ANNALES DE L'INSTITUT FOURIER

PIOTR GRACZYK

A central limit theorem on the space of positive definite symmetric matrices

Annales de l'institut Fourier, tome 42, n° 4 (1992), p. 857-874

<http://www.numdam.org/item?id=AIF_1992__42_4_857_0>

Ann. Inst. Fourier, Grenoble
42, 4 (1992), 857-874

A CENTRAL LIMIT THEOREM ON THE SPACE OF POSITIVE DEFINITE SYMMETRIC MATRICES

by Piotr GRACZYK

0. Introduction.

Central limit theorems for rotation-invariant random variables on the symmetric space $\mathcal{P}_n$ of positive definite symmetric $n \times n$ matrices have been investigated in case $n = 2$ by Karpelevich, Tutubalin and Shur ([6]), Faraut ([1]) and Terras ([9]), in case $n = 3$ by Terras ([10]) and for n arbitrary by Richards ([8]). They find applications in multivariate statistics and in some engineering problems ([9]).

In this paper we prove a central limit theorem of Lindeberg-Feller type on the space $\mathcal{P}_n$. It generalizes a theorem obtained by Faraut ([1]) for $n = 2$. To state and prove this theorem we introduce on $\mathcal{P}_n$ some analogs of the mean and dispersion in the real case.

Sections 1 and 2 provide the basic definitions and facts from the harmonic analysis on $\mathcal{P}_n$ used in the paper. In Section 3 we define and investigate the mean and the dispersions on $\mathcal{P}_n$. In Section 4 we derive in a simple way a Taylor expansion of the spherical functions on $\mathcal{P}_n$. Section 5 contains the main result of the paper.

This paper was written during the author's stay at the University Paris VI. The author is very grateful to Jacques Faraut for his great hospitality, suggesting the problem and many fruitful discussions.

Key words : Symmetric spaces – Central limit theorem – Spherical functions.
A.M.S. Classification : 43A05 – 60B15 – 60F05.

1. Preliminaries.

Throughout this paper $G = GL(n, \mathbb{R})$ will denote the general linear group of $n \times n$ nonsingular matrices and $K = O(n)$ the group of $n \times n$ orthogonal matrices. The symmetric space G/K is identified with $\mathcal{P}_n$, the space of real, positive definite symmetric $n \times n$ matrices.

The transitive action of G on $\mathcal{P}_n$ is defined by

$$X \mapsto X[g] = gXg^t$$

where $g \in G$, $X \in \mathcal{P}_n$ and g^t is the transpose of g. The correspondence of G/K and $\mathcal{P}_n$ is given by $gK \mapsto I[g] = gg^t$, where $I \in \mathcal{P}_n$ is the identity matrix. The space $\mathcal{P}_n$ is a Riemannian manifold with the *arc length* $ds^2 = \mathrm{Tr}\big((X^{-1}dX)^2\big)$ for $X = (x_{ij})_{i,j \le n}$, $dX = (dx_{ij})_{i,j\le n}$.

A differential operator L on $\mathcal{P}_n$ is said to be *G-invariant* if it commutes with the action of G, that is for every $g \in G$ and $f \in C^\infty(\mathcal{P}_n)$

$$(Lf)^g = L(f^g)$$

where $f^g(X) = f(X[g])$ when $X \in \mathcal{P}_n$.

The algebra $\mathbb{D}(\mathcal{P}_n)$ of all G-invariant differential operators on $\mathcal{P}_n$ is commutative and isomorphic with the algebra $I(\mathfrak{a})$ of symmetric polynomials on the Cartan space $\mathfrak{a} = \{H | H \text{diagonal}\} \cong \mathbb{R}^n$. By Newton's theorem the algebra $I(\mathfrak{a})$ is generated by the symmetric polynomials :

$$p_j(\mathbf{x}) = \sum_{i=1}^{n} x_i^j, \qquad j = 0, \ldots, n. \tag{1}$$

We will denote $\gamma : \mathbb{D}(\mathcal{P}_n) \to I(\mathfrak{a})$ the isomorphism of $\mathbb{D}(\mathcal{P}_n)$ and $I(\mathfrak{a})$. Remark that the order of $L \in \mathbb{D}(\mathcal{P}_n)$ equals to the degree of $\gamma(L)$ ([4], p. 306).

A function $h : \mathcal{P}_n \to \mathbb{C}$ is *K-invariant* if $h^k = h$ for all $k \in O(n)$. A K-invariant function h is said to be *spherical* if $h(I) = 1$ and h is an eigenfunction of all G-invariant differential operators on $\mathcal{P}_n$. All the spherical functions are given by

$$\Phi_{\mathbf{s}}(X) = \int_K \Delta_1^{s_1 - s_2}(X[k]) \ldots \Delta_{n-1}^{s_{n-1}-s_n}(X[k]) \Delta_n^{s_n}(X[k]) dk \tag{2}$$

where $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n, \Delta_j(Y)$ is the principal minor of order j of Y and dk denotes the normalised Haar measure on K. Formula (2) corresponds to the classical Harish-Chandra integral formula for spherical functions on G with $\mathbf{s} = \frac{\lambda+\rho}{2}$, where $\rho = (\rho_1, \ldots, \rho_n)$, $\rho_j = \frac{1}{2}(2j-n-1)$, $\lambda \in \mathbb{C}^n$. We will write $\varphi_\lambda = \Phi_{\mathbf{s}}$. By ([4], p. 418) we have then $L\varphi_\lambda = \gamma(L)(\lambda)\varphi_\lambda$ for $L \in \mathbb{D}(\mathcal{P}_n)$.

