A category of modules for the full toroidal Lie algebra.

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Introduction.

Toroidal Lie algebras are very natural multi-variable generalizations of affine Kac-Moody algebras. The theory of affine Lie algebras is rich and beautiful, having connections with diverse areas of mathematics and physics. Toroidal Lie algebras are also proving themselves to be useful for the applications. Frenkel, Jing and Wang [FJW] used representations of toroidal Lie algebras to construct a new form of the McKay correspondence. Inami et al., studied toroidal symmetry in the context of a 4-dimensional conformal field theory [IKUX], [IKU]. There are also applications of toroidal Lie algebras to soliton theory. Using representations of the toroidal algebras one can construct hierarchies of non-linear PDEs [B2], [ISW]. In particular, the toroidal extension of the Korteweg-de Vries hierarchy contains the Bogoyavlensky's equation, which is not in the classical KdV hierarchy [IT]. One can use the vertex operator realizations to construct n-soliton solutions for the PDEs in these hierarchies. We hope that further development of the representation theory of toroidal Lie algebras will help to find new applications of this interesting class of algebras.

The construction of a toroidal Lie algebra is totally parallel to the well-known construction of an (untwisted) affine Kac-Moody algebra [K1]. One starts with a finite-dimensional simple Lie algebra $\dot{\mathfrak{g}}$ and considers Fourier polynomial maps from an $N+1$-dimensional torus into $\dot{\mathfrak{g}}$. Setting $t_k = e^{ix_k}$, we may identify the algebra of Fourier polynomials on a torus with the Laurent polynomial algebra $\mathcal{R} = \mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_N^{\pm}]$, and the Lie algebra of the $\dot{\mathfrak{g}}$-valued maps from a torus with the multi-loop algebra $\mathbb{C}[t_0^{\pm}, t_1^{\pm}, \ldots, t_N^{\pm}] \otimes \dot{\mathfrak{g}}$. When $N = 0$, this yields the usual loop algebra.

Just as for the affine algebras, the next step is to build the universal central extension $(\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K}$ of $\mathcal{R} \otimes \dot{\mathfrak{g}}$. However unlike the affine case, the center $\mathcal{K}$ is infinite-dimensional when $N \geq 1$. The infinite-dimensional center makes this Lie algebra highly degenerate. One can show, for example, that in an irreducible bounded weight module, most of the center should act trivially. To eliminate this degeneracy, we add the Lie algebra of vector fields on a torus, $\mathcal{D} = \mathrm{Der}\,(\mathcal{R})$ to $(\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K}$. The resulting algebra,

$$\mathfrak{g} = (\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K} \oplus \mathcal{D}$$

is called the full toroidal Lie algebra (see Section 1 for details). The action of $\mathcal{D}$ on $\mathcal{K}$ is non-trivial, making the center of the toroidal Lie algebra $\mathfrak{g}$ finite-dimensional. This enlarged algebra will have a much better representation theory.

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The most important class of modules for the affine Lie algebras are the highest weight modules, and one would certainly want to construct their toroidal analogs. The first problem that arises here is that one needs a triangular decomposition for the Lie algebra in order to introduce the notion of the highest weight module. Toroidal Lie algebras are graded by $\mathbb{Z}^{N+1}$, and for $N > 0$, there is no canonical way of dividing this lattice into positive and negative parts. This difficulty is not present for the affine Lie algebras, which are graded by $\mathbb{Z}$, and for $\mathbb{Z}$ such a splitting is natural.

One way to split $\mathbb{Z}^{N+1}$ is to cut it with a hyperplane that intersects with the lattice only at zero. The corresponding class of the highest weight modules was studied by Berman and Cox [BC], where it was found that the Verma modules constructed in this way will have infinite-dimensional weight spaces and do not produce any representations with interesting realizations. Modules corresponding to other decompositions of the lattice were studied in [DFP].

An extremal way of dividing the lattice is to cut it with a hyperplane that intersects $\mathbb{Z}^{N+1}$ at a sublattice of rank N. This approach was taken by Moody, Rao and Yokonuma, who constructed a homogeneous vertex operator realization of the basic module for the universal central extension of the multi-loop algebra [MRY]. In [EM], Rao and Moody showed how to get a representation of a bigger algebra on the same space, adding a subalgebra $\mathcal{D}^* = \mathop{\oplus}\limits_{p=1}^{N} \mathcal{R}\frac{\partial}{\partial t_p}$ of the Lie algebra of vector fields. One unanticipated development in [EM] was the appearance of a $\mathcal{K}$-valued 2-cocycle τ_1 on the Lie algebra of vector fields, which is an abelian generalization of the Virasoro cocycle. A principal realization for the basic module was given in [B1].

Developing further these ideas, Larsson succeeded in constructing a wider class of representations for the toroidal Lie algebras [L]. He showed that the basic module for the affine Lie algebra $\hat{\mathfrak{g}}$ may be replaced with an arbitrary highest weight module. Larsson also discovered that affine $\widehat{gl}_N$-modules can be used as an ingredient in these constructions. In Larsson's paper a combination of 2-cocycles τ_1 and τ_2 had appeared.

Berman and Billig developed a categorical approach to the representation theory of toroidal Lie algebras, introducing the generalized Verma modules for toroidal Lie algebras [BB]. They developed a theory of Lie algebras with polynomial multiplication, which allowed them to prove that simple quotients of the generalized Verma modules always have finite-dimensional weight spaces. Realizations of these irreducible quotients were obtained using a modified version of Larsson's construction.

A vertex algebra interpretation of these results was given by Berman, Billig and Szmigielski in [BBS].

Although the generalized Verma modules could be defined for the full toroidal algebras and the result of [BB] concerning finite-dimensional weight spaces also holds in full generality, there was one substantial difficulty unresolved in all of these publications. It was not known how to construct realizations for the modules over the full toroidal algebra – the piece $\mathcal{R}\frac{\partial}{\partial t_0}$ was always missing. This piece corresponds to the energy-momentum tensor in quantum field theories, so it is important to have it represented.

A class of modules for the full toroidal Lie algebras was constructed in [B4] (unpublished), and in the present paper we completely solve this problem.

Extended affine Lie algebras (EALAs) are another important family of algebras closely related to the toroidal Lie algebras. The main feature of the extended affine Lie algebras is the existence of a non-degenerate symmetric invariant bilinear form. Such a form does not exist on the full toroidal algebra, but it can be defined on its subalgebra $(\mathcal{R}\otimes\dot{\mathfrak{g}})\oplus\mathcal{K}\oplus\mathcal{D}_{\rm div}$, where $\mathcal{D}_{\rm div}$ is the Lie algebra of divergence zero vector fields on a torus. The results of the present paper make it possible to develop the representation theory for the toroidal EALA using restriction from the full toroidal algebras [B4], [B6].

It is shown in [ABFP] that most of the extended affine Lie algebras may be realized as twisted toroidal EALAs. The representation theory of the twisted toroidal EALAs is studied by Billig and Lau in [BL].

In the present paper we define a rather natural category $\mathcal{B}_\chi$ of bounded $\mathfrak{g}$-modules with finite-dimensional weight spaces with the central character χ. Our goal is to study irreducible modules in this category. We show that every irreducible module is characterized by its top T – the highest eigenspace for the operator $d_0 = t_0\frac{\partial}{\partial t_0}$. The space T is a submodule with respect to the subalgebra $\mathfrak{g}_0$ consisting of elements of $\mathfrak{g}$ of degree zero with respect to t_0. Following [BB], we define a generalized Verma module $M(T)$ and its irreducible quotient $L(T)$. We show (Theorem 2.5) that every irreducible module in category $\mathcal{B}_\chi$ is isomorphic to $L(T)$ for some irreducible $\mathfrak{g}_0$-module T with finite-dimensional weight spaces. Using the results of [JM], [E] and [B5], we see that such $\mathfrak{g}_0$-modules are precisely those considered in [BB] – they are multi-loop modules with respect to $\mathcal{R}_N\otimes\dot{\mathfrak{g}}$ and tensor modules with respect to Der $(\mathcal{R}_N)$.

Once we get a description of the tops T, we wish to completely determine the structure of the $\mathfrak{g}$-modules $L(T)$. This is done by constructing the vertex operator realizations of these modules. The crucial observation here is that the full toroidal Lie algebras are vertex Lie algebras. This allows us to construct the universal enveloping VOA $V_\mathfrak{g}$. We show (Proposition 4.2) that for a particular top T_0, the irreducible module $L(T_0)$ is a factor-VOA of $V_\mathfrak{g}$. Using the methods developed in [BB], we study the kernel of the projection $V_\mathfrak{g}\to L(T_0)$. This kernel gives us valuable information about $L(T_0)$. Once we determine that a vector $v\in V_\mathfrak{g}$ belongs to the kernel, we apply the state-field correspondence Y and conclude that $Y_{L(T_0)}(v,z)=0$. In this way we derive important relations that hold in $L(T_0)$. We use these relations to define a toroidal VOA $V(T_0)$ as a tensor product of a sub-VOA V^+_{Hyp} of a lattice VOA and a VOA $V_\mathfrak{f}$ corresponding to the twisted Virasoro-affine Lie algebra with $\dot{\mathfrak{f}}=\dot{\mathfrak{g}}\oplus gl_N$.

In the previous papers on this subject, the vertex operator realization for the toroidal modules had to be essentially guessed. The significant difference in the present approach is that we are able to derive all the properties of the vertex operator realizations from inside, using only the relations in the toroidal Lie algebra $\mathfrak{g}$ and its universal enveloping vertex algebra $V_\mathfrak{g}$.

The VOA $V(T_0)$ controls the representation theory of $\mathfrak{g}$. We show (Theorem 5.3) that every irreducible module $L(T)$ in category B_χ is a simple VOA module for $V(T_0)$ and can be constructed as a tensor product of a simple module $M^+_{Hyp}(\alpha)$ for the VOA V^+_{Hyp} with an irreducible highest weight module $L_\mathfrak{f}$ for the twisted Virasoro-affine algebra $\mathfrak{f}$. For a generic level c, we can further factor $L_\mathfrak{f}$ in a tensor product and get the following decomposition for $L(T)$:

$$L(T)\cong M^+_{Hyp}(\alpha)\otimes L_{\widehat{\dot{\mathfrak{g}}}}\otimes L_{\widehat{sl}_N}\otimes L_{\mathcal{H}ei}\otimes L_{\mathcal{V}ir},$$

where the last four factors are certain irreducible highest weight modules for the affine algebras $\widehat{\dot{\mathfrak{g}}}, \widehat{sl}_N$, the infinite-dimensional Heisenberg algebra and the Virasoro algebra. In this way we reduce the representation theory of toroidal Lie algebras to the representation theory of affine, Heisenberg and Virasoro algebras. Whenever explicit realizations are available for the components in the tensor product decomposition above, we get a realization for the irreducible module for the full toroidal Lie algebra.

This leads to the following open problem: while the explicit expressions for the characters of irreducible modules may be known, there is no Weyl-type character formula for the toroidal Lie algebras. Obtaining such a formula may yield interesting number-theoretic identities.

The structure of the paper is the following. In Section 1 we review the construction of the toroidal Lie algebras. In Section 2 we introduce a category $\mathcal{B}_\chi$ of $\mathfrak{g}$-modules and show that every irreducible module in $\mathcal{B}_\chi$ is characterized by its top T. We also describe the structure of the top T. In Section 3 we recall the definition of the vertex operator algebra and the construction of the universal enveloping vertex algebra of a vertex Lie algebra. We introduce twisted Virasoro-affine Lie algebras, show that these algebras are in fact vertex Lie algebras and describe the structure of the corresponding enveloping vertex algebras and their simple modules. At the end of Section 3 we describe the hyperbolic lattice VOA V_{Hyp} and its sub-VOA V_{Hyp}^+. In Section 4 we show that the full toroidal Lie algebra is vertex Lie algebra and define its enveloping VOA $V_\mathfrak{g}$. Next we establish a series of relations that hold in the simple quotient $L(T_0)$ and use them to decompose $L(T_0)$ into a tensor product of two VOAs, V_{Hyp}^+ and $L_\mathfrak{f}(\gamma_0)$. In the final Section 5 we prove that a slightly bigger VOA $V(T_0) = V_{Hyp}^+ \otimes V_\mathfrak{f}(\gamma_0)$ also admits the structure of a module over the full toroidal algebra $\mathfrak{g}$. We show that every irreducible $\mathfrak{g}$-module in category $\mathcal{B}_\chi$ is a simple VOA module for $V(T_0)$, which then allows us to obtain a complete description of these irreducible $\mathfrak{g}$-modules.

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1. Toroidal Lie algebras.

Toroidal Lie algebras are the natural multi-variable generalizations of affine Lie algebras. In this review of the toroidal Lie algebras we follow the work [BB]. Let $\dot{\mathfrak{g}}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$ with a non-degenerate invariant bilinear form $(\cdot|\cdot)$ and let $N \geq 1$ be an integer. We consider the Lie algebra $\mathcal{R} \otimes \dot{\mathfrak{g}}$ of maps from an $N+1$ dimensional torus into $\dot{\mathfrak{g}}$, where $\mathcal{R} = \mathbb{C}[t_0^\pm, t_1^\pm, \ldots, t_N^\pm]$ is the algebra of Fourier polynomials on a torus. The universal central extension of this Lie algebra may be described by means of the following construction which is due to Kassel [Kas]. Let $\Omega^1_\mathcal{R}$ be the space of 1-forms on a torus: $\Omega^1_\mathcal{R} = \mathop{\oplus}\limits_{p=0}^{N} \mathcal{R} dt_p$. We will choose the forms $\{k_p = t_p^{-1} dt_p | p = 0, \ldots, N\}$ as a basis of this free $\mathcal{R}$ module. There is a natural map d from the space of functions $\mathcal{R}$ into $\Omega^1_\mathcal{R}$: $d(f) = \sum\limits_{p=0}^{N} \frac{\partial f}{\partial t_p} dt_p = \sum\limits_{p=0}^{N} t_p \frac{\partial f}{\partial t_p} k_p$. The center $\mathcal{K}$ for the universal central extension $(\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K}$ of $\mathcal{R} \otimes \dot{\mathfrak{g}}$ is realized as

$$\mathcal{K} = \Omega^1_\mathcal{R} / d(\mathcal{R}),$$

and the Lie bracket is given by the formula

$$[f_1(t) g_1, f_2(t) g_2] = f_1(t) f_2(t) [g_1, g_2] + (g_1 | g_2) f_2 d(f_1).$$

Here and in the rest of the paper we will denote elements of $\mathcal{K}$ by the same symbols as elements of $\Omega^1_{\mathcal{R}}$, keeping in mind the canonical projection $\Omega^1_{\mathcal{R}} \to \Omega^1_{\mathcal{R}}/d(\mathcal{R})$.

Next we add to $(\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K}$ the algebra $\mathcal{D}$ of vector fields on the torus

$$\mathcal{D} = \mathop{\oplus}_{p=0}^{N} \mathcal{R} d_p,$$

where $d_p = t_p \frac{\partial}{\partial t_p}$. We will use the multi-index notation writing $t^r = t_0^{r_0} t_1^{r_1} \dots t_N^{r_N}$ for $r = (r_0, r_1, \dots, r_N)$, etc.

The natural action of $\mathcal{D}$ on $\mathcal{R} \otimes \dot{\mathfrak{g}}$

$$[t^r d_a, t^m g] = m_a t^{r+m} g \tag{1.1}$$

uniquely extends to the action on the universal central extension $(\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K}$ by

$$[t^r d_a, t^m k_b] = m_a t^{r+m} k_b + \delta_{ab} \sum_{p=0}^{N} r_p t^{r+m} k_p. \tag{1.2}$$

This corresponds to the Lie derivative action of the vector fields on 1-forms.

It turns out that there is still an extra degree of freedom in defining the Lie algebra structure on $(\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K} \oplus \mathcal{D}$. The Lie bracket on $\mathcal{D}$ may be twisted with a $\mathcal{K}$-valued 2-cocycle:

$$[t^r d_a, t^m d_b] = m_a t^{r+m} d_b - r_b t^{r+m} d_a + \tau(t^r d_a, t^m d_b). \tag{1.3}$$

In order to compute the second cohomology space $H^2(\mathcal{D}, \mathcal{K})$, one could use the Gelfand-Fuks cohomology theory [F], [T]. Unfortunately this theory only allows one to do the computations in the C^∞ setup, i.e., when $\mathcal{R}$ is replaced with the algebra of infinitely differentiable functions on a torus, and not for the algebra of Fourier polynomials that we consider here. For the C^∞ situation the calculation of $H^2_{C^\infty}(\mathcal{D}, \mathcal{K})$ has been carried out in [BN]. For the $(N+1)$-dimensional torus with $N+1 \geq 2$, the dimension of the second cohomology space is

$$\dim H^2_{C^\infty}(\mathcal{D}, \mathcal{K}) = 2 + \binom{N+1}{3},$$

and the following cocycles form the basis of this space:

$$\tau_1(t^r d_a, t^m d_b) = m_a r_b \sum_{p=0}^{N} m_p t^{r+m} k_p,$$

$$\tau_2(t^r d_a, t^m d_b) = r_a m_b \sum_{p=0}^{N} m_p t^{r+m} k_p,$$

together with a family $\{\eta_{abc} | 0 \leq a < b < c \leq N\}$, where the cocycle η_{abc} is defined by the conditions that

$$\eta_{abc}(t^r d_{\sigma(a)}, t^m d_{\sigma(b)}) = (-1)^\sigma t^{r+m} k_{\sigma(c)},$$

for any permutation $\sigma : \{a,b,c\} \to \{a,b,c\}$ and $\eta_{abc}(t^r d_i, t^m d_j) = 0$ if $i=j$ or $\{i,j\} \not\subset \{a,b,c\}$. It is clear that $H^2(\mathcal{D},\mathcal{K})$ in the algebraic setup contains the space $H^2_{C^\infty}(\mathcal{D},\mathcal{K})$. After twisting with a cocycle η_{abc}, the vector fields d_a and d_b no longer commute. For this reason we will be considering only the cocycles τ_1 and τ_2 in this paper.

