08.002
[11] Pang B., Zhao Y., Characterizations of L-convex spaces, Iran. J. Fuzzy Syst., 2016, 13, 4, 51–61
[12] Pang B., Zhao Y., Xiu Z.-Y., A new definition of order relation for the introduction of algebraic fuzzy closure operators, Int. J. Approx. Reason., 2018, 92, 87–96
[13] Rosa M.V., On fuzzy topology fuzzy convexity spaces and fuzzy local convexity, Fuzzy Sets Syst., 1994, 62, 97–100
[14] Pang B., Xiu Z.-Y., Lattice-valued interval operators and its induced lattice-valued convex structures, IEEE T. Fuzzy Syst., 2018, 26,3, 1525–1534
[15] Shi F.-G., Li E.-Q., The restricted hull operator of M-fuzzifying convex structures, J. Intell. Fuzzy Syst., 2015, 30, 409–421
[16] Shi F.-G., Xiu Z.-Y., A new approach to the fuzzification of convex structures, Journal of Applied Mathematics, (2014), https://dx. doi.org/10.1155/2014/249183
[17] Wu X.-Y., Bai S.-Z., On M-fuzzifying JHC convex structures and M-fuzzifying Peano interval spaces, J. Intell. Fuzzy Syst., 2016, 30, 2447–2458
[18] Wu X.-Y., Li E.-Q., Bai S.-Z., Geometric properties of M-fuzzifying convex structures, J. Intell. Fuzzy Syst., 2017, 32, 4273–4284
[19] Xiu Z.-Y., Pang B., Base axioms and subbase axioms in M-fuzzifying convex spaces, Iran. J. Fuzzy Syst., 2018, 15, 2, 75–87
[20] Fang J.M., Stratified L-ordered convergence structures, Fuzzy Sets Syst., 2010, 161, 2130–2149
[21] Jäger G., A category of L-fuzzy convergence spaces, Quaest. Math., 2001, 24, 501–517
[22] Li L., Jin Q., p-Topologicalness and p-regularity for lattice-valued convergence spaces, Fuzzy Sets Syst., 2014, 238, 26–45
[23] Li L., Jin Q., Hu K., On stratified L-convergence spaces: Fischer's diagonal axiom, Fuzzy Sets Syst., 2015, 267, 31–40
[24] Pang B., On (L, M)-fuzzy convergence spaces, Fuzzy Sets Syst., 2014, 238, 46–70
[25] Pang B., Degrees of separation properties in stratified L-generalized convergence spaces using residual implication, Filomat, 2017, 31, 20, 6293–6305
[26] Pang B., Stratified L-ordered filter spaces, Quaest. Math., 2017, 40, 5, 661–678

[27] Pang B., Xiu Z.-Y., Stratified L-prefilter convergence structures in stratified L-topological spaces, Soft Comput., 2018, 22, 7539–7551
[28] Pang B., Categorical properties of L-fuzzifying convergence spaces, Filomat, 2018, 32, 11, 4021–4036
[29] Preuss G., Foundations of Topology–An Approach to Convenient Topology, Kluwer Academic Publisher, Dordrecht, Boston, London, 2002
[30] Adámek J., Herrlich H., Strecker G.E., Abstract and Concrete Categories, 1990, Wiley, New York