Terras ([10],[11]) and Richards ([8]) use the coordinates $\mathbf{r} = (r_1, \ldots, r_n)$ with $\mathbf{r} = \frac{\lambda}{2}$.

Let W denote the Weyl group of permutations. Then $\varphi_\lambda = \varphi_{w\lambda}$ for all $w \in W$. One has also $\Phi_\rho = \varphi_\rho \equiv 1$. For all the details concerning the spherical functions on $\mathcal{P}_n$ see e.g.[11].

The following lemma describes the relationship between the spherical functions on $\mathcal{P}_n$ and the spherical functions on

$$\mathcal{SP}_n = \{X \in \mathcal{P}_n | \det X = 1\}.$$

The space $\mathcal{SP}_n$ may be identified with the symmetric space $SL(n)/SO(n)$.

LEMMA 1.

$$\Phi_{\mathbf{s}}(X) = (\det X)^{\frac{1}{n}\sum_{i=1}^n s_i}\Psi_{\mathbf{s}}(X_1) \tag{3}$$

where $X = (\det X)^{\frac{1}{n}} X_1$ *with* $X_1 \in \mathcal{SP}_n$ *and*

$$\Psi_{\mathbf{s}}(X_1) = \int_{SO(n)} \Delta_1^{s_1-s_2}(X_1[k]) \ldots \Delta_{n-1}^{s_{n-1}-s_n}(X_1[k]) dk_{SO(n)}$$

is a spherical function on $\mathcal{SP}_n$.

Proof. — By (2) one gets

$$\Phi_{\mathbf{s}}(X) = (\det X)^{\frac{1}{n}\sum_{i=1}^n s_i} \int_K \Delta_1^{s_1-s_2} \ldots \Delta_{n-1}^{s_{n-1}-s_n}(X_1[k]) dk. \tag{4}$$

Using the fact that $dk|_{SO(n)} = \frac{1}{2} dk_{SO(n)}$ and the invariance of dk it follows that the integral in (4) equals $\frac{1}{2}(\Psi_{\mathbf{s}}(X_1) + \Psi_{\mathbf{s}}(X_1[h]))$ for all $h \in O(n)$ such that $\det(h) = -1$. There exist $k_0 \in SO(n)$ and h_0, $\det(h_0) = -1$, such that $X_1[k_0]$ is diagonal and $X_1[h_0 k_0] = X_1[k_0]$. Then $\det(h_0 k_0) = -1$ and $\Psi_{\mathbf{s}}(X_1[h_0 k_0]) = \Psi_{\mathbf{s}}(X_1)$. □

Theorem of Helgason-Johnson ([4], p. 458) and Lemma 1 imply that the spherical function φ_λ is bounded on $\mathcal{P}_n$ if and only if $\mathrm{Re}\left(\sum_{i=1}^n \lambda_i\right) = 0$

and Re$\lambda \in C(\rho)$, where $C(\rho)$ denotes the convex enveloppe of $\{w\rho | w \in W\}$. Then $|\varphi_\lambda| \leq 1$.

In the sequel we will often use the G-invariant differential operators on $\mathcal{P}_n$ of order 1 and 2. Let us state some of their principal properties now.

All the differential operators of order 1 in $\mathbb{D}(\mathcal{P}_n)$ are given up to a multiplicative constant by the *Euler operator* E which is defined by

$$Ef(X) = \frac{d}{dt} f(tX)|_{t=1}. \tag{5}$$

Note that E is homogeneous of degree 0. Lemma 1 shows that a spherical function $\Phi_{\mathbf{s}}$ is homogeneous of order $\sum_{i=1}^{n} s_i$, so

$$E\Phi_{\mathbf{s}} = \left(\sum_{i=1}^{n} s_i\right) \Phi_{\mathbf{s}}.$$

We denote the eigenvalue of E acting on $\Phi_{\mathbf{s}}$ by $\gamma_1(\mathbf{s})$.

If f is K-invariant on $\mathcal{P}_n$ then by the spectral decomposition of $\mathcal{P}_n$ it suffices to know f on

$$A = \{a \in \mathcal{P}_n | a \text{ diagonal with } a_{ii} > 0\}.$$

The Lie algebra of A is given by $\mathfrak{a}$. One denotes the diagonal entries of $H \in \mathfrak{a}$ by $h_1, \ldots, h_n$. Formula (5) implies

$$Ef(\exp H) = \sum_{i=1}^{n} \frac{\partial \tilde{f}}{\partial h_i} \tag{6}$$

where $\tilde{f}(h_1, \ldots, h_n) = f(\exp H)$.