We will write $\tau = \mu\tau_1 + \nu\tau_2$. The resulting algebra (or rather a family of algebras) is called the full toroidal Lie algebra

$$\mathfrak{g} = \mathfrak{g}(\mu,\nu) = (\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K} \oplus \mathcal{D}.$$

Note that after adding the algebra of derivations $\mathcal{D}$, the center $\mathcal{Z}$ of the toroidal Lie $\mathfrak{g}$ becomes finite-dimensional with the basis $\{k_0, k_1, \ldots, k_N\}$. This can be seen from the action (1.2) of $\mathcal{D}$ on $\mathcal{K}$, which is non-trivial.

The study representation theory of toroidal Lie algebras has begun in [MRY] and [EM], with further developments in [B1], [L], [BB], [BBS]. In all of these papers there was one common difficulty that has not been resolved – the representations constructed there were not for the full toroidal algebra $\mathfrak{g}$, but only for a subalgebra

$$\mathfrak{g}^* = (\mathcal{R} \otimes \dot{\mathfrak{g}}) \oplus \mathcal{K} \oplus \left(\mathop{\oplus}_{p=1}^{N} \mathcal{R} d_p \right),$$

where the piece $\mathcal{R}d_0$ that corresponds to the toroidal energy-momentum tensor was missing. This left the theory in a somewhat incomplete form, and the goal of the present paper is to study representations for the full toroidal Lie algebra.

2. A category of bounded modules for toroidal Lie algebras.

In this section we will introduce a category of bounded modules for toroidal Lie algebras that could be regarded as analogs of the highest weight modules for affine Kac-Moody algebras. The difference from the affine case is that the highest weight space is not 1-dimensional, but rather forms a multi-loop module for a smaller toroidal subalgebra.

These bounded modules are quite promising from the point of view of applications. In [B2] a module of this type was used to construct a toroidal extension of the Korteweg-de Vries hierarchy. The vertex operator realizations of the toroidal modules allow one to construct soliton solutions to these non-linear PDEs.

The variable t_0 will play a special role in our construction. From the physics perspective, it may be interpreted as time, whereas $t_1, \ldots t_N$ are the space variables.

The algebra $\mathfrak{g}$ has a $\mathbb{Z}^{N+1}$-grading by the eigenvalues of the adjoint action of $d_0, d_1, \ldots d_N$. We will denote by $\{\epsilon_0, \ldots, \epsilon_N\}$ the standard basis of $\mathbb{Z}^{N+1}$. We will be also considering its $\mathbb{Z}$-grading just with respect to the action of d_0:

$$\mathfrak{g} = \mathop{\oplus}_{n\in\mathbb{Z}} \mathfrak{g}_n,$$

and define subalgebras $\mathfrak{g}_\pm = \mathop{\oplus}_{n \gtrless 0} \mathfrak{g}_n$, which yields the decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$.

We recall that the center $\mathcal{Z}$ of $\mathfrak{g}$ is $N+1$-dimensional: $\mathcal{Z} = \mathrm{Span}\,\langle k_0, k_1, \ldots, k_N \rangle$. Clearly, in any irreducible weight module with finite-dimensional weight spaces, these central elements will act as multiplications by scalars. Let us fix a non-zero central character $\chi : \mathcal{Z} \to \mathbb{C}$ and define a category of bounded modules with central character χ for the toroidal Lie algebra.

Definition. A category $\mathcal{B}_\chi$ of bounded modules for the toroidal Lie algebra is a category whose objects are $\mathfrak{g}$-modules B satisfying the following axioms:

(B1) B has a weight decomposition with respect to the subalgebra $\langle d_0, d_1, \ldots, d_N \rangle$:

$$B = \bigoplus_{m \in \mathbb{C}^{N+1}} B_m,$$

where $B_m = \{ v \in B \,|\, d_j(v) = m_j v, \ j = 0, \ldots, N \}$;

(B2) All weight spaces B_m are finite-dimensional;

(B3) The action of the center $\mathcal{Z}$ on B is given by the central character χ: $k_j v = \chi(k_j) v$ for all $v \in B, j = 0, \ldots, N$;

(B4) Real parts of eigenvalues of d_0 on B are bounded from above.

The physical meaning of the last axiom is that the spectrum of the energy operator $E = -d_0$ has a lower bound, i.e., there exist states of the lowest energy.

The goal of this paper is to describe irreducible modules in category $\mathcal{B}_\chi$ and find their characters.

First of all we are going to show that in order for $\mathcal{B}_\chi$ to be non-trivial, the central character must satisfy the condition $\chi(k_1) = 0, \ldots, \chi(k_N) = 0$. Clearly, χ must vanish on an N-dimensional subspace in $\mathcal{Z}$, and it turns out that this N-dimensional nullspace must be "aligned" with the choice of the operator d_0 in axiom (B4).

Lemma 2.1. *Suppose that $\mathcal{B}_\chi$ is a non-trivial category. Then $\chi(k_j) = 0$ for all $j = 1, \ldots, N$.*

Proof. Let B be a non-zero module in $\mathcal{B}_\chi$. Let us reason by contradiction and assume that $\chi(k_j) = c_j \neq 0$ for some $j, \quad 1 \leq j \leq N$.

Since the spectrum of d_0 is bounded, we can find a weight space B_m such that $m_0 + 1$ is not an eigenvalue of d_0 on B. Let v be a non-zero vector in B_m, and consider the following family of vectors:

$$(t_0^{-1} t_j^{-n} k_j)(t_0^{-1} t_j^n k_j) v, \quad n = 1, 2, \ldots$$

Clearly, all these vectors belong to the same weight space $B_{m - 2\epsilon_0}$, and we claim that they are all linearly independent. Indeed, suppose

$$\sum_{n > 0} a_n (t_0^{-1} t_j^{-n} k_j)(t_0^{-1} t_j^n k_j) v = 0.$$

Since

$$d_0 (t_0 t_j^r d_0) v = (m_0 + 1)(t_0 t_j^r d_0) v,$$

and $m_0 + 1$ is not an eigenvalue of d_0 on B, we conclude that $(t_0 t_j^r d_0) v = 0$ for all $r \in \mathbb{Z}$. We also note that

$$[t_0 t_j^r d_0, t_0^{-1} t_j^s k_j] = -t_j^{r+s} k_j = -\delta_{r,-s} k_j.$$

Taking these two facts into account, we get that for $r > 0$,

$$0 = (t_0 t_j^r d_0)(t_0 t_j^{-r} d_0) \sum_{n>0} a_n (t_0^{-1} t_j^{-n} k_j)(t_0^{-1} t_j^n k_j) v = a_r c_j^2 v.$$

Since $c_j \neq 0$, we conclude that $a_r = 0$ for all $r > 0$. Thus the vectors $\left\{ (t_0^{-1} t_j^{-n} k_j)(t_0^{-1} t_j^n k_j) v \right\}$ with $n > 0$ are linearly independent, which contradicts (B2). This proves that $\chi(k_1) = 0, \ldots, \chi(k_N) = 0$.

For the rest of this paper we fix a non-zero constant $c \in \mathbb{C}$ and let $\chi = (c, 0, \ldots, 0)$. From now on, the multivariable t will not include t_0, that is t^r will stand for $t_1^{r_1} \ldots t_N^{r_N}$, etc.

Consider an irreducible module L in category $\mathcal{B}_\chi$. It is clear that the eigenvalues of d_0 on L belong to a single $\mathbb{Z}$-coset in $\mathbb{C}$. Let d be the eigenvalue of d_0 with the highest real part, and let T be the corresponding eigenspace.

Obviously, T is a $\mathfrak{g}_0$-module and $\mathfrak{g}_+ T = 0$. It is easy to see that irreducibility of L implies the irreducibility of T as a $\mathfrak{g}_0$-module. We will call the subspace T the *top* of L. Next we are going to describe the structure of T. We will be using a result of [JM] for this.

Theorem 2.2 ([JM]). *Suppose $\chi(k_0) = c \neq 0$, $\chi(k_1) = 0, \ldots, \chi(k_N) = 0$. Let L be an irreducible module in category $\mathcal{B}_\chi$ with the top T. Then*

$$T \cong \mathbb{C}[q_1^{\pm}, \ldots q_N^{\pm}] \otimes U,$$

where U is a finite-dimensional space, and the action of $\mathfrak{g}_0$ on T satisfies

$$(t^r k_0)(q^m \otimes u) = c q^{m+r} \otimes u, \qquad (t^r k_j)(q^m \otimes u) = 0, \tag{2.1}$$

$$d_0(q^m \otimes u) = d q^m \otimes u, \quad d_j(q^m \otimes u) = (m_j + \alpha_j) q^m \otimes u, \quad u \in U, j = 1, \ldots, N, \tag{2.2}$$

for some fixed $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{C}^N, d \in \mathbb{C}$.

If we take the quotient of $\mathfrak{g}_0$ by the ideal $J = \mathrm{Span}\left\{ t^r k_j | r \in \mathbb{Z}^N, j = 1, \ldots, N \right\}$, which annihilates T, then we will get a semi-direct product of the Lie algebra of vector fields $\mathcal{D}_N = \mathrm{Der}\, \mathbb{C}[t_1^{\pm}, \ldots, t_N^{\pm}]$ on N-dimensional torus with a multi-loop algebra:

$$\mathfrak{g}_0 / J \cong \mathcal{D}_N \ltimes \mathbb{C}[t_1^{\pm}, \ldots, t_N^{\pm}] \otimes (\dot{\mathfrak{g}} \oplus \mathbb{C} d_0 \oplus \mathbb{C} k_0).$$

Since $\frac{1}{c}(t^r k_0)$ acts on T as multiplication by q^r, we can derive from (1.2) the following compatibility relations between the action of $\mathfrak{g}_0$ and the operators of multiplication by q^r:

$$(t^s d_j) q^r - q^r (t^s d_j) = r_j q^{s+r}, \tag{2.3}$$

$$(t^s d_0) q^r = q^r (t^s d_0), \qquad (t^s g) q^r = q^r (t^s g), \quad g \in \dot{\mathfrak{g}}. \tag{2.4}$$

Eswara Rao [E] classified irreducible $\mathcal{D}_N$-modules with a compatible action of the algebra of Laurent polynomials, proving that any such module is a tensor module. We will use a version of this result for the semidirect product of $\mathcal{D}_N$ with a multi-loop algebra given in [B5], Theorem 4(c):

Theorem 2.3 ([E], [B5]). *Let $\alpha \in \mathbb{C}^N, c, d \in \mathbb{C}$, $c \neq 0$. Let T be an irreducible $\mathfrak{g}_0$-module satisfying the conclusion of Theorem 2.2, as well as* (2.3), (2.4). *Then there exist a finite-dimensional irreducible $\dot{\mathfrak{g}}$-module V and a finite-dimensional irreducible gl_N-module W, such that*

$$T \cong \mathbb{C}[q_1^{\pm}, \dots q_N^{\pm}] \otimes V \otimes W, \tag{2.5}$$

and the action of $\mathfrak{g}_0$ on T is given by (2.1) and

$$(t^r d_j)(q^m \otimes v \otimes w) = (m_j + \alpha_j) q^{m+r} \otimes v \otimes w + \sum_{p=1}^{N} r_p q^{m+r} \otimes v \otimes E_{pj} w, \quad j = 1, \dots, N, \tag{2.6}$$

$$(t^r d_0)(q^m \otimes v \otimes w) = d q^{m+r} \otimes v \otimes w, \tag{2.7}$$

$$(t^r g)(q^m \otimes v \otimes w) = q^{m+r} \otimes g v \otimes w, \quad g \in \dot{\mathfrak{g}}. \tag{2.8}$$

Here in (2.6) E_{pj} denotes a matrix with 1 in position (p, j), and zeros elsewhere.

Combining these two theorems, we conclude that an irreducible module in category $\mathcal{B}_\chi$ yields the following data – a finite-dimensional irreducible $\dot{\mathfrak{g}}$-module V, a finite-dimensional irreducible gl_N-module W, a constant $d \in \mathbb{C}$ and $\alpha \in \mathbb{C}^N$. It is easy to see that the choice of α is not canonical and α can be changed to any value in the coset $\alpha + \mathbb{Z}^N$ by choosing a different weight space for the generators of T as a free $\mathbb{C}[q_1^{\pm}, \dots, q_N^{\pm}]$-module. An irreducible gl_N-module W is determined by the action of sl_N and a scalar h, by which the identity matrix acts on W.

It was shown in [BB] that for any $\mathfrak{g}_0$-module T, corresponding to the data (V, W, h, d, α) as above, there exists an irreducible module in $\mathcal{B}_\chi$ with T as a top. Let us review this construction.

First we let $\mathfrak{g}_+$ act on T trivially, and define the generalized Verma module as the induced module

$$M(T) = \mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_+}^{\mathfrak{g}} (T).$$

Note that the module $M(T)$ does not belong to category $\mathcal{B}_\chi$ since its weight spaces lying below T are infinite-dimensional. Nonetheless, the following result holds:

Theorem 2.4 ([BB]). *(a) The $\mathfrak{g}$-module $M(T)$ has a unique maximal submodule M^{rad}.*

(b) The factor-module $L(T) = M(T)/M^{rad}$ is an irreducible $\mathfrak{g}$-module.

(c) All weight spaces of $L(T)$ are finite-dimensional, and $L(T)$ belongs to the category $\mathcal{B}_\chi$.

Summarizing, we get the following

Theorem 2.5. *(a) Let χ be a non-zero central character $\chi : \mathcal{Z} \to \mathbb{C}$. A category $\mathcal{B}_\chi$ is non-trivial if and only if $\chi(k_0) = c$, $\chi(k_1) = 0, \dots, \chi(k_N) = 0$ for some non-zero $c \in \mathbb{C}$.*

Let now $\chi = (c, 0, \dots, 0)$ with $c \neq 0$.

(b) Irreducible $\mathfrak{g}$-modules in category $\mathcal{B}_\chi$ are in 1-1 correspondence with the data (V, W, h, d, α), where V is a finite-dimensional irreducible $\dot{\mathfrak{g}}$-module, W is a finite-dimensional irreducible sl_N-module, $\alpha \in \mathbb{C}^N/\mathbb{Z}^N$, $h, d \in \mathbb{C}$.

(c) Every irreducible module in category $\mathcal{B}_\chi$ is isomorphic to $L(T)$ where

$$T = \mathbb{C}[q_1^{\pm}, \dots q_N^{\pm}] \otimes V \otimes W$$

with the action of $\mathfrak{g}_0$ given by (2.1) and (2.6)-(2.8).

Proof. Part (a) has been already proved in Lemma 2.1. Let us prove part (c). As we have seen above, an irreducible module L in category $\mathcal{B}_\chi$ has a top T, the structure of which is described by Theorem 2.3. Thus L is a factor-module of $M(T)$. However $M(T)$ has a unique irreducible factor, which is isomorphic to $L(T)$. This proves $L \cong L(T)$. Part (b) follows from (c) and Theorem 2.4.

Our main goal will be to completely determine the structure of the irreducible modules $L(T)$, and in particular, to find their characters. This will be done using the theory of vertex operator algebras (VOAs). We will show that for a particular choice of the data (V, W, h, d, α), namely, with V and W being trivial 1-dimensional modules for $\dot{\mathfrak{g}}$ and sl_N, $\alpha = 0$, $h = N\nu c$ and $d = \frac{1}{2}(\mu + \nu)c$, yielding the top

$$T_0 = \mathbb{C}[q_1^{\pm}, \ldots, q_N^{\pm}],$$

the module $L(T_0)$ is a vertex operator algebra, while all irreducible modules $L(T)$ are VOA-modules for a slightly bigger VOA $V(T_0)$. Once we determine the structure of $V(T_0)$ as a VOA, we will immediately get the structure of all the modules $L(T)$ using the principle of preservation of identities in the VOA theory.

3. Vertex operator algebras and vertex Lie algebras.

3.1. Definitions and properties of a VOA.

Let us recall the basic notions of the theory of the vertex operator algebras. Here we are following [K2] and [Li].