The G-invariant differential operators of order 2 (without terms of lower order) are given by the linear combinations of E^2 and the *Laplace-Beltrami operator* Δ on $\mathcal{P}_n$. If f is K-invariant then by [2]VI,4.2

$$\Delta f(\exp H) = \sum_j \frac{\partial^2 \tilde{f}}{\partial h_j^2} + \frac{1}{2} \sum_{i<j} \coth\left(\frac{h_i - h_j}{2}\right) \left(\frac{\partial}{\partial h_i} - \frac{\partial}{\partial h_j}\right) \tilde{f}. \tag{7}$$

In order to find the eigenvalues of Δ acting on $\Phi_{\mathbf{s}}$ one may use the horospherical part of Δ (see [2]VI,4.4). If N denotes the nilpotent group of lower triangular matrices with 1 on the diagonal and if f is N-invariant on

$\mathcal{P}_n$, then $f(X) = F(a_1, \ldots, a_n)$ where $(a_i)_{1\le i\le n}$ are the eigenvalues of X and

$$\Delta f(X) = \sum_{i=1}^{n} \left(a_i \frac{\partial}{\partial a_i} \right)^2 F - \sum_{i=1}^{n} \rho_i a_i \frac{\partial F}{\partial a_i}.$$

Remark that the function under the integral in (2) is N-invariant. It follows that

$$\Delta \Phi_{\mathbf{s}} = \gamma_2(\mathbf{s}) \Phi_{\mathbf{s}}$$

with $\gamma_2(\mathbf{s}) = (\mathbf{s} - \rho | \mathbf{s}) = \frac{1}{4}(\|\lambda\|^2 - \|\rho\|^2)$.

It is easy to check that Δ is elliptic and E^2 is semi-elliptic on $\mathcal{P}_n$. All the elliptic second order operators in $\mathbb{D}(\mathcal{P}_n)$ are given by

$$L = a\left(\Delta - \frac{1}{n} E^2 \right) + bE^2 + cE + d, \quad a, b > 0. \tag{8}$$

Observe that the operator $\Omega = \Delta - \frac{1}{n} E^2$ restrained to $\mathcal{SP}_n$ is the Laplace-Beltrami operator on $\mathcal{SP}_n$. Ω is semi-elliptic on $\mathcal{P}_n$.

2. K-invariant probability measures on $\mathcal{P}_n$.

A probability measure μ on $\mathcal{P}_n$ is said to be *K-invariant* if for any measurable subset B of $\mathcal{P}_n$ and for all $k \in K$ we have $\mu(kBk^t) = \mu(B)$. We shall then write $\mu \in M^\natural(\mathcal{P}_n)$.

In this paper we consider only K-invariant measures on $\mathcal{P}_n$. We can identify such measures with K-biinvariant measures on G. Then the *convolution* $\mu_1 * \mu_2$ of two measures in $M^\natural(\mathcal{P}_n)$ is defined by the convolution of the corresponding measures on G and then projecting on $\mathcal{P}_n$. This convolution is commutative.

The *spherical Fourier transform* of a measure $\mu \in M^\natural(\mathcal{P}_n)$ is defined by

$$\hat{\mu}(\lambda) = \int_{\mathcal{P}_n} \varphi_\lambda(X) d\mu(X)$$

for λ such that φ_λ is bounded. In Section 1 we have formulated sufficient and necessary conditions for such λ. We also write

$$\hat{\mu}(\mathbf{s}) = \int_{\mathcal{P}_n} \Phi_{\mathbf{s}}(X) d\mu(X).$$

The spherical Fourier transform carries the convolution of K-invariant measures on $\mathcal{P}_n$ into the usual product

$$\widehat{\mu_1 * \mu_2}(\mathrm{s}) = \hat{\mu}_1(\mathrm{s})\hat{\mu}_2(\mathrm{s}).$$

If $\mu \in M^\natural(\mathcal{P}_n)$ is infinitely divisible, the infinitesimal generator of the continuous semigroup of measures $(\mu_t)_{t>0}$ with $\mu_1 = \mu$ is given by the Hunt formula ([5]). It is then natural to call μ *Gaussian* if the generator of $(\mu_t)_{t>0}$ is a second order G-invariant elliptic differential operator on $\mathcal{P}_n$ which annihilates constants (cf.[3]). If the generator is semi-elliptic we say that μ is *Gaussian degenerate.* By (8) the Fourier transform of a Gaussian measure $\mu \in M^\natural(\mathcal{P}_n)$ has the following form

$$\hat{\mu}(\mathrm{s}) = \exp\left[a\gamma_2(\mathrm{s}) + \left(b - \frac{1}{n}a\right)\gamma_1^2(\mathrm{s}) + c\gamma_1(\mathrm{s})\right]$$

with $a, b > 0$, $c \in \mathbb{R}$.

Examples. — (*i*) The Laplace-Beltrami operator Δ is the generator of the *heat semigroup* $(\kappa_t)_{t>0}$ on $\mathcal{P}_n$ (cf.[11]). We have

$$\hat{\kappa}_t(\mathrm{s}) = \exp[t(\mathrm{s} - \rho|\mathrm{s})].$$

Certainly, the measures κ_t are Gaussian on $\mathcal{P}_n$.