Definition. A vertex algebra is a vector space V with a distinguished vector $\mathbf{1}$ (vacuum vector) in V, an operator D (infinitesimal translation) on the space V, and a linear map Y (state-field correspondence)

$$\begin{aligned} Y(\cdot, z): \quad & V \to (\mathrm{End} V)[[z, z^{-1}]], \\ & a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \quad (\text{where } \ a_{(n)} \in \mathrm{End} V), \end{aligned}$$

such that the following axioms hold:

(V1) For any $a, b \in V$, $\quad a_{(n)} b = 0$ for n sufficiently large;

(V2) $[D, Y(a, z)] = Y(D(a), z) = \frac{d}{dz} Y(a, z)$ for any $a \in V$;

(V3) $Y(\mathbf{1}, z) = \mathrm{Id}_V$;

(V4) $Y(a, z)\mathbf{1} \in V[[z]]$ and $Y(a, z)\mathbf{1}|_{z=0} = a$ for any $a \in V$ (self-replication);

(V5) For any $a, b \in V$, the fields $Y(a, z)$ and $Y(b, z)$ are mutually local, that is,

$$(z - w)^n \, [Y(a, z), Y(b, w)] = 0, \quad \text{for } \ n \text{ sufficiently large.}$$

A vertex algebra V is called a vertex operator algebra (VOA) if, in addition, V contains a vector ω (Virasoro element) such that

(V6) The components $L(n) = \omega_{(n+1)}$ of the field

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} \omega_{(n)} z^{-n-1} = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$$

satisfy the Virasoro algebra relations:

$$[L(n), L(m)] = (n-m)L(n+m) + \delta_{n,-m}\frac{n^3-n}{12}(\mathrm{rank}\, V)\mathrm{Id}, \quad \text{where } \ \mathrm{rank}\, V \in \mathbb{C}; \tag{3.1}$$

(V7) $D = L(-1)$;
(V8) V is graded by the eigenvalues of $L(0)$: $V = \underset{n\in\mathbb{Z}}{\oplus} V_n$ with $L(0)\big|_{V_n} = n\mathrm{Id}$.

This completes the definition of a VOA.

As a consequence of the axioms of the vertex algebra we have the following important commutator formula:

$$[Y(a, z_1), Y(b, z_2)] = \sum_{n\geq 0} \frac{1}{n!} Y(a_{(n)}b, z_2) \left[z_1^{-1} \left(\frac{\partial}{\partial z_2}\right)^n \delta\left(\frac{z_2}{z_1}\right)\right]. \tag{3.2}$$

As usual, the delta function is

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n.$$

By (V1), the sum in the right hand side of the commutator formula is actually finite.

All the vertex operator algebras that appear in this paper have the gradings by non-negative integers (degree): $V = \overset{\infty}{\underset{n=0}{\oplus}} V_n$. In this case the sum in the right hand side of the commutator formula (3.2) runs from $n = 0$ to $n = \deg(a) + \deg(b) - 1$, because

$$\deg(a_{(n)}b) = \deg(a) + \deg(b) - n - 1, \tag{3.3}$$

and the elements of negative degree vanish.

It follows from (V7) and (V8) that

$$\omega_{(0)}a = D(a), \quad \omega_{(1)}a = \deg(a)a \qquad \text{for } a \text{ homogeneous.} \tag{3.4}$$

Another consequence of the axioms of a vertex algebra is the Borcherds' identity:

$$\sum_{j\geq 0} \binom{m}{j} (a_{(k+j)}b)_{(m+n-j)}c$$

$$= \sum_{j\geq 0}(-1)^{k+j+1}\binom{k}{j} b_{(n+k-j)}a_{(m+j)}c + \sum_{j\geq 0}(-1)^j \binom{k}{j} a_{(m+k-j)}b_{(n+j)}c, \qquad k, m, n \in \mathbb{Z}. \tag{3.5}$$

We will particularly need its special case when $m = 0$:

$$(a_{(k)}b)_{(n)}c = \sum_{j\geq 0}(-1)^{k+j+1}\binom{k}{j} b_{(n+k-j)}a_{(j)}c + \sum_{j\geq 0}(-1)^{j}\binom{k}{j} a_{(k-j)}b_{(n+j)}c, \qquad k,n\in\mathbb{Z}. \tag{3.6}$$

The last formula that we quote here is the skew-symmetry identity:

$$a_{(n)}b = \sum_{j\geq 0}(-1)^{n+j+1}\frac{1}{j!}D^j(b_{(n+j)}a). \tag{3.7}$$

3.2. Tensor products of VOAs.

Let us review here the definition of the tensor product of two VOAs $(V', Y', \omega', \mathbf{1})$ and $(V'', Y'', \omega'', \mathbf{1})$ (the case of an arbitrary number of factors is a trivial generalization). The tensor product space $V = V' \otimes V''$ has the VOA structure under

$$Y(a\otimes b, z) = Y'(a,z)\otimes Y''(b,z), \tag{3.8}$$

$$\omega = \omega'\otimes\mathbf{1} + \mathbf{1}\otimes\omega'', \tag{3.9}$$

and $\mathbf{1} = \mathbf{1}\otimes\mathbf{1}$ being the identity element.

It follows from (3.9) that the rank of V (see V6) is the sum of the ranks of the tensor factors.

3.3. Vertex Lie algebras.

An important source of the vertex algebras is provided by the vertex Lie algebras. In presenting this construction we will be following [DLM] (see also [P], [R], [K2], [FKRW]).

Let $\mathcal{L}$ be a Lie algebra with the basis $\{u(n), c(-1) \big| u\in\mathcal{U}, c\in\mathcal{C}, n\in\mathbb{Z}\}$ ($\mathcal{U}$, $\mathcal{C}$ are some index sets). Define the corresponding fields in $\mathcal{L}[[z,z^{-1}]]$:

$$u(z) = \sum_{n\in\mathbb{Z}} u(n)z^{-n-1}, \quad c(z) = c(-1)z^0, \quad u\in\mathcal{U}, c\in\mathcal{C}.$$

Let $\mathcal{F}$ be a subspace in $\mathcal{L}[[z,z^{-1}]]$ spanned by all the fields $u(z), c(z)$ and their derivatives of all orders.

Definition. *A Lie algebra $\mathcal{L}$ with the basis as above is called a vertex Lie algebra if the following two conditions hold:*

(VL1) for all $u_1, u_2 \in \mathcal{U}$,

$$[u_1(z_1), u_2(z_2)] = \sum_{j=0}^{n} f_j(z_2)\left[z_1^{-1}\left(\frac{\partial}{\partial z_2}\right)^j \delta\left(\frac{z_2}{z_1}\right)\right], \tag{3.10}$$

where $f_j(z)\in\mathcal{F}, n\geq 0$ and depend on u_1, u_2,

(VL2) for all $c\in\mathcal{C}$, the elements $c(-1)$ are central in $\mathcal{L}$.

Let $\mathcal{L}^{(+)}$ be a subspace in $\mathcal{L}$ with the basis $\{u(n) \big| u \in \mathcal{U}, n \geq 0\}$ and let $\mathcal{L}^{(-)}$ be a subspace with the basis $\{u(n), c(-1) \big| u \in \mathcal{U}, c \in \mathcal{C}, n < 0\}$. Then $\mathcal{L} = \mathcal{L}^{(+)} \oplus \mathcal{L}^{(-)}$ and $\mathcal{L}^{(+)}, \mathcal{L}^{(-)}$ are in fact subalgebras in $\mathcal{L}$.

The universal enveloping vertex algebra $V_{\mathcal{L}}$ of a vertex Lie algebra $\mathcal{L}$ is defined as an induced module

$$V_{\mathcal{L}} = \mathrm{Ind}_{\mathcal{L}^{(+)}}^{\mathcal{L}}(\mathbb{C}\mathbf{1}) = U(\mathcal{L}^{(-)}) \otimes \mathbf{1},$$

where $\mathbb{C}\mathbf{1}$ is a trivial 1-dimensional $\mathcal{L}^{(+)}$ module.

Theorem 3.1. ([DLM], Theorem 4.8) *Let $\mathcal{L}$ be a vertex Lie algebra. Then*

(a) $V_{\mathcal{L}}$ has a structure of a vertex algebra with the vacuum vector $\mathbf{1}$, infinitesimal translation D being a natural extension of the derivation of $\mathcal{L}$ given by $D(u(n)) = -nu(n-1)$, $D(c(-1)) = 0$, $u \in \mathcal{U}$, $c \in \mathcal{C}$, and the state-field correspondence map Y defined by the formula:

$$Y\left(a_1(-1-n_1)\dots a_{k-1}(-1-n_{k-1})a_k(-1-n_k)\mathbf{1}, z\right)$$

$$=: \left(\frac{1}{n_1!}\left(\frac{\partial}{\partial z}\right)^{n_1} a_1(z)\right) \dots : \left(\frac{1}{n_{k-1}!}\left(\frac{\partial}{\partial z}\right)^{n_{k-1}} a_{k-1}(z)\right)\left(\frac{1}{n_k!}\left(\frac{\partial}{\partial z}\right)^{n_k} a_k(z)\right) : \dots : \quad , \tag{3.11}$$

where $a_j \in \mathcal{U}, n_j \geq 0$ or $a_j \in \mathcal{C}, n_j = 0$.

(b) Any bounded $\mathcal{L}$-module is a vertex algebra module for $V_{\mathcal{L}}$.

(c) For an arbitrary character $\gamma : \mathcal{C} \to \mathbb{C}$, the factor module

$$V_{\mathcal{L}}(\gamma) = U(\mathcal{L}^{(-)})\mathbf{1}/U(\mathcal{L}^{(-)})\big\langle (c(-1) - \gamma(c))\mathbf{1} \big\rangle_{c \in \mathcal{C}}$$

is a quotient vertex algebra.

(d) Any bounded $\mathcal{L}$-module in which $c(-1)$ act as $\gamma(c)\mathrm{Id}$, for all $c \in \mathcal{C}$, is a vertex algebra module for $V_{\mathcal{L}}(\gamma)$.

In the formula (3.11) above, the normal ordering of two fields $: a(z)b(z) :$ is defined as

$$: a(z)b(z) := \sum_{n<0} a_{(n)}z^{-n-1}b(z) + \sum_{n \geq 0} b(z)a_{(n)}z^{-n-1}.$$

Note the following relation that we will implicitly use throughout the paper: $(a(-1)\mathbf{1})_{(n)} = a(n)$ for $a \in \mathcal{U}$, $n \in \mathbb{Z}$.

Theorem 3.2. *Any D-invariant $\mathcal{L}$-submodule U in $V_{\mathcal{L}}$ is a vertex algebra ideal in $V_{\mathcal{L}}$. Conversely, every vertex algebra ideal in $V_{\mathcal{L}}$ is a D-invariant $\mathcal{L}$-submodule.*

Proof. Let us prove the first part of the Theorem. We need to show that for any $u \in U$, $a \in V_{\mathcal{L}}$, $n \in \mathbb{Z}$, we have $a_{(n)}u \in U$ and $u_{(n)}a \in U$. By (3.7), it is enough to prove that $a_{(n)}u \in U$. It is sufficient to consider a of the form $a = a_1(-1-n_1)\dots a_k(-1-n_k)\mathbf{1}$, where $a_j \in \mathcal{U} \cup \mathcal{C}$, $n_j \geq 0$. We use induction on k. For $k = 0$ we get that $a = \mathbf{1}$ and $\mathbf{1}_{(n)}u = \delta_{n,-1}u$. The inductive step follows from the Borcherds' formula (3.6) and $\mathcal{L}$-invariance of U. The second part of the Theorem follows immediately from the definition of $V_{\mathcal{L}}$.

Corollary 3.3. *If U is a maximal D-invariant $\mathcal{L}$-submodule in $V_{\mathcal{L}}$ then the quotient $L_{\mathcal{L}} = V_{\mathcal{L}}/U$ is a simple vertex algebra.*

Remark 3.4. In case when the set $\mathcal{U}$ contains an element ω generating the Virasoro field $\omega(z)$ in $\mathcal{L}$, satisfying $[\omega(0), a(n)] = -na(n-1)$ for all $a \in \mathcal{U}$, the vertex algebra $V_{\mathcal{L}}$ becomes a VOA, and in the statement of Theorem 3.2 the condition of D-invariance of U will automatically follow from its $\mathcal{L}$-invariance.

3.4. VOA associated with the twisted Virasoro-affine algebra.

The toroidal VOA that will be constructed in Section 4, decomposes into a tensor product of two VOAs. One of these factors is a VOA associated with a twisted Virasoro-affine Lie algebra, which we introduce here.

Let $\dot{\mathfrak{f}}$ be a finite-dimensional reductive Lie algebra. Consider a semi-direct product of the Lie algebra of vector fields on a circle with a loop algebra:

$$\tilde{\mathfrak{f}} = \mathrm{Der}\, \mathbb{C}[t_0, t_0^{-1}] \ltimes \mathbb{C}[t_0, t_0^{-1}] \otimes \dot{\mathfrak{f}}.$$

A *twisted Virasoro-affine algebra* $\mathfrak{f}$ is the universal central extension of the Lie algebra $\tilde{\mathfrak{f}}$. Using the results on the central extensions of the loop algebras and the Lie algebra of vector fields on a circle, one can show that the second cohomology of $\tilde{\mathfrak{f}}$ has the following description:

$$H^2(\tilde{\mathfrak{f}}) = S^2(\dot{\mathfrak{f}})^{inv} \oplus \dot{\mathfrak{f}}^{inv} \oplus \mathbb{C},$$

where the last 1-dimensional component corresponds to the Virasoro cocycle on $\mathrm{Der}\, \mathbb{C}[t_0, t_0^{-1}]$, and its generator will be denoted by $C_{\mathcal{V}ir}$. Since $\dot{\mathfrak{f}}$ is reductive, we have the following canonical projections of $\dot{\mathfrak{f}}$-modules:

$$\varphi : \quad \dot{\mathfrak{f}} \otimes \dot{\mathfrak{f}} \to S^2(\dot{\mathfrak{f}})^{inv}$$

and

$$\psi : \quad \dot{\mathfrak{f}} \to Z(\dot{\mathfrak{f}}) = \dot{\mathfrak{f}}^{inv}.$$

We will use this maps to write down the Lie bracket in the twisted Virasoro-affine algebra $\mathfrak{f} = \tilde{\mathfrak{f}} \oplus S^2(\dot{\mathfrak{f}})^{inv} \oplus \dot{\mathfrak{f}}^{inv} \oplus \mathbb{C}$:

$$[L(n), L(m)] = (n-m)L(n+m) + \frac{n^3 - n}{12}\delta_{n,-m} C_{\mathcal{V}ir}, \tag{3.12}$$

$$[L(n), f(m)] = -mf(n+m) - (n^2+n)\delta_{n,-m}\psi(f), \tag{3.13}$$

$$[f(n), g(m)] = [f,g](n+m) + n\delta_{n,-m}\varphi(f \otimes g), \quad f, g \in \dot{\mathfrak{f}}. \tag{3.14}$$

Here and below we are using the notations $L(n) = -t_0^{n+1}\frac{d}{dt_0}$ and $f(n) = t_0^n \otimes f$ for $f \in \dot{\mathfrak{f}}$.

Consider the following fields in $\mathfrak{f}$:

$$\omega(z) = \sum_{n \in \mathbb{Z}} \omega(n) z^{-n-1} = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$$

and

$$f(z) = \sum_{n \in \mathbb{Z}} f(n) z^{-n-1}, \quad \text{for} \quad f \in \dot{\mathfrak{f}}.$$

Proposition 3.5. *Twisted Virasoro-affine Lie algebra $\mathfrak{f}$ is a vertex Lie algebra.*

Proof. Take for a set $\mathcal{U}$ the element ω together with a basis of $\dot{\mathfrak{f}}$, and for a set $\mathcal{C}$ a basis of $S^2(\dot{\mathfrak{f}})^{inv} \oplus \dot{\mathfrak{f}}^{inv} \oplus \mathbb{C}$. Then the defining relations (3.12)-(3.14) may be rewritten as follows:

$$\begin{aligned}[][\omega(z_1), \omega(z_2)] = \left(\frac{\partial}{\partial z_2}\omega(z_2)\right)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right] + 2\omega(z_2)\left[z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)\right] \\ + \frac{C_{\mathcal{V}ir}}{12}\left[z_1^{-1}\left(\frac{\partial}{\partial z_2}\right)^3\delta\left(\frac{z_2}{z_1}\right)\right],\end{aligned} \tag{3.15}$$

$$\begin{aligned}[][\omega(z_1), f(z_2)] = \left(\frac{\partial}{\partial z_2}f(z_2)\right)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right] + f(z_2)\left[z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)\right] \\ - \psi(f)\left[z_1^{-1}\left(\frac{\partial}{\partial z_2}\right)^2\delta\left(\frac{z_2}{z_1}\right)\right],\end{aligned} \tag{3.16}$$

$$[f(z_1), g(z_2)] = [f,g](z_2)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right] + \varphi(f\otimes g)\left[z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)\right]. \tag{3.17}$$

This shows that $\mathfrak{f}$ is indeed a vertex Lie algebra, and the claim of the Proposition is established.

From now on we fix $\dot{\mathfrak{f}}$ to be $\dot{\mathfrak{f}} = \dot{\mathfrak{g}} \oplus gl_N$.

A linear map $S^2(\dot{\mathfrak{f}})^{inv} \to \mathbb{C}$ defines a symmetric invariant bilinear form on $\dot{\mathfrak{f}}$. Thus $S^2(\dot{\mathfrak{f}})^{inv}$ is the dual space to the space of symmetric invariant forms on $\dot{\mathfrak{f}}$ and has dimension 3. Let us fix an invariant form on $\dot{\mathfrak{g}}$ normalized by the condition that $(\alpha|\alpha) = 2$ for the long roots of $\dot{\mathfrak{g}}$, an invariant form on sl_N with the same normalization, and a form on the space of scalar matrices normalized by its value on the identity matrix: $(I|I) = 1$. Denote by $\{C_{\dot{\mathfrak{g}}}, C_{sl_N}, C_{\mathcal{H}ei}\}$ the dual basis in $S^2(\dot{\mathfrak{f}})^{inv}$.

The space $\dot{\mathfrak{f}}^{inv} \cong Z(\dot{\mathfrak{f}})$ is one-dimensional, and we will denote its generator $\psi(I)$ by $C_{\mathcal{VH}}$. Hence the space $H^2(\tilde{\mathfrak{f}})$ is 5-dimensional with the basis $\{C_{\dot{\mathfrak{g}}}, C_{sl_N}, C_{\mathcal{H}ei}, C_{\mathcal{VH}}, C_{\mathcal{V}ir}\}$.

The Lie algebra $\mathfrak{f}$ contains four subalgebras – a Virasoro algebra, two affine algebras, $\widehat{\dot{\mathfrak{g}}} = \mathbb{C}[t_0, t_0^{-1}] \otimes \dot{\mathfrak{g}} \oplus \mathbb{C}C_{\dot{\mathfrak{g}}}$ and $\widehat{sl}_N = \mathbb{C}[t_0, t_0^{-1}] \otimes sl_N \oplus \mathbb{C}C_{sl_N}$, and an infinite-dimensional Heisenberg algebra $\mathcal{H}ei = \mathbb{C}[t_0, t_0^{-1}] \otimes I \oplus \mathbb{C}C_{\mathcal{H}ei}$.

Fix a central character $\gamma : \quad H^2(\tilde{\mathfrak{f}}) \to \mathbb{C}$:

$$\gamma(C_{\dot{\mathfrak{g}}}) = c_{\dot{\mathfrak{g}}}, \quad \gamma(C_{sl_N}) = c_{sl_N}, \quad \gamma(C_{\mathcal{H}ei}) = c_{\mathcal{H}ei}, \quad \gamma(C_{\mathcal{VH}}) = c_{\mathcal{VH}}, \quad \gamma(C_{\mathcal{V}ir}) = c_{\mathcal{V}ir},$$

and consider the corresponding quotient $V_{\mathfrak{f}}(\gamma)$ of the universal enveloping vertex algebra. Using the commutator formula (3.2), we derive from (3.16), (3.17) the following relations for the n-th products.

Lemma 3.6. *The following relations hold in $V_{\mathfrak{f}}(\gamma)$:*

$$\begin{aligned}(a) \qquad & E_{ab}(0)E_{cd}(-1)\mathbf{1} = \delta_{bc}E_{ad}(-1)\mathbf{1} - \delta_{ad}E_{cb}(-1)\mathbf{1}, \\ & E_{ab}(1)E_{cd}(-1)\mathbf{1} = \delta_{ad}\delta_{bc}c_{sl_N}\mathbf{1} + \delta_{ab}\delta_{cd}\left(\frac{c_{\mathcal{H}ei}}{N^2} - \frac{c_{sl_N}}{N}\right)\mathbf{1}, \\ & E_{ab}(n)E_{cd}(-1)\mathbf{1} = 0 \text{ for } \ n \geq 2.\end{aligned}$$

$$(b) \qquad \omega_{(0)}E_{ab}(-1)\mathbf{1} = D(E_{ab}(-1)\mathbf{1}), \quad \omega_{(1)}E_{ab}(-1)\mathbf{1} = E_{ab}(-1)\mathbf{1},$$
$$\omega_{(2)}E_{ab}(-1)\mathbf{1} = -\delta_{ab}\frac{2c_{\mathcal{VH}}}{N}\mathbf{1}, \quad \omega_{(n)}E_{ab}(-1)\mathbf{1} = 0 \text{ for } n \geq 3.$$

Let us now discuss bounded weight modules for $\mathfrak{f}$. This Lie algebra is $\mathbb{Z}$-graded by degree in t_0. We associate with this grading a decomposition $\mathfrak{f} = \mathfrak{f}_- \oplus \mathfrak{f}_0 \oplus \mathfrak{f}_+$, where $\mathfrak{f}_0 = \mathbb{C}d_0 \oplus \dot{\mathfrak{g}} \oplus gl_N \oplus H^2(\tilde{\mathfrak{f}})$. Let V be a finite-dimensional irreducible $\dot{\mathfrak{g}}$-module, and W be a finite-dimensional irreducible module for sl_N. Fix a central character $\gamma : H^2(\tilde{\mathfrak{f}}) \to \mathbb{C}$ and two constants $h_{\mathcal{V}ir}, h_{\mathcal{H}ei} \in \mathbb{C}$. We define on $V \otimes W$ the structure of an irreducible $\mathfrak{f}_0$-module on which $L(0) = -d_0$ acts as multiplication by $h_{\mathcal{V}ir}$, I acts as multiplication by $h_{\mathcal{H}ei}$, and the action of $H^2(\tilde{\mathfrak{f}})$ is determined by γ. Let $\mathfrak{f}_+$ act on $V \otimes W$ trivially and consider the induced module

$$M_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma) = \mathrm{Ind}_{\mathfrak{f}_0 \oplus \mathfrak{f}_+}^{\mathfrak{f}} (V \otimes W).$$

This module has a unique maximal submodule, and the factor-module by the maximal submodule is an irreducible $\mathfrak{f}$-module, which we denote as $L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma)$.

Remark 3.7. Similar constructions may be applied to Virasoro, affine, and Heisenberg algebras, yielding the corresponding vertex algebras $V_{\mathcal{V}ir}(c_{\mathcal{V}ir})$, $V_{\hat{\dot{\mathfrak{g}}}}(c_{\dot{\mathfrak{g}}})$, $V_{\widehat{sl}_N}(c_{sl_N})$, $V_{\mathcal{H}ei}(c_{\mathcal{H}ei})$ and irreducible highest weight modules $L_{\mathcal{V}ir}(h_{\mathcal{V}ir}, c_{\mathcal{V}ir})$ for the Virasoro algebra, $L_{\hat{\dot{\mathfrak{g}}}}(V, c_{\dot{\mathfrak{g}}})$ for the affine algebra $\hat{\dot{\mathfrak{g}}}$, $L_{\widehat{sl}_N}(W, c_{sl_N})$ for the affine algebra $\widehat{sl}_N$ and $L_{\mathcal{H}ei}(h_{\mathcal{H}ei}, c_{\mathcal{H}ei})$ for the infinite-dimensional Heisenberg algebra.

Note that for the trivial 1-dimensional modules $V = \mathbb{C}$, $W = \mathbb{C}$ and $h_{\mathcal{V}ir} = h_{\mathcal{H}ei} = 0$, the irreducible module $L_{\mathfrak{f}}(\mathbb{C}, \mathbb{C}, 0, 0, \gamma)$ is precisely the simple VOA $L_{\mathfrak{f}}(\gamma)$.

For a generic γ (γ not at a critical level), we may apply the Sugawara construction to decompose the irreducible module $L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma)$ into a tensor product of irreducible Virasoro, affine and Heisenberg modules.

Proposition 3.8. Let $c_{\dot{\mathfrak{g}}} \neq -h^\vee, c_{sl_N} \neq -N, c_{\mathcal{H}ei} \neq 0$, where $h^\vee$ is the dual Coxeter number for $\hat{\dot{\mathfrak{g}}}$. Then the VOA $V_{\mathfrak{f}}(\gamma)$ decomposes into a tensor product of four VOAs:

$$V_{\mathfrak{f}}(\gamma) \cong V_{\hat{\dot{\mathfrak{g}}}}(c_{\dot{\mathfrak{g}}}) \otimes V_{\widehat{sl}_N}(c_{sl_N}) \otimes V_{\mathcal{H}ei}(c_{\mathcal{H}ei}) \otimes V_{\mathcal{V}ir}(c'_{\mathcal{V}ir}),$$

where

$$c'_{\mathcal{V}ir} = c_{\mathcal{V}ir} - \frac{c_{\dot{\mathfrak{g}}}\dim(\dot{\mathfrak{g}})}{c_{\dot{\mathfrak{g}}} + h^\vee} - \frac{c_{sl_N}(N^2-1)}{c_{sl_N} + N} - 1 + 12\frac{c_{\mathcal{VH}}^2}{c_{\mathcal{H}ei}}, \tag{3.18}$$

and the Heisenberg VOA $V_{\mathcal{H}ei}(c_{\mathcal{H}ei})$ is taken with a non-standard Virasoro element

$$\omega_{\mathcal{H}ei} = \frac{1}{2c_{\mathcal{H}ei}}I(-1)I(-1)\mathbf{1} + \frac{c_{\mathcal{VH}}}{c_{\mathcal{H}ei}}I(-2)\mathbf{1}, \tag{3.19}$$

so that its rank is $1 - 12\frac{c_{\mathcal{VH}}^2}{c_{\mathcal{H}ei}}$.

Proof. This result is obtained by applying the Sugawara construction three times – to the affine $\hat{\dot{\mathfrak{g}}}$-subalgebra, affine $\widehat{sl}_N$-subalgebra, and the twisted Virasoro-Heisenberg subalgebra (see e.g. [FLM] and [ACKP] for details). Since this construction is well-known, we only sketch the proof.

Let $\{u_i\}, \{u^i\}$ be dual bases of $\dot{\mathfrak{g}}$, and $\{v_j\}, \{v^j\}$ be dual bases of sl_N with respect to the chosen invariant bilinear forms. Consider a new Virasoro field

$$\begin{aligned} \omega'(z) =&\omega(z) - \frac{1}{2(c_{\dot{\mathfrak{g}}} + h^\vee)} \sum_i : u_i(z) u^i(z) : - \frac{1}{2(c_{sl_N} + N)} \sum_j : v_j(z) v^j(z) : \\ &- \frac{1}{2c_{\mathcal{H}ei}} : I(z) I(z) : - \frac{c_{\mathcal{VH}}}{c_{\mathcal{H}ei}} \frac{\partial}{\partial z} I(z). \end{aligned} \tag{3.20}$$

It is possible to verify that the moments of $\omega'(z)$ satisfy the Virasoro algebra relations, with the action of the central element given by (3.18). Moreover, this new Virasoro field $\omega'(z)$ commutes with the fields of the affine $\widehat{\dot{\mathfrak{g}}}$ and $\widehat{sl}_N$ subalgebras, as well the Heisenberg subalgebra field.

The formula (3.20) defines a homomorphism of vertex algebras

$$V_{\widehat{\dot{\mathfrak{g}}}}(c_{\dot{\mathfrak{g}}}) \otimes V_{\widehat{sl}_N}(c_{sl_N}) \otimes V_{\mathcal{H}ei}(c_{\mathcal{H}ei}) \otimes V_{\mathcal{V}ir}(c'_{\mathcal{V}ir}) \to V_{\mathfrak{f}}(\gamma),$$

which is in fact an isomorphism. Moreover, if we choose the Virasoro element in $V_{\mathcal{H}ei}(c_{\mathcal{H}ei})$ to be given by (3.19), the above map becomes the isomorphism of the VOAs.

Corollary 3.9. *Under the same restriction on the central charges as in Proposition 3.8, the irreducible highest weight $\mathfrak{f}$-module $L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma)$ decomposes into a tensor product of irreducible highest weight modules for the affine $\widehat{\dot{\mathfrak{g}}}$, $\widehat{sl}_N$, infinite-dimensional Heisenberg and the Virasoro modules:*

$$L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma) \cong L_{\widehat{\dot{\mathfrak{g}}}}(V, c_{\dot{\mathfrak{g}}}) \otimes L_{\widehat{sl}_N}(W, c_{sl_N}) \otimes L_{\mathcal{H}ei}(h_{\mathcal{H}ei}, c_{\mathcal{H}ei}) \otimes L_{\mathcal{V}ir}(h'_{\mathcal{V}ir}, c'_{\mathcal{V}ir}),$$

where $c'_{\mathcal{V}ir}$ is given by (3.18) and

$$h'_{\mathcal{V}ir} = h_{\mathcal{V}ir} - \frac{\Omega_V}{2(c_{\dot{\mathfrak{g}}} + h^\vee)} - \frac{\Omega_W}{2(c_{sl_N} + N)} - \frac{h^2_{\mathcal{H}ei} - 2c_{\mathcal{VH}} h_{\mathcal{H}ei}}{2c_{\mathcal{H}ei}}. \tag{3.21}$$

Here Ω_V and Ω_W are the eigenvalues of the Casimir operators of $\dot{\mathfrak{g}}$ and sl_N on V and W respectively. These are given by $\Omega_V = (\lambda_V | \lambda_V + 2\rho)$, $\Omega_W = (\lambda_W | \lambda_W + 2\rho)$, where λ_V is the highest weight of the irreducible $\dot{\mathfrak{g}}$-module V, and λ_W is the highest weight of the sl_N-module W [K1].

Remark 3.10. If one of the inequalities in the statement of the above proposition fails, we still can apply a partial Sugawara construction to the remaining components. For example, if $c_{\dot{\mathfrak{g}}} \neq -h^\vee, c_{sl_N} \neq -N$, but $c_{\mathcal{H}ei} = 0$, the irreducible highest weight $\mathfrak{f}$-module is isomorphic to the tensor product of the affine $\widehat{\dot{\mathfrak{g}}}$ and $\widehat{sl}_N$-modules and an irreducible highest weight module for the twisted Heisenberg-Virasoro algebra at level zero. The characters of such modules for the twisted Heisenberg-Virasoro algebra were computed in [B3].

As we mentioned earlier, the toroidal VOA decomposes in a tensor product, where one factor is a twisted Virasoro-affine VOA. The other factor in this decomposition is a sub-VOA of a lattice VOA, which we will describe next.

3.5. Hyperbolic lattice VOA.

Here we present the construction of a hyperbolic lattice VOA. The general construction of a VOA corresponding to an arbitrary even lattice may be found in [FLM] or [K2].

Consider a hyperbolic lattice Hyp, which is a free abelian group on $2N$ generators $\{u_i, v_i | i = 1, \dots, N\}$ with the symmetric bilinear form

$$(\cdot|\cdot): \quad Hyp \times Hyp \to \mathbb{Z},$$

defined by

$$(u_i|v_j) = \delta_{ij}, \quad (u_i|u_j) = (v_i|v_j) = 0.$$

Note that the form $(\cdot|\cdot)$ is non-degenerate and Hyp is an even lattice, i.e., $(x|x) \in 2\mathbb{Z}$.

The construction of the VOA associated to Hyp proceeds as follows.

First we complexify Hyp:

$$H = Hyp \otimes_{\mathbb{Z}} \mathbb{C},$$

and extend $(\cdot|\cdot)$ by linearity on H. Next, we "affinize" H by defining a Lie algebra $\widehat{H} = \mathbb{C}[t, t^{-1}] \otimes H \oplus \mathbb{C}K$ with the bracket

$$[x(n), y(m)] = n(x|y)\delta_{n,-m}K, \quad x, y \in H, \quad [\widehat{H}, K] = 0. \tag{3.22}$$

Here and in what follows, we are using the notation $x(n) = t^n \otimes x$. The algebra $\widehat{H}$ has a triangular decomposition $\widehat{H} = \widehat{H}_- \oplus \widehat{H}_0 \oplus \widehat{H}_+$, where $\widehat{H}_0 = \langle 1 \otimes H, K \rangle$ and $\widehat{H}_\pm = t^{\pm 1}\mathbb{C}[t^{\pm 1}] \otimes H$.

We also need a twisted group algebra of Hyp, denoted by $\mathbb{C}[Hyp]$, which we now describe. The basis of $\mathbb{C}[Hyp]$ is $\{e^x | x \in Hyp\}$, and the multiplication is twisted with the 2-cocycle ϵ:

$$e^x e^y = \epsilon(x, y) e^{x+y}, \quad x, y \in Hyp, \tag{3.23}$$

where ϵ is a multiplicatively bilinear map

$$\epsilon : Hyp \times Hyp \to \{\pm 1\},$$

defined on the generators by $\epsilon(v_i, u_j) = (-1)^{\delta_{ij}}, \epsilon(u_i, v_j) = \epsilon(u_i, u_j) = \epsilon(v_i, v_j) = 1, \quad i, j = 1, \dots, N$.

We define the structure of $\widehat{H}_0 \oplus \widehat{H}_+$-module on $\mathbb{C}[Hyp]$, letting $\widehat{H}_+$ act on $\mathbb{C}[Hyp]$ trivially and $\widehat{H}_0$ act by

$$x(0)e^y = (x|y)e^y, \quad Ke^y = e^y. \tag{3.24}$$

Finally let V_{Hyp} be the induced $\widehat{H}$ module:

$$V_{Hyp} = \mathrm{Ind}_{\widehat{H}_0 \oplus \widehat{H}_+}^{\widehat{H}} \left(\mathbb{C}[Hyp] \right).$$

This is the VOA attached to the lattice Hyp. As a space V_{Hyp} is isomorphic to the tensor product of the symmetric algebra $S(\widehat{H}_-)$ with the twisted group algebra $\mathbb{C}[Hyp]$:

$$V_{Hyp} = S(\widehat{H}_-) \otimes \mathbb{C}[Hyp].$$

The Y-map is defined on the basis elements of $\mathbb{C}[Hyp]$ by

$$Y(e^x, z) = \exp\left(\sum_{j\geq 1} \frac{x(-j)}{j} z^j\right) \exp\left(-\sum_{j\geq 1} \frac{x(j)}{j} z^{-j}\right) e^x z^x, \tag{3.25}$$

where e^x acts by twisted multiplication (3.23) and $z^x e^y = z^{(x|y)} e^y$. For a general basis element $a = x_1(-1-n_1)\dots x_k(-1-n_k) \otimes e^y$, with $x_i, y \in Hyp, n_i \geq 0$, one defines (cf. (3.11))

$$Y(a,z) =: \left(\frac{1}{n_1!}\left(\frac{\partial}{\partial z}\right)^{n_1} x_1(z)\right) \dots \left(\frac{1}{n_k!}\left(\frac{\partial}{\partial z}\right)^{n_k} x_k(z)\right) Y(e^y, z) :, \tag{3.26}$$

where $x(z) = \sum\limits_{j\in\mathbb{Z}} x(j) z^{-j-1}$.