(*ii*) The operator $\Omega = \Delta - \frac{1}{n}E^2$ is the generator of the semigroup $(\nu_t)_{t>0}$ on $\mathcal{P}_n$ which extends naturally the heat semigroup $(\tilde{\nu}_t)_{t>0}$ on $\mathcal{SP}_n$, i.e. $\nu_t(B) = \tilde{\nu}_t(B \cap \mathcal{SP}_n)$. We have

$$\hat{\nu}_t(\mathrm{s}) = \exp\left\{t\left[(\mathrm{s} - \rho|\mathrm{s}) - \frac{1}{n}\left(\sum_{i=1}^{n} s_i\right)^2\right]\right\}$$

and ν_t are Gaussian degenerate.

(*iii*) Let $\mathfrak{n}_t$ be a random variable on $\mathbb{R}$ with normal distribution of mean 0 and variance t. Let η_t be the probability distribution of the random variable $\exp(\mathfrak{n}_t I)$ on $\mathcal{P}_n$. We have then

$$\hat{\eta}_t(\mathrm{s}) = \exp\left[\frac{t}{2}\left(\sum_{i=1}^{n} s_i\right)^2\right].$$

The generator of the semigroup $(\eta_t)_{t>0}$ equals $\frac{1}{2}E^2$. This operator corresponds to the Laplacian in the $\mathbb{R}^+I$-direction of the space $\mathcal{P}_n$ considered as a product $\mathcal{SP}_n \times \mathbb{R}^+$. The measures η_t are Gaussian degenerate on $\mathcal{P}_n$.

(*iv*) If δ_t is the Dirac delta in $\exp(tI)$ then

$$\hat{\delta}_t(\mathbf{s}) = \exp\left(t\sum_{i=1}^{n} s_i\right)$$

and the generator of $(\delta_t)_{t>0}$ is E.

Note that $\kappa_t = \nu_t * \eta_{2tn^{-1}}$. All the Gaussian measures on $\mathcal{P}_n$ have the form $\nu_t * \eta_u * \delta_w$ for some t, u positive and $w \in \mathbb{R}$.

3. The mean and the dispersions on $\mathcal{P}_n$.

In order to analyse the asymptotic behavior of K-invariant measures on $\mathcal{P}_n$ we need some analogs of the mean and the covariance of a measure on $\mathbb{R}^n$. In this section we introduce in a natural way such analogs and prove some properties of them.

3.1. Choice of dispersions on $\mathcal{P}_n$.

To have an analog of the covariance on $\mathcal{P}_n$ one seeks an application

$$D : M^{\natural}(\mathcal{P}_n) \to [0, \infty]$$

satisfying for all $\mu_1, \mu_2 \in M^{\natural}(\mathcal{P}_n)$ the condition

$$D(\mu_1 * \mu_2) = D(\mu_1) + D(\mu_2). \tag{9}$$

One also assumes that there exists an analytic, K-invariant function Q on $\mathcal{P}_n$ such that

$$D(\mu) = \int Q(X) d\mu(X) \tag{10}$$

for all $\mu \in M^{\natural}(\mathcal{P}_n)$. The function D will be called a *dispersion* on $\mathcal{P}_n$. Observe that the condition (9) is equivalent to

$$\int_K Q(Y[xk])dk = Q(I[x]) + Q(Y) \tag{11}$$

for all $x \in G$ and $Y \in \mathcal{P}_n$.

In the real case the covariance of a centralised measure may be represented by a second order derivative of its Fourier transform in 0. In the case of $\mathcal{P}_n$ the spherical function $\varphi_\lambda \equiv 1$ if $\lambda = \rho$. It turns out that on $\mathcal{P}_n$ we have the following analogous property.

THEOREM 1. — *If Q is an analytic, K-invariant function on $\mathcal{P}_n$ verifying (11) then there exists a vector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ such that*

$$Q(X) = \frac{\partial \varphi_\lambda(X)}{\partial \mathbf{v}}\Big|_{\lambda=\rho}, \quad X \in \mathcal{P}_n. \tag{12}$$

Proof. — First observe that if Q satisfies (11) then for all $L \in \mathbb{D}(\mathcal{P}_n)$ annihilating constants $LQ = \text{const}$. Indeed, let us apply L to (11) for x fixed and then put $Y = I$. One gets $LQ(xx^t) = LQ(I)$ for all $x \in G$, so LQ is constant on $\mathcal{P}_n$.

Next remark that if Q_1 and Q_2 verify (11) and $LQ_1 = LQ_2$ for all $L \in \mathbb{D}(\mathcal{P}_n)$ annihilating constants then $Q_1 = Q_2$. This property comes out from the analyticity of Q_1 and Q_2 and the fact that (11) implies $Q_1(I) = Q_2(I) = 0$.

If $Q = \frac{\partial \varphi_\lambda}{\partial \mathbf{v}}|_{\lambda=\rho}$ for any vector $\mathbf{v}$ then by the convolution property of the spherical Fourier transform on $\mathcal{P}_n$ the condition (9) holds for D corresponding to such Q.