The Virasoro element in V_{Hyp} is $\omega_{Hyp} = \sum\limits_{p=1}^{N} u_p(-1)v_p(-1)\mathbf{1}$, where $\mathbf{1} = e^0$ is the identity element of V_{Hyp}. The rank of V_{Hyp} is $2N$.

In the construction of the toroidal VOAs we would need not V_{Hyp} itself, but its sub-VOA V^+_{Hyp}:

$$V^+_{Hyp} = S(\widehat{H}_-) \otimes \mathbb{C}[Hyp^+],$$

where Hyp^+ (resp. Hyp^-) is the isotropic sublattice of Hyp generated by $\{u_i | i = 1, \dots, N\}$ (resp. $\{v_i | i = 1, \dots, N\}$). One can verify immediately by inspecting (3.25) and (3.26) that V^+_{Hyp} is indeed a sub-VOA of V_{Hyp}. Also note that the cocycle ϵ trivializes on $\mathbb{C}[Hyp^+]$, making $\mathbb{C}[Hyp^+]$ the usual (untwisted) group algebra. The Virasoro element of V^+_{Hyp} is the same as in V_{Hyp}, and so the rank of V^+_{Hyp} is also $2N$.

Let us describe a class of modules for V^+_{Hyp}. Consider the group algebra $\mathbb{C}[H^+]$ of the vector space $H^+ = Hyp^+ \otimes_{\mathbb{Z}} \mathbb{C}$. The space $S(\widehat{H}_-) \otimes \mathbb{C}[H^+] \otimes \mathbb{C}[Hyp^-]$ has a structure of a VOA module for V^+_{Hyp}, where the action of V^+_{Hyp} is still given by (3.24),(3.25) and (3.26). Fix $\alpha \in \mathbb{C}^N, \beta \in \mathbb{Z}^N$. Then the subspace

$$M^+_{Hyp}(\alpha, \beta) = S(\widehat{H}_-) \otimes e^{\alpha u + \beta v} \mathbb{C}[Hyp^+]$$

in $S(\widehat{H}_-) \otimes \mathbb{C}[H^+] \otimes \mathbb{C}[Hyp^-]$ is an irreducible VOA module for V^+_{Hyp}. Here we are using the notations $\alpha u = \alpha_1 u_1 + \ldots + \alpha_N u_N$, etc. For $\beta = 0$ we will denote the module $M^+_{Hyp}(\alpha, 0)$ simply by $M^+_{Hyp}(\alpha)$.

4. Toroidal vertex operator algebras.

In this section we will construct several VOAs associated with the toroidal Lie algebras. We will construct a universal enveloping VOA $V_{\mathfrak{g}}$, its "level c" quotient $V_{\mathfrak{g}}(c)$ and the simple quotient $L(T_0)$. As $\mathfrak{g}$-modules, $V_{\mathfrak{g}}(c)$ does not belong to category $\mathcal{B}_\chi$, but $L(T_0)$ does. We will establish several important relations that hold in $L(T_0)$, which will allow us to show that $L(T_0)$ factors into the tensor product of two VOAs discussed in the previous section, V^+_{Hyp} and the twisted Virasoro-affine VOA $L_{\mathfrak{f}}(\gamma_0)$.

The key fact which makes it possible to construct these VOAs, is the observation that toroidal Lie algebras $\mathfrak{g}(\mu,\nu)$ are in fact vertex Lie algebras for all values of μ,ν.

This observation is not quite trivial, since it requires a rather delicate choice of a basis in $\mathfrak{g}(\mu,\nu)$, in order to exhibit the vertex Lie algebra structure.

Theorem 4.1. *Toroidal Lie algebras $\mathfrak{g}(\mu,\nu)$ are vertex Lie algebras.*
Proof. Consider the following generating series in $\mathfrak{g}[[z,z^{-1}]]$:

$$k_0(r,z) = \sum_{j=-\infty}^{\infty} t_0^j t^r k_0 z^{-j}, \qquad k_p(r,z) = \sum_{j=-\infty}^{\infty} t_0^j t^r k_p z^{-j-1}, \tag{4.1}$$

$$g(r,z) = \sum_{j=-\infty}^{\infty} t_0^j t^r g z^{-j-1}, \quad g \in \dot{\mathfrak{g}}, \tag{4.2}$$

$$\tilde{d}_p(r,z) = \sum_{j=-\infty}^{\infty} t_0^j t^r \tilde{d}_p z^{-j-1}, \qquad \tilde{d}_0(r,z) = \sum_{j=-\infty}^{\infty} t_0^j t^r \tilde{d}_0 z^{-j-2}, \tag{4.3}$$

where for $p = 1,\ldots,n$,

$$t_0^j t^r \tilde{d}_p = t_0^j t^r d_p - \nu r_p t_0^j t^r k_0, \tag{4.4}$$

and

$$t_0^j t^r \tilde{d}_0 = -t_0^j t^r d_0 + (\mu+\nu)(j+\frac{1}{2}) t_0^j t^r k_0. \tag{4.5}$$

Although the moments of the above series are not linearly independent, all linear dependencies may be encoded as relations between the fields:

$$\frac{\partial}{\partial z} k_0(r,z) = \sum_{p=1}^{N} r_p k_p(r,z).$$

Using these relations we can eliminate from the above list for each non-zero r the field $k_p(r,z)$ with the smallest p such that $r_p \neq 0$. Non-zero moments of the remaining fields will form a basis of $\mathfrak{g}(\mu,\nu)$.

Verification of the axioms of a vertex Lie algebra is now quite straightforward. The set $\mathcal{C}$ consists of a single element that corresponds to the central field $k_0(0,z) = k_0 z^0$, so (VL2) holds.

Before we check the property (VL1), let us record the commutator relations between the newly introduced elements $t_0^j t^r \tilde{d}_p, t_0^j t^r \tilde{d}_0$. Note that their brackets with the elements of $\mathcal{R}\otimes\dot{\mathfrak{g}}$

and $\mathcal{K}$ are essentially given by (1.1) and (1.2) (with an obvious change of sign for $t_0^j t^r \tilde{d}_0$), while the rest of the commutator relations are given by the following formulas with $a, b = 1, \dots, N$:

$$\begin{aligned}[t_0^i t^r \tilde{d}_a, t_0^j t^s \tilde{d}_b] =& s_a t_0^{i+j} t^{r+s} \tilde{d}_b - r_b t_0^{i+j} t^{r+s} \tilde{d}_a \\ & + (\mu s_a r_b + \nu r_a s_b) j t_0^{i+j} t^{r+s} k_0 + (\mu s_a r_b + \nu r_a s_b) \sum_{p=1}^{N} s_p t_0^{i+j} t^{r+s} k_p.\end{aligned} \tag{4.6}$$

$$\begin{aligned}[t_0^i t^r \tilde{d}_0, t_0^j t^s \tilde{d}_b] =& - j t_0^{i+j} t^{r+s} \tilde{d}_b - r_b t_0^{i+j} t^{r+s} \tilde{d}_0 - (\mu r_b (j-1) + \nu s_b (i+1)) j t_0^{i+j} t^{r+s} k_0 \\ & - (\mu r_b j + \nu s_b (i+1)) \sum_{p=1}^{N} s_p t_0^{i+j} t^{r+s} k_p.\end{aligned} \tag{4.7}$$

$$\begin{aligned}[t_0^i t^r \tilde{d}_0, t_0^j t^s \tilde{d}_0] =& (i-j) t_0^{i+j} t^{r+s} \tilde{d}_0 + (\mu+\nu) j (j+1)(i+1) t_0^{i+j} t^{r+s} k_0 \\ & + (\mu+\nu)(j+1)(i+1) \sum_{p=1}^{N} s_p t_0^{i+j} t^{r+s} k_p.\end{aligned} \tag{4.8}$$

Using these formulas together with (1.1) and (1.2), we can derive the commutator relations for the fields in $\mathfrak{g}$:

$$[k_a(r, z_1), k_b(m, z_2)] = 0, \tag{4.9}$$

$$[g(r, z_1), k_a(m, z_2)] = 0, \tag{4.10}$$

$$\begin{aligned}[g_1(r, z_1), g_2(m, z_2)] = [g_1, g_2](z_2) \left[z_1^{-1} \delta\left(\frac{z_2}{z_1}\right) \right] \\ + (g_1|g_2) k_0(r+m, z_2) \left[z_1^{-1} \frac{\partial}{\partial z_2} \delta\left(\frac{z_2}{z_1}\right) \right] + (g_1|g_2) \sum_{p=1}^{N} r_p k_p(r+m, z_2) \left[z_1^{-1} \delta\left(\frac{z_2}{z_1}\right) \right],\end{aligned} \tag{4.11}$$

$$\left[\tilde{d}_j(r, z_1), g(m, z_2) \right] = m_j g(r+m, z_2) \left[z_1^{-1} \delta\left(\frac{z_2}{z_1}\right) \right], \tag{4.12}$$

$$\left[\tilde{d}_0(r, z_1), g(m, z_2) \right] = \frac{\partial}{\partial z_2} \left\{ g(r+m, z_2) \left[z_1^{-1} \delta\left(\frac{z_2}{z_1}\right) \right] \right\}, \tag{4.13}$$

$$\begin{aligned}\left[\tilde{d}_i(r, z_1), \tilde{d}_j(m, z_2) \right] =& \left(m_i \tilde{d}_j(r+m, z_2) - r_j \tilde{d}_i(r+m, z_2) \right) \left[z_1^{-1} \delta\left(\frac{z_2}{z_1}\right) \right] \\ & - (\mu m_i r_j + \nu r_i m_j) \sum_{p=1}^{N} r_p k_p(r+m, z_2) \left[z_1^{-1} \delta\left(\frac{z_2}{z_1}\right) \right] \\ & - (\mu m_i r_j + \nu r_i m_j) k_0(r+m, z_2) \left[z_1^{-1} \frac{\partial}{\partial z_2} \delta\left(\frac{z_2}{z_1}\right) \right],\end{aligned} \tag{4.14}$$

$$\left[\tilde{d}_0(r, z_1), \tilde{d}_j(m, z_2) \right] = \frac{\partial}{\partial z_2} \left\{ \tilde{d}_j(r+m, z_2) \left[z_1^{-1} \delta\left(\frac{z_2}{z_1}\right) \right] \right\} - r_j \tilde{d}_0(r+m, z_2) \left[z_1^{-1} \delta\left(\frac{z_2}{z_1}\right) \right]$$

$$+\nu m_j \sum_{p=1}^{N} r_p k_p(r+m, z_2) \left[z_1^{-1} \frac{\partial}{\partial z_2} \delta\left(\frac{z_2}{z_1}\right) \right] + \nu m_j k_0(r+m, z_2) \left[z_1^{-1} \left(\frac{\partial}{\partial z_2}\right)^2 \delta\left(\frac{z_2}{z_1}\right) \right]$$

$$-\mu r_j \frac{\partial}{\partial z_2} \left\{ \sum_{p=1}^{N} r_p k_p(r+m, z_2) \left[z_1^{-1} \delta\left(\frac{z_2}{z_1}\right) \right] + k_0(r+m, z_2) \left[z_1^{-1} \frac{\partial}{\partial z_2} \delta\left(\frac{z_2}{z_1}\right) \right] \right\}, \qquad (4.15)$$

$$\left[\tilde{d}_0(r, z_1), \tilde{d}_0(m, z_2) \right] = \left\{ \frac{\partial}{\partial z_2} \tilde{d}_0(r+m, z_2) \right\} \left[z_1^{-1} \delta\left(\frac{z_2}{z_1}\right) \right] + 2\tilde{d}_0(r+m, z_2) \left[z_1^{-1} \frac{\partial}{\partial z_2} \delta\left(\frac{z_2}{z_1}\right) \right]$$

$$+(\mu+\nu) \frac{\partial}{\partial z_2} \left\{ \sum_{p=1}^{N} r_p k_p(r+m, z_2) \left[z_1^{-1} \frac{\partial}{\partial z_2} \delta\left(\frac{z_2}{z_1}\right) \right] + k_0(r+m, z_2) \left[z_1^{-1} \left(\frac{\partial}{\partial z_2}\right)^2 \delta\left(\frac{z_2}{z_1}\right) \right] \right\}, \qquad (4.16)$$

where $g, g_1, g_2 \in \dot{\mathfrak{g}}$, $a, b = 0, 1, \ldots, N$, $i, j = 1, \ldots, N$.

Now, the right-hand sides of the above commutators are precisely in the format required by (VL1). Thus both (VL1) and (VL2) hold, we conclude that $\mathfrak{g}(\mu, \nu)$ is a vertex Lie algebra, which allows us to consider its universal enveloping vertex algebra $V_{\mathfrak{g}}$. Moreover, $\tilde{d}_0(0, z)$ is the Virasoro field with the central element $C_{\mathcal{V}ir} = 12(\mu+\nu)k_0$ and $D = t_0^{-1}\tilde{d}_0$.

The subalgebra $\mathfrak{g}^{(-)}$ of the vertex Lie algebra $\mathfrak{g}$ (not to be confused with its subalgebra $\mathfrak{g}_-$ with respect to $\mathbb{Z}$-grading) is generated by the following elements: $t_0^j t^r k_0$ with $j \leq 0$, $t_0^j t^r k_p$, $t_0^j t^r g$, $t_0^j t^r \tilde{d}_p$ with $j \leq -1, p = 1, \ldots, N$, and $t_0^j t^r \tilde{d}_0$ with $j \leq -2$. The subalgebra $\mathfrak{g}^{(+)}$ is spanned by the complementary moments of the fields (4.1)-(4.3). We recall that $\mathfrak{g}^{(+)}\mathbf{1} = 0$ in $V_{\mathfrak{g}}$.

It follows from Theorem 3.1 that $Y(t^r k_0, z) = k_0(r, z)$, $Y(t_0^{-1} t^r k_j, z) = k_j(r, z)$, $Y(t_0^{-1} t^r g, z) = g(r, z)$, $Y(t_0^{-1} t^r \tilde{d}_j, z) = \tilde{d}_j(r, z)$, $Y(t_0^{-2} t^r \tilde{d}_0, z) = \tilde{d}_0(r, z)$. For the sake of simplicity of notations, we are writing $Y(t^r k_0, z)$ for $Y((t^r k_0)\mathbf{1}, z)$, etc. Also, when $r = 0$, we will simply write $g(z)$ for $g(0, z)$, etc.

We are going to show that for a particular irreducible $\mathfrak{g}_0$-module T_0, the irreducible $\mathfrak{g}$-module $L(T_0)$ is a factor-VOA of $V_{\mathfrak{g}}$. This will be done in two steps. First we will construct a factor-VOA of $V_{\mathfrak{g}}$ that has an irreducible $\mathfrak{g}_0$-module as its top. After that we will show that the irreducible quotient of this $\mathfrak{g}$-module is a VOA, and determine the structure of this vertex algebra.

The operator $\tilde{d}_0 = -d_0 + \frac{1}{2}(\mu+\nu)k_0$ induces the $\mathbb{Z}$-grading of the universal enveloping vertex algebra $V_{\mathcal{L}}$. Since $V_{\mathfrak{g}} = U(\mathfrak{g}^{(-)}) \otimes \mathbf{1}$, we see that its zero component is spanned by the elements $(t^{r_1} k_0) \ldots (t^{r_s} k_0)\mathbf{1}$.

Proposition 4.2. Fix a non-zero $c \in \mathbb{C}$ and consider a $\mathfrak{g}$-submodule $R(S)$ in $V_{\mathfrak{g}}$ generated by the set $S = \left\{ k_0\mathbf{1} - c\mathbf{1},\ (t^r k_0)(t^m k_0)\mathbf{1} - c(t^{r+m} k_0)\mathbf{1} \mid r, m \in \mathbb{Z}^N \right\}$.

(a) The quotient $V_{\mathfrak{g}}(c) = V_{\mathfrak{g}}/R(S)$ is a factor-module of the generalized Verma module $M(T_0)$ with the top

$$T_0 = \mathbb{C}[q_1^{\pm}, \ldots, q_N^{\pm}] \otimes V \otimes W = \mathbb{C}[q_1^{\pm}, \ldots, q_N^{\pm}]$$

defined as in (2.5) with $\alpha = 0$, $d = \frac{1}{2}(\mu+\nu)c$, V being the trivial 1-dimensional $\dot{\mathfrak{g}}$-module, W being a 1-dimensional gl_N-module on which sl_N acts trivially and I acts a multiplication by $h = N\nu c$.

(b) $V_{\mathfrak{g}}(c)$ inherits a vertex algebra structure from $V_{\mathfrak{g}}$.

(c) $V_{\mathfrak{g}}(c)$ is a VOA of rank $12(\mu+\nu)c$ with the Virasoro field $\omega(z) = \tilde{d}_0(z)$.