Let now Q be arbitrary satisfying (11). To prove the theorem it suffices to show that there exists a vector $\mathbf{v} = (v_1, \ldots, v_n)$ such that

$$L_j Q = L_j \left(\frac{\partial \varphi_\lambda}{\partial \mathbf{v}}\Big|_{\lambda=\rho} \right) = \mathbf{v} \cdot \operatorname{grad}(L_j \varphi_\lambda)|_{\lambda=\rho}$$

for $L_j = \gamma^{-1}(p_j)$ with p_j as in (1), $j = 1, \ldots, n$. We have $\mathbf{l}_j = \operatorname{grad}(L_j\varphi_\lambda)|_{\lambda=\rho} = \operatorname{grad}\, \gamma(L_j)|_{\lambda=\rho} = \operatorname{grad}\, p_j|_{\lambda=\rho} = j(\rho_1^{j-1}, \ldots, \rho_n^{j-1})$. It follows that $\mathbf{l}_1, \ldots, \mathbf{l}_n$ are independent and the system of equations

$$L_j Q = \mathbf{v} \cdot \mathbf{l}_j, \quad j = 1, \ldots, n$$

has a solution. □

Now we want to choose the directions $\mathbf{v}$ of derivation in (12) so as the function Q was nonnegative. The decomposition (3) shows that this is possible only for $\sum_{i=1}^{n} v_i = 0$.

By reasons of convexity (or more generally by the Helgason-Johnson theorem) the spherical functions verify $0 < \varphi_\lambda \leq 1$ for $\lambda \in C(\rho)$. Thus, if

$\rho+t\mathbf{v} \in C(\rho)$ for t positive sufficiently small then $-\frac{\partial \varphi_\lambda}{\partial \mathbf{v}}|_{\lambda=\rho}$ is nonnegative. One will differentiate in the directions of neighbour vertices of ρ in $C(\rho)$. They are given by the permutations of neighbour entries of ρ :

$$\beta_1 = (\rho_2, \rho_1, \rho_3, \dots, \rho_n)$$

$$\dots$$

$$\beta_{n-1} = (\rho_1, \rho_2, \dots, \rho_n, \rho_{n-1}).$$

Then the vectors $\mathbf{v}_j = \beta_j - \rho = (0, \dots, 1, -1, \dots, 0)$ lie on the edges of $C(\rho)$ beginning in ρ. Observe that $\mathbf{v}_j = -\alpha_j$, $j = 1, \dots, n-1$, where α_j are the simple positive roots corresponding to the Iwasawa decomposition $G = NAK$ with N lower triangular. The vectors $\mathbf{v}_j$, $j = 1, \dots, n-1$, are independent and span all the hyperplane $\{\lambda | \sum \lambda_i = 0\}$. By reasons of normalisation we will differentiate with respect to the vectors $2\mathbf{v}_j$.

DEFINITION. — *The dispersions D_j on $\mathcal{P}_n$, $j = 1, \dots, n-1$, are defined by*

$$D_j(\mu) = \int Q_j(X) d\mu(X)$$

where

$$Q_j(X) = -2\frac{\partial \varphi_\lambda(X)}{\partial \mathbf{v}_j}|_{\lambda=\rho}.$$

Then one has

(13)
$$Q_j(X) = 2\left(\frac{\partial}{\partial \lambda_{j+1}} - \frac{\partial}{\partial \lambda_j}\right) \varphi_\lambda(X)\big|_{\lambda=\rho} = \left(\frac{\partial}{\partial s_{j+1}} - \frac{\partial}{\partial s_j}\right) \Phi_{\mathbf{s}}(X)\big|_{\mathbf{s}=\rho}$$

and for $\mu \in M^\natural(\mathcal{P}_n)$

$$D_j(\mu) = 2\left(\frac{\partial}{\partial \lambda_{j+1}} - \frac{\partial}{\partial \lambda_j}\right) \hat{\mu}(\lambda)\big|_{\lambda=\rho} = \left(\frac{\partial}{\partial s_{j+1}} - \frac{\partial}{\partial s_j}\right) \hat{\mu}(\mathbf{s})\big|_{\mathbf{s}=\rho}. \quad (14)$$

Example. — A direct calculation using (14) allows to find the dispersions of the measures considered in Section 2 :

$$D_j(\kappa_t) = D_j(\nu_t) = t\ ;$$
$$D_j(\eta_t) = 0\ ;$$
$$D_j(\delta_t) = 0.$$

We shall give now some properties of the functions Q_j.

THEOREM 2.

(i) $Q_j(X) = Q_j(tX)$, $j = 1, \dots, n-1$, *for all* $X \in \mathcal{P}_n$ *and* $t > 0$.

(ii) $Q_j(I) = 0$, $j = 1, \dots, n-1$.

(iii) $Q_1(X) + \dots + Q_{n-1}(X) > 0$ *for all* $X \neq tI$.

Proof. — (*i*) follows obviously from (3) and (13). (*ii*) follows from $\Phi_{\mathbf{s}}(I) = 1$ for all $\mathbf{s}$. To prove (*iii*) it suffices to consider $X \in \mathcal{S}P_n$, $X \neq I$. Suppose that $Q_1(X) = \dots = Q_{n-1}(X) = 0$. Then $\frac{\partial \varphi_\lambda}{\partial \mathbf{u}}|_{\lambda=\rho} = 0$ for all the directions $\mathbf{u}$ such that $\sum u_i = 0$. If $\sum \lambda_i = 0$, by (3) $\varphi_\lambda(X) = \psi_\lambda(X)$ where ψ_λ denotes the spherical function on $\mathcal{SP}_n$ in the Harish-Chandra notation. Thus the application $\lambda \mapsto \psi_\lambda(X)$, $\sum \lambda_i = 0$, has a critical point in $\lambda = \rho$, and by W-invariance also in $\lambda = w\rho$ for $w \in W$.

On the other hand ψ_λ is given by the formula of Harish-Chandra :

$$\psi_\lambda(X) = \int_{SO(n)} e^{(\lambda - \rho | \mathcal{H}(ak))} dk_{SO(n)} \tag{15}$$

where $a \in A \cap SL(n)$ is such that $X = a^2[k_0]$ for some $k_0 \in SO(n)$ and $g = k \exp \mathcal{H}(g).n$ is the Iwasawa decomposition of $SL(n)$. Denote by μ_X the image of $dk_{SO(n)}$ by the mapping $k \mapsto \mathcal{H}(ak)$. Then μ_X is a probability measure on $\tilde{\mathfrak{a}} = \{H | H$ diagonal and $\mathrm{Tr} H = 0\}$. By (15) we have then for any $\mathbf{u} \in \mathbb{R}^n$

$$\psi_\lambda(X) = \int_{\tilde{\mathfrak{a}}} e^{(\lambda - \rho | H)} d\mu_X(H)$$

$$\frac{\partial \psi_\lambda}{\partial \mathbf{u}}(X) = \int_{\tilde{\mathfrak{a}}} (\mathbf{u}|H) e^{(\lambda - \rho | H)} d\mu_X(H)$$

$$\frac{\partial^2 \psi_\lambda}{\partial \mathbf{u}^2}(X) = \int_{\tilde{\mathfrak{a}}} (\mathbf{u}|H)^2 e^{(\lambda - \rho | H)} d\mu_X(H). \tag{16}$$

By a theorem of Kostant ([7]) supp $\mu_X = C(\log a)$. Since $X \neq I$ we have $a \neq I$ and $\log a \neq 0$. The space $\mathcal{SP}_n$ is irreducible so by [4]IV,10.11 $\dim C(\log a) = \dim \tilde{\mathfrak{a}}$ and (16) implies $\frac{\partial^2 \psi_\lambda}{\partial \mathbf{u}^2}(X) > 0$ for all λ and $\mathbf{u} \neq 0$. It means that $\psi_\lambda(X)$ is strictly convex and in particular it has at most one critical point on $\tilde{\mathfrak{a}}$. For $w \in W$ different from identity $w\rho \neq \rho$. That gives a contradiction. □

COROLLARY 1. — *Let $\mu \in M^\natural(\mathcal{P}_n)$. Then*

$$D_1(\mu) = \cdots = D_{n-1}(\mu) = 0$$

if and only if μ is concentrated on $\{tI|t > 0\}$. □

Example. — In the case $n = 2$ the explicit form of the dispersion density is known ([1]) :

$$Q(a_r) = 2\log\left(\mathrm{ch}\frac{r}{2}\right)$$

where

$$a_r = \begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix}.$$

For $n \geq 3$ one may give the following integral formula

$$Q_j(X) = \int_K \{\log \Delta_{j-1}(X[k]) - 2\log \Delta_j(X[k]) + \log \Delta_{j+1}(X[k])\}dk$$

but the explicit form of $Q_j(X)$ is not known.

3.2. The mean and the variance in $\mathbb{R}^+I$-direction.

Theorem 2 and Corollary 1 show that the dispersions D_j do not control the behaviour of measures in $M^\natural(\mathcal{P}_n)$ in the direction $\mathbb{R}^+I$. In fact, via (3) one may say that D_j are the dispersions in the direction of $\mathcal{SP}_n$. We will introduce now some complementary characteristics of $\mu \in M^\natural(\mathcal{P}_n)$.

Having in mind the decomposition (3) of a spherical function on $\mathcal{P}_n$ and denoting $w = \sum_{i=1}^{n} s_i$ it is natural to have the following definition.

DEFINITION. — *Let $\mu \in M^\natural(\mathcal{P}_n)$. We define :*

the mean of μ by

$$M(\mu) = \frac{\partial}{\partial w}\hat{\mu}\big|_{\mathbf{s}=\rho} = \frac{1}{n}\int \log(\det X)d\mu(X)$$

the second moment of μ by

$$M_2(\mu) = \frac{\partial^2}{\partial w^2}\hat{\mu}\big|_{\mathbf{s}=\rho} = \frac{1}{n^2}\int \log^2(\det X)d\mu(X)$$

the variance of μ *by* $d^2(\mu) = M_2(\mu) - M^2(\mu)$.