(d) The projection from $M(T_0)$ to $L(T_0)$ factors through $V_{\mathfrak{g}}(c)$:

$$M(T_0) \to V_{\mathfrak{g}}(c) \to L(T_0). \tag{4.17}$$

This defines a VOA structure on $L(T_0)$ as a factor-VOA of $V_{\mathfrak{g}}(c)$.

Proof. Let us prove part (a). Consider a $\mathbb{Z}$-grading on $V_{\mathfrak{g}}$. We claim that the zero component $R(S)_0$ coincides with

$$\mathrm{Span}\left\langle (t^{r_1}k_0)\dots(t^{r_s}k_0)\mathbf{1} - c^{s-1}(t^{r_1+\dots+r_s}k_0)\mathbf{1}, \quad k_0\mathbf{1} - c\mathbf{1} \mid r_1,\dots,r_s \in \mathbb{Z}^N \right\rangle. \tag{4.18}$$

First of all it is easy to see by repeated multiplication of the elements in S by $t^{r_j}k_0$, that the elements (4.18) are indeed in $R(S)$. Next, we write $R(S) = U(\mathfrak{g})S$. We have a triangular decomposition $U(\mathfrak{g}) = U(\mathfrak{g}_-) \otimes U(\mathfrak{g}_0) \otimes U(\mathfrak{g}_+)$, and $\mathfrak{g}_+$ acts on S trivially because the elements of S are of degree zero. This implies that $R(S)_0 = U(\mathfrak{g}_0)S$. Let us show that (4.18) is invariant under the action of $\mathfrak{g}_0$. The subalgebra $\mathfrak{g}_0$ is spanned by the elements $t^m k_0$, $t^m k_p$, $t^m g$, $t^m \tilde{d}_p$, $t^m \tilde{d}_0$, $\quad m \in \mathbb{Z}^N, g \in \dot{\mathfrak{g}}, p = 1,\dots,N$. We have already verified the invariance of (4.18) under the action of $t^m k_0$. We note that the remaining generators of $\mathfrak{g}_0$ belong to $\mathfrak{g}^{(+)}$ and thus annihilate $\mathbf{1}$. In addition to this, the elements $t^m k_p$ and $t^m g$ commute with $t^r k_0$, which implies that they also annihilate (4.18). It follows from (1.2) that $t^m \tilde{d}_0$ annihilate (4.18) as well.

Using the commutator relation $\left[t^m \tilde{d}_p, t^r k_0\right] = r_p t^{r+m} k_0$, we get that

$$(t^m \tilde{d}_p)\left((t^{r_1}k_0)\dots(t^{r_s}k_0)\mathbf{1} - c^{s-1}(t^{r_1+\dots+r_s}k_0)\mathbf{1}\right)$$

$$= \sum_{j=1}^{s} r_j \left((t^{r_1}k_0)\dots(t^{r_j+m}k_0)\dots(t^{r_s}k_0)\mathbf{1} - c^{s-1}(t^{r_1+\dots+r_s+m}k_0)\mathbf{1}\right),$$

and the right hand side is in (4.18). This proves our claim that $R(S)_0$ is given by (4.18).

It follows from this that the top of the module $V_{\mathfrak{g}}(c) = V_{\mathfrak{g}}/R(S)$ may be identified with the space of Laurent polynomials $T_0 = \mathbb{C}[q_1^{\pm},\dots,q_N^{\pm}]$, under the isomorphism

$$(t^r k_0)\mathbf{1} \mapsto c q^r.$$

Let us describe the action of the subalgebra $\mathfrak{g}_0$ on the top T_0. It follows from the relations (4.18) that

$$(t^m k_0) q^r = c q^{r+m}. \tag{4.19}$$

Next, we have seen that $t^m k_p, t^m g$ and $t^m \tilde{d}_0$ annihilate T_0:

$$(t^m k_p) q^r = 0, \quad (t^m g) q^r = 0, \quad (t^m \tilde{d}_0) q^r = 0. \tag{4.20}$$

Since $t^m \tilde{d}_0 = -t^m d_0 + \frac{1}{2}(\mu+\nu) t^m k_0$, we get that

$$(t^m d_0) q^r = \frac{1}{2}(\mu+\nu) c q^{r+m}. \tag{4.21}$$

Finally,

$$(t^m\tilde{d}_p)q^r = \frac{1}{c}(t^m\tilde{d}_p)(t^r k_0)\mathbf{1} = \frac{1}{c}\left[t^m\tilde{d}_p, t^r k_0\right]\mathbf{1} = \frac{1}{c}r_p(t^{r+m}k_0)\mathbf{1} = r_p q^{r+m}. \tag{4.22}$$

Taking into account that $t^m\tilde{d}_p = t^m d_p - m_p\nu t^m k_0$, we obtain

$$(t^m d_p)q^r = (r_p + \nu c m_p)q^{r+m}, \tag{4.23}$$

which corresponds to the tensor module action (2.6) with the trivial action of sl_N and I acting as multiplication by $N\nu c$. This completes the proof of part (a). Part (b) follows from Theorem 3.2. The claim of part (c) has been already established and finally, for part (d) we note that $L(T_0)$ is a unique irreducible factor of $M(T_0)$, thus the projection $M(T_0) \to L(T_0)$ factors through $V_{\mathfrak{g}}(c)$ as in (4.17). Let us point out that the kernel of the homomorphism $M(T_0) \to V_{\mathfrak{g}}(c)$ is the submodule generated by $\left\{(t_0^{-1}t^m\tilde{d}_0)\mathbf{1} | m \in \mathbb{Z}^N\right\}$. Applying Theorem 3.2 again, we conclude that $L(T_0)$ inherits the vertex operator algebra structure from $V_{\mathfrak{g}}(c)$ given by formula (3.11). This completes the proof of the Proposition.

Next we are going to study the structure of the VOA $L(T_0)$.

Theorem 4.3.

(a) The VOA $L(T_0)$ is generated by the following elements: $q^m = \frac{1}{c}(t^m k_0)\mathbf{1}$, $(t_0^{-1}g)\mathbf{1}$, $(t_0^{-1}k_a)\mathbf{1}$, $(t_0^{-1}\tilde{d}_a)\mathbf{1}$, E_{ab}, $(t_0^{-2}\tilde{d}_0)\mathbf{1}$, with $m \in \mathbb{Z}^N$, $g \in \dot{\mathfrak{g}}$, $a, b = 1, \dots, N$, where

$$E_{ab} = \frac{1}{c}(t_0^{-1}t^{\epsilon_a}\tilde{d}_b)(t^{-\epsilon_a}k_0)\mathbf{1} - (t_0^{-1}\tilde{d}_b)\mathbf{1} + \frac{1}{c}\delta_{ab}(t_0^{-1}k_a)\mathbf{1}, \tag{4.24}$$

with ϵ_a being a standard basis vector in $\mathbb{Z}^N$ with 1 in a-th position.

(b) The module $L(T_0)$ has a structure of a $\mathbb{C}[q_1^{\pm}, \dots, q_N^{\pm}]$-module, which is compatible with the action of the algebra of Laurent polynomials on T_0. The action of the field $k_0(m, z)$ is given by the following vertex operator:

$$\frac{1}{c}k_0(m, z) = Y(q^m, z) = q^m \exp\left(\sum_{p=1}^N m_p \sum_{j\in\mathbb{Z}\backslash\{0\}} \frac{1}{cj}(t_0^{-j}k_p)z^j\right). \tag{4.25}$$

(c) The action on $L(T_0)$ of the remaining fields in (4.1)-(4.3) is expressed in the following way

$$g(m, z) = g(z)Y(q^m, z), \tag{4.26}$$

$$k_a(m, z) = k_a(z)Y(q^m, z), \tag{4.27}$$

$$\tilde{d}_a(m, z) =: \tilde{d}_a(z)Y(q^m, z) : + \sum_{p=1}^N m_p Y(E_{pa}, z)Y(q^m, z), \tag{4.28}$$

$$\tilde{d}_0(m, z) =: \tilde{d}_0(z)Y(q^m, z) : + \frac{1}{c}\sum_{a,b=1}^N m_a k_b(z)Y(E_{ab}, z)Y(q^m, z)$$

$$+(\mu - \frac{1}{c}) \sum_{p=1}^{N} m_p \left(\frac{\partial}{\partial z} k_p(z) \right) Y(q^m, z). \tag{4.29}$$

(d) The vertex operator algebra $L(T_0)$ is isomorphic to the tensor product of two VOAs:

$$L(T_0) = V_{Hyp}^{+} \otimes L_{\mathfrak{f}}(\gamma_0),$$

where V_{Hyp}^{+} is a sub-VOA of the lattice VOA described in section 3.5, and $L_{\mathfrak{f}}(\gamma_0)$ is the simple VOA corresponding to the twisted Virasoro-affine Lie algebra constructed from the reductive Lie algebra $\dot{\mathfrak{f}} = \dot{\mathfrak{g}} \oplus gl_N$, and the central character γ_0 given by the following values:

$$c_{\dot{\mathfrak{g}}} = c, \qquad c_{sl_N} = 1 - \mu c,$$

$$c_{\mathcal{H}ei} = N(1 - \mu c) - N^2 \nu c, \qquad c_{\mathcal{VH}} = N(\frac{1}{2} - \nu c),$$

$$c_{\mathcal{V}ir} = 12c(\mu + \nu) - 2N. \tag{4.30}$$

The fields $k_0(m, z)$, $k_p(z)$, $\tilde{d}_p(z)$, $p = 1, \ldots, N$, act on V_{Hyp}^{+} by

$$k_0(m, z) = cY(e^{mu}, z), \quad k_p(z) = cu_p(z), \quad \tilde{d}_p(z) = v_p(z),$$

while the fields $g(z)$ and $Y(E_{ab}, z)$ act on $L_{\mathfrak{f}}(\gamma_0)$. The field $\tilde{d}_0(z)$ is the Virasoro field of the tensor product $V_{Hyp}^{+} \otimes L_{\mathfrak{f}}(\gamma_0)$.

Proof. We are going to determine the structure of the VOA $L(T_0)$ using the following strategy. The technique developed in [BB] provides a method to calculate any given homogeneous component of the kernel of the epimorphism $\pi: V_{\mathfrak{g}}(c) \to L(T_0)$, though of course such a computation is feasible only for the components of low degrees. Any $v \in \mathrm{Ker}\ \pi$ yields a relation between the fields in $L(T_0)$:

$$Y_{L(T_0)}(v, z) = 0.$$

It turns out that it is sufficient to know the elements of Ker (π) of degrees 1 and 2 in order to completely determine the structure of $L(T_0)$ as a VOA.

Let us illustrate the technique of [BB] with the following example. Fix $m \in \mathbb{Z}^N$ and $g \in \dot{\mathfrak{g}}$. Consider the subspace $P_1 = \mathrm{Span} \left\langle (t_0^{-1} t^r g) q^{m-r} \mid r \in \mathbb{Z}^N \right\rangle \subset V_{\mathfrak{g}}(c)$. This subspace belongs to the homogeneous component of weight $(-1, m)$ in $V_{\mathfrak{g}}(c)$. We are going to find the intersection of P_1 with Ker π. We note that a vector v in component $(-1, m)$ of $V_{\mathfrak{g}}(c)$ belongs to Ker π if and only if $U_1(\mathfrak{g}_+) v = 0$.

Lemma 4.4. *Let $g, g' \in \dot{\mathfrak{g}}$, $b = 1, \ldots, N$. Then*

$$(t_0 t^s k_0)(t_0^{-1} t^r g) q^{m-r} = 0, \quad (t_0 t^s k_b)(t_0^{-1} t^r g) q^{m-r} = 0, \tag{4.31}$$

$$(t_0 t^s g')(t_0^{-1} t^r g) q^{m-r} = (g'|g) c q^{m+s}, \tag{4.32}$$

$$(t_0 t^s \tilde{d}_0)(t_0^{-1} t^r g) q^{m-r} = 0, \quad (t_0 t^s \tilde{d}_b)(t_0^{-1} t^r g) q^{m-r} = 0. \tag{4.33}$$

Proof. Let us prove (4.32):

$$(t_0 t^s g')(t_0^{-1} t^r g) q^{m-r} = [t_0 t^s g', t_0^{-1} t^r g] q^{m-r} + (t_0^{-1} t^r g)(t_0 t^s g') q^{m-r}.$$

The second term vanishes because $\mathfrak{g}_+$ acts trivially on the top of $V_{\mathfrak{g}}(c)$. For the first term we use the relations in $\mathfrak{g}$, (4.19) and (4.20):

$$\begin{aligned}[t_0 t^s g', t_0^{-1} t^r g] q^{m-r} &= (t^{r+s}[g', g]) q^{m-r} + (g'|g)(t^{r+s} k_0) q^{m-r} + (g'|g) \sum_{p=1}^{N} s_p (t^{r+s} k_p) q^{m-r} \\ &= (g'|g) c q^{m+s}\end{aligned}.$$

All other equalities in the statement of this Lemma are obtained in a similar way.

Let us analyze the results of this Lemma. We see that the right hand sides in (4.31)-(4.33) are independent of r. Thus $U_1(\mathfrak{g}_+) \left((t_0^{-1} t^r g) q^{m-r} - (t_0^{-1} g) q^m \right) = 0$, which implies that $(t_0^{-1} t^r g) q^{m-r} - (t_0^{-1} g) q^m \in \operatorname{Ker} \pi$. Applying the state-field correspondence, we get that the following relation holds in $L(T_0)$:

$$: Y(t_0^{-1} t^r g, z) Y(q^{m-r}, z) : = : Y(t_0^{-1} g, z) Y(q^m, z) : . \tag{4.34}$$

Since these vertex operators commute, we may drop the normal ordering symbol. In particular, for $m = r$, we get that

$$Y(t_0^{-1} t^m g, z) = Y(t_0^{-1} g, z) Y(q^m, z). \tag{4.35}$$

This factorization property means that the fields $Y(t_0^{-1} t^m g, z) = g(m, z)$ reduce to more elementary fields $Y(t_0^{-1} g, z) = g(z)$ and $Y(q^m, z)$. The fields $g(z)$ correspond to the affine subalgebra $\mathbb{C}[t_0, t_0^{-1}] \otimes \dot{\mathfrak{g}} \oplus \mathbb{C} k_0 \subset \mathfrak{g}$. Below we establish analogous factorization formulas for other fields. We shall also see that the affine fields $Y(t_0^{-1} g, z)$ commute with other elementary fields, except for $Y(t_0^{-2} \tilde{d}_0, z)$. This will imply that the affine VOA generated by the affine fields splits off as a tensor factor in $L(T_0)$ when c is not the critical level for this affine subalgebra. In this way we will obtain a tensor product decomposition of $L(T_0)$.

Now let us get the formula for the vertex operators $Y(t_0^{-1} t^r k_a, z)$ and $Y(t^r k_0, z)$. For a fixed $m \in \mathbb{Z}^N$ and $1 \leq a \leq N$, consider the subspace $P_2 = \operatorname{Span} \left\langle (t_0^{-1} t^r k_a) q^{m-r} | r \in \mathbb{Z}^N \right\rangle \subset V_{\mathfrak{g}}(c)$. Again we are going to find the intersection of P_2 with $\operatorname{Ker} \pi$.

Lemma 4.5. *Let* $1 \leq a, b \leq N$. *Then*

$$(t_0 t^s k_0)(t_0^{-1} t^r k_a) q^{m-r} = 0, \quad (t_0 t^s k_b)(t_0^{-1} t^r k_a) q^{m-r} = 0, \tag{4.36}$$

$$(t_0 t^s g)(t_0^{-1} t^r k_a) q^{m-r} = 0, \quad g \in \dot{\mathfrak{g}}, \quad (t_0 t^s \tilde{d}_0)(t_0^{-1} t^r k_a) q^{m-r} = 0, \tag{4.37}$$

$$(t_0 t^s \tilde{d}_b)(t_0^{-1} t^r k_a) q^{m-r} = c \delta_{ab} q^{m+s}. \tag{4.38}$$

The proof of this Lemma is the same as for Lemma 4.4, and we omit these calculations.

We see again that the right hand sides in (4.36)-(4.38) are independent of r. Thus $(t_0^{-1}t^r k_a)q^{m-r} - (t_0^{-1}k_a)q^m \in \mathrm{Ker}\,\pi$, or equivalently,

$$(t_0^{-1}t^r k_a)q^{m-r} = (t_0^{-1}k_a)q^m \quad \text{in} \quad L(T_0). \tag{4.39}$$

Taking $r = m$ and applying the state-field correspondence, we get that the following relation holds in $L(T_0)$:

$$Y(t_0^{-1}t^m k_a, z) = Y(t_0^{-1}k_a, z)Y(q^m, z). \tag{4.40}$$

Since these vertex operators commute, we dropped the normal ordering symbol in the right hand side. Also, taking into account that $t_0^{-1}t^m k_0 = \sum\limits_{p=1}^{N} m_p t_0^{-1}t^m k_p$, we obtain

$$c\frac{\partial}{\partial z}Y(q^m, z) = Y(t_0^{-1}t^m k_0, z) = \sum_{p=1}^{N} m_p Y(t_0^{-1}k_p, z)Y(q^m, z). \tag{4.41}$$

Next we will use the results of Section 3 of [BB]. It is proved there that $L(T_0)$ is a module over a commutative associative algebra $\mathbb{C}[q_1^{\pm}, \ldots, q_N^{\pm}]$ and the vertex operator $Y(q^m, z)$ is given by the expression (4.25). We can see that this formula is compatible with (4.41), and in fact it is not too difficult to derive (4.25) from (4.41).