Note that any measure $\mu \in M^\natural(\mathcal{P}_n)$ may be centralised by putting

$$\tilde{\mu}(B) = \mu(e^{M(\mu)}B)$$

for B measurable. Then $M(\tilde{\mu}) = 0$ and $d^2(\mu) = M_2(\tilde{\mu})$. Observe that

$$M(\mu_1 * \mu_2) = M(\mu_1) + M(\mu_2)$$
$$d^2(\mu_1 * \mu_2) = d^2(\mu_1) + d^2(\mu_2)$$

but $d^2(\mu) = \int q(X)d\mu(X)$ with $q(X) = \frac{1}{n^2}\log^2(\det X)$ only for centralised measures μ.

The derivative $\frac{\partial}{\partial w}$ in the definition of M and d^2 equals in the (s_i)-coordinates

$$\frac{\partial}{\partial w} = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial s_i}.$$

This makes possible to calculate $M(\mu)$ and $d^2(\mu)$ if one knows the spherical Fourier transform of μ.

Example. — For the measures considered in Section 2 we have :

$$\begin{array}{rclrcl} M(\kappa_t) & = & 0, & d^2(\kappa_t) & = & 2tn^{-1}; \\ M(\nu_t) & = & 0, & d^2(\nu_t) & = & 0; \\ M(\eta_t) & = & 0, & d^2(\eta_t) & = & t; \\ M(\delta_t) & = & t, & d^2(\delta_t) & = & 0. \end{array}$$

COROLLARY 2. — *If* $\mu \in M^\natural(\mathcal{P}_n)$ *and* $M(\mu) = d^2(\mu) = 0$, $D_j(\mu) = 0$ *for* $j = 1, \dots, n-1$, *then* $\mu = \delta_I$. □

Note that a Gaussian measure μ on $\mathcal{P}_n$ is fully characterized by its mean $M(\mu)$, variance $d^2(\mu)$ and dispersion $D_1(\mu)$.

4. Taylor expansion of spherical functions.

In this section we will derive a useful Taylor expansion of spherical functions on $\mathcal{P}_n$. This expansion will be more detailed than that of Richards ([8]) and we will prove it in a simpler way.

A spherical function $\Phi_{\mathbf{s}}$ is K-invariant so it is enough to consider $\Phi_{\mathbf{s}}(\exp H)$, $H \in \mathfrak{a}$. One may treat $\Phi_{\mathbf{s}}(\exp H)$ as a function of $h_1, \dots, h_n$. It is then symmetric in $h_1, \dots, h_n$. Remark that $\Phi_{\mathbf{s}}(\exp H)$ is real analytic since $\Phi_{\mathbf{s}}$ is a solution of an elliptic differential equation with analytic coefficients : $\Delta\Phi_{\mathbf{s}} = \gamma_2(\mathbf{s})\Phi_{\mathbf{s}}$. Making use of the symmetry and analyticity of the function in the h_j we get the following Taylor expansion at $H = 0$:

$$\Phi_{\mathbf{s}}(\exp H) = 1 + a(\mathbf{s})\sum h_i + b(\mathbf{s})\sum h_i^2 + c(\mathbf{s})(\sum h_i)^2 + R_{\mathbf{s}}(H) \tag{17}$$

with

$$R_{\mathbf{s}}(H) = \sum f_\alpha(\mathbf{s})P_\alpha(H) \tag{18}$$

where $P_\alpha(H)$ are symmetric polynomials in $h_1, \ldots, h_n$ homogeneous of order greater or equal to 3.

In order to find the functions a, b, c in (17) let us apply the operators E, E^2 and Δ to (17) at $H = 0$.

For the operators E and E^2 one uses (6). E considered as a differential operator on functions of $h_1, \ldots, h_n$ is homogeneous of order 1 while the polynomials P_α are homogeneous of order at least 3. That implies $ER_{\mathbf{s}}(0) = E^2R_{\mathbf{s}}(0) = 0$ and

$$\begin{aligned} \gamma_1(\mathbf{s}) &= na(\mathbf{s}) \\ \gamma_1^2(\mathbf{s}) &= 2nb(\mathbf{s}) + 2n^2c(\mathbf{s}). \end{aligned}$$

For the operator Δ one applies (7). By an argument of homogeneity one obtains $\Delta R_{\mathbf{s}}(0) = 0$. We have then

$$\gamma_2(\mathbf{s}) = (n^2 + n)b(\mathbf{s}) + 2nc(\mathbf{s}).$$

Solving these equations we get

THEOREM 3.

$$\Phi_{\mathbf{s}}(\exp H) = 1 + a(\mathbf{s})\sum h_i + b(\mathbf{s})\sum h_i^2 + c(\mathbf{s})(\sum h_i)^2 + R_{\mathbf{s}}(H)$$

with

$$\begin{aligned} a(\mathbf{s}) &= \frac{1}{n}\gamma_1(\mathbf{s}) \\ b(\mathbf{s}) &= \frac{n\gamma_2(\mathbf{s}) - \gamma_1^2(\mathbf{s})}{n(n-1)(n+2)} \\ c(\mathbf{s}) &= \frac{(n+1)\gamma_1^2(\mathbf{s}) - 2\gamma_2(\mathbf{s})}{2n(n-1)(n+2)} \end{aligned} \tag{19}$$

where $\gamma_1(\mathbf{s}) = \sum s_i$, $\gamma_2(\mathbf{s}) = (\mathbf{s} - \rho|\mathbf{s})$ *and* $R_{\mathbf{s}}(H)$ *is as in (18).* □