We will later need another relation in $L(T_0)$ which can be derived either from (4.39) or (4.25):

$$(t_0^{-1}t^r k_0)q^s = \sum_{p=1}^{N} r_p (t_0^{-1}k_p)q^{r+s}. \tag{4.42}$$

Our next goal is to derive a formula for the vertex operator $Y(t_0^{-1}t^m \tilde{d}_a, z)$. We will use the same strategy as above.

Lemma 4.6. *Let $1 \leq a, b \leq N$. Then*

$$(t_0 t^s k_0)(t_0^{-1}t^r \tilde{d}_a)q^{m-r} = -s_a c q^{m+s}, \tag{4.43}$$

$$(t_0 t^s k_b)(t_0^{-1}t^r \tilde{d}_a)q^{m-r} = \delta_{ab} c q^{m+s}, \tag{4.44}$$

$$(t_0 t^s g)(t_0^{-1}t^r \tilde{d}_a)q^{m-r} = 0, \quad g \in \dot{\mathfrak{g}}, \tag{4.45}$$

$$(t_0 t^s \tilde{d}_0)(t_0^{-1}t^r \tilde{d}_a)q^{m-r} = \left((m_a - r_a) - 2(\mu s_a - \nu r_a)c\right) q^{m+s}, \tag{4.46}$$

$$(t_0 t^s \tilde{d}_b)(t_0^{-1}t^r \tilde{d}_a)q^{m-r} = \left(r_b(m_a - r_a) - s_a(m_b - r_b) - (\mu r_b s_a + \nu r_a s_b)c\right) q^{m+s}. \tag{4.47}$$

Proof. Let us show the calculations for (4.46) and leave rest as an exercise to the reader. We are going to use (4.7), (4.19) and (4.22):

$$\begin{aligned}(t_0 t^s \tilde{d}_0)(t_0^{-1}t^r \tilde{d}_a)q^{m-r} &= [t_0 t^s \tilde{d}_0, t_0^{-1}t^r \tilde{d}_a]q^{m-r} \\ &= (t^{r+s}\tilde{d}_a)q^{m-r} - s_a(t^{r+s}\tilde{d}_0)q^{m-r}\end{aligned}$$

$$+(-2\mu s_a + 2\nu r_a)(t^{r+s}k_0)q^{m-r} - (-\mu s_a + 2\nu r_a)\sum_{p=1}^{N} r_p (t^{r+s}k_p)q^{m-r}$$

$$= (m_a - r_a - 2\mu c s_a + 2\nu c r_a)q^{m+s}.$$

Unlike the previous cases, the right hand sides in Lemma 4.6 do depend on r. Note, however, that this dependence is polynomial, and we may separate constant, linear and quadratic components. Clearly, the constant term is obtained by setting $r = 0$, and is produced by the element $(t_0^{-1}\tilde{d}_a)q^m$.

Next, comparing (4.43)-(4.47) with (4.36)-(4.38) we notice that the quadratic term is given by the vector $-\frac{r_a}{c}\sum\limits_{p=1}^{N} r_p(t_0^{-1}k_p)q^m$:

$$(t_0 t^s \tilde{d}_b)\left(-\frac{r_a}{c}\sum_{p=1}^{N} r_p (t_0^{-1}k_p)q^m\right) = -r_a r_b q^{m+s},$$

while the other raising operators annihilate this vector.

Finally, the linear in r component is given by the vector $\sum\limits_{p=1}^{N} r_p E_{pa}^m$, where

$$E_{pa}^m = (t_0^{-1}t^{\epsilon_p}\tilde{d}_a)q^{m-\epsilon_p} - (t_0^{-1}\tilde{d}_a)q^m + \frac{1}{c}\delta_{ap}(t_0^{-1}k_p)q^m. \tag{4.48}$$

Using Lemmas 4.6 and 4.5, we can easily see that

$$(t_0 t^s k_0)\sum_{p=1}^{N} r_p E_{pa}^m = 0, \quad (t_0 t^s k_b)\sum_{p=1}^{N} r_p E_{pa}^m = 0, \quad (t_0 t^s g)\sum_{p=1}^{N} r_p E_{pa}^m = 0, \tag{4.49}$$

$$(t_0 t^s \tilde{d}_0)\sum_{p=1}^{N} r_p E_{pa}^m = r_a(-1 + 2\nu c)q^{m+s}, \tag{4.50}$$

$$(t_0 t^s \tilde{d}_b)\sum_{p=1}^{N} r_p E_{pa}^m = (r_b m_a + (1-\mu c)s_a r_b - \nu c r_a s_b)q^{m+s}. \tag{4.51}$$

It follows from the above computations that the following vector is annihilated by $U_1(\mathfrak{g}_+)$ and thus vanishes in $L(T_0)$:

$$(t_0^{-1}t^r\tilde{d}_a)q^{m-r} - (t_0^{-1}\tilde{d}_a)q^m - \sum_{p=1}^{N} r_p E_{pa}^m + \frac{r_a}{c}\sum_{p=1}^{N} r_p (t_0^{-1}k_p)q^m = 0. \tag{4.52}$$

Setting $r = m$, and applying the map Y we obtain a relation for the vertex operators in $L(T_0)$:

$$\tilde{d}_a(m,z) =: \tilde{d}_a(z)Y(q^m,z) : + \sum_{p=1}^{N} m_p Y(E_{pa}^m, z) - \frac{m_a}{c}\sum_{p=1}^{N} m_p k_p(z)Y(q^m,z). \tag{4.53}$$

Let us now establish a relation between E_{ab}^m and

$$E_{ab} = E_{ab}^0 = (t_0^{-1} t^{\epsilon_a} \tilde{d}_b) q^{-\epsilon_a} - (t_0^{-1} \tilde{d}_b)\mathbf{1} + \frac{1}{c}\delta_{ab}(t_0^{-1} k_a)\mathbf{1}. \tag{4.54}$$

Lemma 4.7. *(a) The fields $Y(q^m, z), g(z), k_p(z), \tilde{d}_p(z)$ commute with $Y(E_{ab}, z)$, $p, a, b = 1, \dots, N$.*

(b) The following relation holds:

$$E_{ab}^m = (E_{ab})_{(-1)} q^m + \frac{m_b}{c}(t_0^{-1} k_a) q^m. \tag{4.55}$$

Proof. Let us show that

$$(q^m)_{(n)} E_{ab} = 0 \text{ for all } \; n \geq 0. \tag{4.56}$$

Since $\deg\left((q^m)_{(n)} E_{ab}\right) = -n$, we only need to consider the case of $n = 0$. But $(q^m)_{(0)} = \frac{1}{c}(t_0 t^m k_0)$, and we get the desired claim from (4.49). Now applying the commutator formula (3.2) we get that the fields $Y(q^m, z)$ and $Y(E_{ab}, z)$ commute. Using a similar argument we can derive from (4.49)-(4.51) that $g(z), k_p(z), \tilde{d}_p(z)$ also commute with $Y(E_{ab}, z)$.

Taking into account the skew symmetry identity (3.7) we obtain as a consequence of (4.56) the equality

$$(E_{ab})_{(-1)} q^m = (q^m)_{(-1)} E_{ab}.$$

If we substitute (4.54) in the right hand side of this equality, we will get

$$(E_{ab})_{(-1)} q^m = \frac{1}{c}(t^m k_0)\left((t_0^{-1} t^{\epsilon_a} \tilde{d}_b) q^{-\epsilon_a} - (t_0^{-1}\tilde{d}_b)\mathbf{1} + \frac{1}{c}\delta_{ab}(t_0^{-1} k_a)\mathbf{1} \right)$$

$$= \frac{1}{c}(t_0^{-1} t^{\epsilon_a} \tilde{d}_b)(t^m k_0) q^{-\epsilon_a} - \frac{1}{c}(t_0^{-1}\tilde{d}_b)(t^m k_0)\mathbf{1} + \frac{1}{c^2}\delta_{ab}(t_0^{-1} k_a)(t^m k_0)\mathbf{1}$$

$$-\frac{1}{c}[t_0^{-1} t^{\epsilon_a}\tilde{d}_b, t^m k_0] q^{-\epsilon_a} + \frac{1}{c}[t_0^{-1}\tilde{d}_b, t^m k_0]\mathbf{1}$$

$$= (t_0^{-1} t^{\epsilon_a}\tilde{d}_b) q^{m-\epsilon_a} - (t_0^{-1}\tilde{d}_b) q^m + \frac{1}{c}\delta_{ab}(t_0^{-1} k_a) q^m - \frac{m_b}{c}(t_0^{-1} t^{m+\epsilon_a} k_0) q^{-\epsilon_a} + \frac{m_b}{c}(t_0^{-1} t^m k_0)\mathbf{1}$$

$$= E_{ab}^m - \frac{m_b}{c}(t_0^{-1} k_a) q^m.$$

To get the last equality we used (4.48) and (4.42). This completes the proof of the Lemma.

Combining (4.53) with (4.55), we obtain (4.28). We also get from (4.52) and (4.55) that

$$(t_0^{-1} t^r \tilde{d}_a) q^m = (t_0^{-1}\tilde{d}_a) q^{m+r} + \sum_{p=1}^{N} r_p (E_{pa})_{(-1)} q^{m+r} + \frac{m_a}{c}\sum_{p=1}^{N} r_p (t_0^{-1} k_p) q^{m+r}. \tag{4.57}$$

Next we are going to determine the commutator relations between $Y(E_{ab}, z)$, $1 \leq a, b \leq N$.

Lemma 4.8. *(a)* $(E_{ab})_{(0)}E_{sp} = \delta_{bs}E_{ap} - \delta_{ap}E_{sb}$,

(b) $(E_{ab})_{(1)}E_{sp} = (1-\mu c)\delta_{bs}\delta_{ap}\mathbf{1} - \nu c\delta_{ab}\delta_{sp}\mathbf{1}$,

(c) $(E_{ab})_{(n)}E_{sp} = 0$ *for* $n \geq 2$.

Proof. Since $\deg(E_{ab}) = 1$, we have that $\deg\left((E_{ab})_{(n)}E_{sp}\right) = 1-n$, from which part (c) immediately follows. Let us carry out the computations for part (b) of the Lemma. We have

$$(E_{ab})_{(1)}E_{sp} = \left((t_0^{-1}t^{\epsilon_a}\tilde{d}_b)_{(-1)}q^{-\epsilon_a}\right)_{(1)} E_{sp} - (t_0\tilde{d}_b)E_{sp} + \frac{1}{c}\delta_{ab}(t_0k_a)E_{sp}. \tag{4.58}$$

The last two terms vanish by (4.49) and (4.51). To evaluate the first term in the right hand side of (4.58), we use the Borcherds' identity (3.6). Noting that by (4.56), $(q^{-\epsilon_a})_{(n)}E_{sp} = 0$ for all $n \geq 0$, we get

$$\left((t_0^{-1}t^{\epsilon_a}\tilde{d}_b)_{(-1)}q^{-\epsilon_a}\right)_{(1)} E_{sp} = \frac{1}{c}(t_0t^{-\epsilon_a}k_0)(t^{\epsilon_a}\tilde{d}_b)E_{sp} + \frac{1}{c}(t^{-\epsilon_a}k_0)(t_0t^{\epsilon_a}\tilde{d}_b)E_{sp}.$$

The first term is equal to zero since

$$\frac{1}{c}(t_0t^{-\epsilon_a}k_0)(t^{\epsilon_a}\tilde{d}_b)E_{sp} = \frac{1}{c}(t^{\epsilon_a}\tilde{d}_b)(t_0t^{-\epsilon_a}k_0)E_{sp} - \frac{1}{c}[t^{\epsilon_a}\tilde{d}_b, t_0t^{-\epsilon_a}k_0]E_{sp}$$

and we may apply (4.49) to the right hand side. Finally, using (4.51) and (4.19), we get

$$\frac{1}{c}(t^{-\epsilon_a}k_0)(t_0t^{\epsilon_a}\tilde{d}_b)E_{sp} = \left((1-\mu c)\delta_{bs}\delta_{ap} - \nu c\delta_{ab}\delta_{sp}\right)\mathbf{1}.$$

This completes the proof of part (b) of the Lemma, and we leave part (a) as an exercise for the reader.

Comparing Lemmas 4.8 and 3.6 (a), we conclude that the operators $(E_{ab})_{(n)}$ produce a representation of affine $\widehat{gl}_N$.

Lemma 4.9. *The following relations hold in* $L(T_0)$*:*

(a) $$(t_0^{-1}t^r\tilde{d}_0)q^m = \frac{1}{c}\sum_{p=1}^{N} m_p(t_0^{-1}k_p)q^{m+r},$$

(b) $$(t_0t^r\tilde{d}_0)(E_{ab})_{(-1)}q^m = \delta_{ab}(-1+2\nu c).$$

Proof.

$$(t_0^{-1}t^r\tilde{d}_0)q^m = \frac{1}{c}(t_0^{-1}t^r\tilde{d}_0)(t^mk_0)\mathbf{1} = \frac{1}{c}[t_0^{-1}t^r\tilde{d}_0, t^mk_0]\mathbf{1}$$

$$= \frac{1}{c}\sum_{p=1}^{N} m_p(t_0^{-1}t^{r+m}k_p)\mathbf{1} = \frac{1}{c}\sum_{p=1}^{N} m_p(t_0^{-1}k_p)q^{r+m}.$$

This proves the claim (a). Part (b) follows from Lemma 4.7(b), (4.50) and (4.37).

Finally let us study the properties of the field $\tilde{d}_0(r,z)$. This will require carrying out calculations with certain elements of degree 2. Let $v \in L(T_0), \deg(v) = 2$. Since $L(T_0)$ is an irreducible $\mathfrak{g}$-module, it is generated by any non-zero vector. If $v \neq 0$ then $U_2(\mathfrak{g}_+)v = T_0$. However it is easy to see that $\mathfrak{g}_+$ is generated by $\mathfrak{g}_1$. Thus $\mathfrak{g}_1 v = 0$ implies that $v = 0$. This observation will help us find a relation in $L(T_0)$ involving $(t_0^{-2}t^m\tilde{d}_0)\mathbf{1}$, which in turn will yield an expression for the field $\tilde{d}_0(m,z)$.

Lemma 4.10. *The following relation holds in $L(T_0)$:*

$$(t_0^{-2}t^m\tilde{d}_0)\mathbf{1} = (t_0^{-2}\tilde{d}_0)q^m + \frac{1}{c}\sum_{p,j=1}^{N} m_p(t_0^{-1}k_j)(E_{pj})_{(-1)}q^m - \frac{1}{c}(1-\mu c)\sum_{p=1}^{N} m_p(t_0^{-2}k_p)q^m.$$

Proof. We shall prove this Lemma by showing that the vector

$$v = (t_0^{-2}t^m\tilde{d}_0)\mathbf{1} - (t_0^{-2}\tilde{d}_0)q^m - \frac{1}{c}\sum_{p,j=1}^{N} m_p(t_0^{-1}k_j)(E_{pj})_{(-1)}q^m + \frac{1}{c}(1-\mu c)\sum_{p=1}^{N} m_p(t_0^{-2}k_p)q^m$$

is annihilated in $L(T_0)$ by $\mathfrak{g}_1$. Let us show that $(t_0t^s\tilde{d}_0)v = 0$ in $L(T_0)$:

$$(t_0t^s\tilde{d}_0)v = [t_0t^s\tilde{d}_0, t_0^{-2}t^m\tilde{d}_0]\mathbf{1} - [t_0t^s\tilde{d}_0, t_0^{-2}\tilde{d}_0]q^m$$

$$-\frac{1}{c}\sum_{p,j=1}^{N} m_p[t_0t^s\tilde{d}_0, t_0^{-1}k_j](E_{pj})_{(-1)}q^m - \frac{1}{c}\sum_{p,j=1}^{N} m_p(t_0^{-1}k_j)(t_0t^s\tilde{d}_0)(E_{pj})_{(-1)}q^m$$

$$+\frac{1}{c}(1-\mu c)\sum_{p=1}^{N} m_p[t_0t^s\tilde{d}_0, t_0^{-2}k_p]q^m$$

$$= 3(t_0^{-1}t^{m+s}\tilde{d}_0)\mathbf{1} + 4(\mu+\nu)(t_0^{-1}t^{m+s}k_0)\mathbf{1} - 2(\mu+\nu)\sum_{p=1}^{N} m_p(t_0^{-1}t^{m+s}k_p)\mathbf{1}$$

$$-3(t_0^{-1}t^s\tilde{d}_0)q^m - 4(\mu+\nu)(t_0^{-1}t^sk_0)q^m$$

$$-\frac{1}{c}\sum_{p,j=1}^{N} m_p(t^sk_j)(E_{pj})_{(-1)}q^m - \frac{1}{c}(-1+2\nu c)\sum_{p=1}^{N} m_p(t_0^{-1}k_p)q^{m+s}$$

$$+\frac{2}{c}(1-\mu c)\sum_{p=1}^{N} m_p(t_0^{-1}t^sk_p)q^m$$

$$= 4(\mu+\nu)\sum_{p=1}^{N}(m_p+s_p)(t_0^{-1}k_p)q^{m+s} - 2(\mu+\nu)\sum_{p=1}^{N} m_p(t_0^{-1}k_p)q^{m+s} - \frac{3}{c}\sum_{p=1}^{N} m_p(t_0^{-1}k_p)q^{m+s}$$

$$-4(\mu+\nu)\sum_{p=1}^{N} s_p(t_0^{-1}k_p)q^{m+s} - \frac{1}{c}\sum_{p,j=1}^{N} m_p(E_{pj})_{(-1)}(t^s k_j)q^m$$

$$-\frac{1}{c}(-1+2\nu c)\sum_{p,j=1}^{N} m_p(t_0^{-1}k_p)q^{m+s} + \frac{2}{c}(1-\mu c)\sum_{p=1}^{N} m_p(t_0^{-1}k_p)q^{m+s} = 0.$$

The cases of other elements spanning $\mathfrak{g}_1$ are treated similarly, and are left as an exercise.

Applying the state-field correspondence Y to both sides of the equality in Lemma 4.10, we will get the formula (4.29).

Let us complete the proof of Theorem 4.3. We have now established all the relations between the fields stated in part (c) of the Theorem. The universal enveloping vertex algebra $V_{\mathfrak{g}}$ is generated by the fields (4.1)-(4.3). The same is true for $L(T_0)$, since $L(T_0)$ is a factor vertex algebra of $V_{\mathfrak{g}}$. Taking into account the relations of part (c), the claim of part (a) of the Theorem follows. As we mentioned above, the claim of part (b) follows from the results of [BB].

Let us establish the claim of part (d) of the Theorem. First we construct a homomorphism of vertex algebras

$$\varphi : \quad V^+_{Hyp} \to L(T_0),$$

defined by $\varphi(e^{mu}) = q^m$, $\varphi(u_p(-1)\mathbf{1}) = \frac{1}{c}(t_0^{-1}k_p)\mathbf{1}$, $\varphi(v_p(-1)\mathbf{1}) = (t_0^{-1}\tilde{d}_p)\mathbf{1}$. Using (1.2), (4.20), (4.23) and (4.25), we can see that the images of the generators of V^+_{Hyp} satisfy the required relations (3.22)-(3.25), thus φ is indeed a homomorphism of vertex algebras. Since V^+_{Hyp} is a simple vertex algebra, the map φ is injective. The image of the Virasoro field of V^+_{Hyp} is

$$\varphi(\omega_{Hyp}(z)) = \frac{1}{c}\sum_{p=1}^{N} : d_p(z)k_p(z) :,$$

and the central charge of this Virasoro field is equal to rank $(V^+_{Hyp}) = 2N$.

We know that the fields $g(z)$ generate an affine subalgebra $\widehat{\dot{\mathfrak{g}}}$ in the toroidal algebra $\mathfrak{g}$, which commutes with the fields generated by the image of φ. This affine subalgebra has the central charge $c_{\dot{\mathfrak{g}}} = c$ on $L(T_0)$. Next, comparing Lemmas 4.8 and 3.6(a), we conclude that the fields $Y(E_{ab}, z)$ yield a representation of the affine $\widehat{gl}_N$ on $L(T_0)$ with central charges $c_{sl_N} = 1 - \mu c$ and $c_{\mathcal{H}ei} = N(1-\mu c) - N^2\nu c$. It follows from Lemma 4.7 (a) that the fields $Y(E_{ab}, z)$ also commute with the image of φ.

The relation (4.8) implies that the field $\tilde{d}_0(z)$ generates a Virasoro algebra with the central charge $12(\mu+\nu)c$. The fact that the element $t_0^{-1}\tilde{d}_0$ is an infinitesimal translation operator D follows from the relations (1.1), (1.2), (4.7) and (4.8). Thus $L(T_0)$ is a VOA of rank $12(\mu+\nu)c$ with the Virasoro field $\tilde{d}_0(z)$. The field $\tilde{d}_0(z) - \varphi(\omega_{Hyp}(z))$ yields another Virasoro algebra with central charge $12(\mu+\nu)c - 2N$. Comparing Lemmas 3.6 (b) and 4.9 (b), and noting that $\varphi(\omega_{Hyp}(z))$ commutes with $g(z)$ and $Y(E_{ab}, z)$, we get that the fields $\tilde{d}_0(z) - \varphi(\omega_{Hyp}(z))$, $g(z)$ and $Y(E_{ab}, z)$ yield a representation of the twisted Virasoro-affine algebra $\mathfrak{f}$ with $\dot{\mathfrak{f}} = \dot{\mathfrak{g}} \oplus gl_N$ and central character given by (4.30).

This allows us to define a homomorphism of vertex algebras

$$\psi : V_{\mathfrak{f}}(\gamma_0) \to L(T_0)$$

by $\psi(g(-1)\mathbf{1}) = (t_0^{-1}g)\mathbf{1}$, $\psi(E_{ab}(-1)\mathbf{1}) = E_{ab}$, $\psi(\omega_{\mathfrak{f}}) = (t_0^{-2}\tilde{d}_0)\mathbf{1} - \varphi(\omega_{Hyp})$. The sub-VOAs $\varphi(V_{Hyp}^+)$ and $\psi(V_{\mathfrak{f}})$ commute in $L(T_0)$. Thus we have a homomorphism

$$\theta : \quad V_{Hyp}^+ \otimes V_{\mathfrak{f}}(\gamma_0) \to L(T_0).$$

Since $\theta(\omega_{Hyp} + \omega_{\mathfrak{f}}) = (t_0^{-2}\tilde{d}_0)\mathbf{1}$, this is in fact a homomorphism of VOAs. Moreover, by part (a), θ is an epimorphism.

The $\mathfrak{f}$-module $V_{\mathfrak{f}}(\gamma_0)$ has a unique maximal submodule and a unique irreducible quotient $L_{\mathfrak{f}}(\gamma_0)$. By Theorem 3.2, $L_{\mathfrak{f}}(\gamma_0)$ is a unique quotient vertex algebra of $V_{\mathfrak{f}}(\gamma_0)$. Since V_{Hyp}^+ is a simple vertex algebra, we conclude that $V_{Hyp}^+ \otimes L_{\mathfrak{f}}(\gamma_0)$ is a unique simple quotient vertex algebra of $V_{Hyp}^+ \otimes V_{\mathfrak{f}}(\gamma_0)$. However $L(T_0)$ is also a simple quotient of $V_{Hyp}^+ \otimes V_{\mathfrak{f}}(\gamma_0)$. Thus

$$L(T_0) \cong V_{Hyp}^+ \otimes L_{\mathfrak{f}}(\gamma_0).$$

This completes the proof of Theorem 4.3.

5. Realizations for the irreducible modules in category $\mathcal{B}_\chi$.

In this section we are going to use the theory of VOA modules to give realizations for all irreducible $\mathfrak{g}(\mu,\nu)$-modules in category $\mathcal{B}_\chi$. By the principle of preservation of identities [Li], every VOA module for $V_{Hyp}^+ \otimes L_{\mathfrak{f}}(\gamma_0)$ is also a module for the Lie algebra $\mathfrak{g}(\mu,\nu)$. However in order to get all irreducible modules in $\mathcal{B}_\chi$, we need to use a larger VOA

$$V(T_0) = V_{Hyp}^+ \otimes V_{\mathfrak{f}}(\gamma_0),$$

which has more irreducible modules than $V_{Hyp}^+ \otimes L_{\mathfrak{f}}(\gamma_0)$. In order to carry out this plan, we should first prove that $V_{Hyp}^+ \otimes V_{\mathfrak{f}}(\gamma_0)$ also admits a structure of a $\mathfrak{g}(\mu,\nu)$-module.

First, we need to establish the following technical lemma:

Lemma 5.1. *For a Zariski dense set of triples (c,μ,ν), the modules $V_{\mathfrak{f}}(\gamma_0)$ and $L_{\mathfrak{f}}(\gamma_0)$ coincide.*

Proof. It follows from Proposition 3.8 and (4.30) that whenever $c \neq 0$, $c \neq -h^\vee$, $c_{sl_N} = 1 - \mu c \neq -N$, $c_{\mathcal{H}ei} = N(1-\mu c) - N^2\nu c \neq 0$, the VOA $V_{\mathfrak{f}}(\gamma_0)$ factors into a tensor product of four VOAs: $V_{\widehat{\dot{\mathfrak{g}}}}(c_{\dot{\mathfrak{g}}})$, $V_{\widehat{sl}_N}(c_{sl_N})$, $V_{\mathcal{H}ei}(c_{\mathcal{H}ei})$ and $V_{\mathcal{V}ir}(c'_{\mathcal{V}ir})$. It is clear that $V_{\mathfrak{f}}(\gamma_0)$ is a simple VOA if and only if each of these four VOAs is simple. First of all, under the assumption that $c_{\mathcal{H}ei} \neq 0$, the Heisenberg VOA is simple. We are going to show that the remaining affine and Virasoro VOAs are simple for (c,μ,ν) in a dense subset of $\mathbb{C}^3$.

We note that affine and Virasoro VOAs are the generalized Verma modules for the respective Lie algebras, and a generalized Verma module admits a Shapovalov form [J]. Using the Shapovalov determinant argument, it is not difficult to see that the VOA $V_{\widehat{\dot{\mathfrak{g}}}}(c_{\dot{\mathfrak{g}}})$ (resp. $V_{\widehat{sl}_N}(c_{sl_N})$, $V_{\mathcal{V}ir}(c'_{\mathcal{V}ir})$) is simple outside a countable set of values of the central charge $c_{\dot{\mathfrak{g}}}$ (resp. c_{sl_N}, $c'_{\mathcal{V}ir}$). In fact, an explicit formula for the Shapovalov determinant for the generalized Verma modules for the affine algebras may be found in [KK], while for the Virasoro VOA, it follows from the description of the irreducible modules for the Virasoro algebra [FF] that

$V_{\mathcal{V}ir}(c'_{\mathcal{V}ir})$ is simple if and only if $c'_{\mathcal{V}ir} \neq 1 - 6\frac{(r-s)^2}{rs}$, where r and s are relatively prime integers, $r, s > 1$.

Each inequality on the values of the central charges defines a dense Zariski open subset of values of (c, μ, ν). Since a countable intersection of dense Zariski open subsets in $\mathbb{C}^3$ is Zariski dense, we establish the claim of the Lemma.

Proposition 5.2. Let $c \neq 0$, and let γ_0 be given by (4.30). Then $V(T_0) = V^+_{Hyp} \otimes V_{\mathfrak{f}}(\gamma_0)$ has a structure of a $\mathfrak{g}(\mu, \nu)$-module given by the formulas of Theorem 4.3 (b)-(d).

Proof. We proved in Theorem 4.3 (d) that $V^+_{Hyp} \otimes L_{\mathfrak{f}}(\gamma_0)$ is a $\mathfrak{g}(\mu, \nu)$-module. Now we want to prove the same for $V^+_{Hyp} \otimes V_{\mathfrak{f}}(\gamma_0)$. This amounts to verifying relations (4.9)-(4.16) in this vertex algebra. It is possible to do this directly, and this was the approach taken, for example, in [B4], but we are going to present here an alternative argument that allows us to circumvent these rather tedious computations.

By Lemma 5.1, for a Zariski dense set of triples (c, μ, ν), the modules $V_{\mathfrak{f}}(\gamma_0)$ and $L_{\mathfrak{f}}(\gamma_0)$ coincide. Thus for the generic values of (c, μ, ν) the VOAs $V^+_{Hyp} \otimes V_{\mathfrak{f}}(\gamma_0)$ are indeed $\mathfrak{g}(\mu, \nu)$-module, and the relations (4.9)-(4.16) hold. However the commutator formula (3.2) applied to the left hand sides of the relations (4.9)-(4.16) in $V^+_{Hyp} \otimes V_{\mathfrak{f}}(\gamma_0)$ will yield expressions with coefficients that are polynomials in $c^{\pm 1}, \mu, \nu$. Since these agree with the right hand sides of (4.9)-(4.16) on a Zariski dense set of parameters, the equalities must hold for all values of (c, μ, ν) with $c \neq 0$. This concludes the proof of the Proposition.

Now we are ready to give realizations for all irreducible $\mathfrak{g}$-modules in category $\mathcal{B}_\chi$ using highest weight $\mathfrak{f}$-modules.

Theorem 5.3. *Let $c \neq 0$, and let $L(T)$ be an irreducible module in category $\mathcal{B}_\chi$ determined by the data (V, W, h, d, α) as in Theorem 2.5, where V is a finite-dimensional irreducible $\dot{\mathfrak{g}}$-module, W is a finite-dimensional irreducible sl_N-module, $\alpha \in \mathbb{C}^N$ and $h, d \in \mathbb{C}$. Then*

$$L(T) \cong M^+_{Hyp}(\alpha) \otimes L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma_0),$$

where γ_0 is the same as in Theorem 4.3,

$$h_{\mathcal{H}ei} = h - N\nu c, \quad h_{\mathcal{V}ir} = -d + \frac{1}{2}(\mu + \nu)c. \tag{5.1}$$

Proof. First of all, we are going to show that $M^+_{Hyp}(\alpha) \otimes L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma_0)$ is a $\mathfrak{g}(\mu, \nu)$-module. Indeed, it is a module for the vertex algebra $V^+_{Hyp} \otimes V_{\mathfrak{f}}(\gamma_0)$, and by principle of preservation of identities [Li], the $\mathfrak{g}(\mu, \nu)$-module structure on $V^+_{Hyp} \otimes V_{\mathfrak{f}}(\gamma_0)$ gets transferred to its VOA module $M^+_{Hyp}(\alpha) \otimes L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma_0)$.

Next, let us show that $M^+_{Hyp}(\alpha) \otimes L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma_0)$ is irreducible as a $\mathfrak{g}$-module. This is not difficult to see. The fields that define the $\mathfrak{g}$-module structure on $V^+_{Hyp} \otimes V_{\mathfrak{f}}(\gamma_0)$ generate this VOA. Thus any $\mathfrak{g}$-submodule in $M^+_{Hyp}(\alpha) \otimes L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma_0)$ is also a VOA submodule. However as a VOA module, $M^+_{Hyp}(\alpha) \otimes L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma_0)$ is irreducible. Thus it is also irreducible as a $\mathfrak{g}$-module.

It is also easy to see that the $\mathfrak{g}(\mu,\nu)$-module $M^+_{Hyp}(\alpha)\otimes L_{\mathfrak{f}}(V,W,h_{\mathcal{H}ei},h_{\mathcal{V}ir},\gamma_0)$ belongs to the category $\mathcal{B}_\chi$, and its top is

$$\mathbb{C}[q_1^{\pm},\ldots,q_N^{\pm}]\otimes V\otimes W.$$

We can derive from (4.25)-(4.29) that $\mathfrak{g}_0$ acts on this top according to (2.1), (2.6)-(2.8). Taking into account the relations (4.4) and (4.5), we conclude that the top of $M^+_{Hyp}(\alpha)\otimes L_{\mathfrak{f}}(V,W,h_{\mathcal{H}ei},h_{\mathcal{V}ir},\gamma_0)$ is isomorphic to T as a $\mathfrak{g}_0$-module. Since two simple modules with the same top are isomorphic, we obtain the claim of the Theorem.

Finally, applying Corollary 3.9, we obtain the following result:

Theorem 5.4. *Let $L(T)$ be the irreducible $\mathfrak{g}(\mu,\nu)$-module in category $\mathcal{B}_\chi$ determined by the data (V,W,h,d,α) as in the statement of Theorem 2.5. Suppose*

$$c\neq 0,\quad c\neq -h^\vee,\quad c_{sl_N}=1-\mu c\neq -N,$$

$$c_{\mathcal{H}ei}=N(1-\mu c)-N^2\nu c\neq 0,$$

$$c'_{\mathcal{V}ir}=12c(\mu+\nu)-2N-\frac{c\dim(\dot{\mathfrak{g}})}{c+h^\vee}-\frac{c_{sl_N}(N^2-1)}{c_{sl_N}+N}-1+12\frac{N^2(\frac{1}{2}-\nu c)^2}{c_{\mathcal{H}ei}}.$$

Let

$$h_{\mathcal{H}ei}=h-N\nu c,$$

$$h'_{\mathcal{V}ir}=-d+\frac{1}{2}(\mu+\nu)c-\frac{\Omega_V}{2(c+h^\vee)}-\frac{\Omega_W}{2(c_{sl_N}+N)}-\frac{h_{\mathcal{H}ei}(h_{\mathcal{H}ei}-N(1-2\nu c))}{2c_{\mathcal{H}ei}}.$$

Then

$$(a)\qquad L(T)\cong M_{Hyp}(\alpha)\otimes L_{\widehat{\dot{\mathfrak{g}}}}(V,c)\otimes L_{\widehat{sl}_N}(W,c_{sl_N})\otimes L_{\mathcal{H}ei}(h_{\mathcal{H}ei},c_{\mathcal{H}ei})\otimes L_{\mathcal{V}ir}(h'_{\mathcal{V}ir},c'_{\mathcal{V}ir}),$$

$$(b)\quad \mathrm{char}\, L(T)=\mathrm{char}\, q^\alpha\mathbb{C}[q_1^{\pm},\ldots,q_N^{\pm}]\times\prod_{j\geq 1}(1-t^j)^{-(2N+1)}$$
$$\times\,\mathrm{char}\, L_{\widehat{\dot{\mathfrak{g}}}}(V,c)\times\mathrm{char}\, L_{\widehat{sl}_N}(W,c_{sl_N})\times\mathrm{char}\, L_{\mathcal{V}ir}(h'_{\mathcal{V}ir},c'_{\mathcal{V}ir}).$$

Remark 5.5. In case of 2-toroidal Lie algebras ($N=1$), the sl_N piece will not be present, and should be omitted from all the statements in this paper.

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