By (13) and by differentiating of (19) one obtains the following expansion of the functions Q_j at $H = 0$:

COROLLARY 3.

$$Q_j(\exp H) = \frac{1}{(n-1)(n+2)} \sum h_i^2 - \frac{1}{n(n-1)(n+2)} (\sum h_i)^2 + R'_j(H) \tag{20}$$

where $R'_j(H) = \sum c_{\alpha j} P_\alpha(H)$ *with* P_α *as in (18),* $j = 1, \ldots, n-1$. □

Writing as in Section 3 $q(X) = \frac{1}{n^2} \log^2(\det X)$ we have $(\sum h_i)^2 = n^2 q(X)$. Replacing $\sum h_i^2$ in (19) by the expression for $\sum h_i^2$ obtained from (20) we get :

$$\Phi_{\mathbf{s}}(\exp H) = 1 + \frac{1}{n}\gamma_1(\mathbf{s}) \sum h_i + \left(\gamma_2 - \frac{1}{n}\gamma_1^2\right) Q_j(\exp H) + \frac{1}{2}\gamma_1^2 q(\exp H) + R_{j,\mathbf{s}}(H) \tag{21}$$

where

$$R_{j,\mathbf{s}}(H) = \sum f_{j\alpha}(\mathbf{s}) P_\alpha(H) \tag{22}$$

with $P_\alpha(H)$ as in (18).

For $H = (h_1, \ldots, h_n)$ we put $\|H\| = \sum |h_i|$. Then we have

$$R_{j,\mathbf{s}}(H) = \mathcal{O}(\|H\|^3) \quad \text{if } H \to 0. \tag{23}$$

In order to estimate $R_{j,\mathbf{s}}(H)$ when $\|H\| \to \infty$ and $\Phi_{\mathbf{s}}$ is bounded one has to estimate Q_j in infinity.

LEMMA 2. — $Q_j(\exp H) \leq \|H\|$ for all $H \in \mathfrak{a}$ and $j = 1, \ldots, n-1$.

Proof. — By Theorem 2(i) $Q_j(\exp H) = Q_j(\exp H')$ with $H' = (h_1 - \frac{1}{n}\sum h_i, \ldots, h_n - \frac{1}{n}\sum h_i) \in \tilde{\mathfrak{a}}$. Then $Q_j(\exp H') = -2\frac{\partial \psi_\lambda}{\partial \mathbf{v}_j}|_{\lambda=\rho}$ where ψ_λ is spherical on $\mathcal{SP}_n$. By the Harish-Chandra formula

$$\begin{aligned} Q_j(\exp H') &= 2\int_{SO(n)} \left(-\mathbf{v}_j | \mathcal{H}\left(\exp \frac{1}{2} H'.k\right)\right) dk_{SO(n)} \\ &= 2\int_{SO(n)} \left(\alpha_j | \mathcal{H}\left(\exp \frac{1}{2} H'.k\right)\right) dk_{SO(n)}. \end{aligned}$$

By K-invariance of Q_j one may assume that $H' \in \tilde{\mathfrak{a}}^+$, i.e. $h'_1 \leq \cdots \leq h'_n$. By [4]IV,6.5 $\mathcal{H}(ak) \leq \mathcal{H}(a)$ for $a \in \exp(\tilde{\mathfrak{a}}^+), k \in SO(n)$, so

$$Q_j(\exp H') \leq 2\left(\alpha_j|\mathcal{H}\left(\exp\frac{1}{2}H'\right)\right) = h'_{j+1} - h'_j = h_{j+1} - h_j.$$

Finally

$$Q_j(\exp H) \leq |h_j| + |h_{j+1}| \leq \|H\|.$$

□

COROLLARY 4. — *For every $j = 1, \ldots, n-1$ and $\mathbf{s}$ such that $\Phi_{\mathbf{s}}$ is bounded*

$$R_{j,\mathbf{s}}(H) = \mathcal{O}(\|H\|) + \mathcal{O}((\sum h_i)^2)$$

when $\|H\| \to \infty$. □

5. Central limit theorem.

Let $\{\mu_{mj}\}$, $m \in \mathbb{N}$, $1 \leq j \leq k_m$ be a family of K-invariant probability measures on $\mathcal{P}_n$. Put

$$\mu_m = \mu_{m1} * \mu_{m2} * \cdots * \mu_{mk_m}.$$

Denote by $H(X)$ the diagonal matrix of logarithms of eigenvalues of $X \in \mathcal{P}_n$. Then we have the following central limit theorem :

THEOREM 4. — *Suppose that the measures $\{\mu_{mj}\}_{m\in\mathbb{N},1\leq j\leq k_m}$ satisfy the following conditions :*

$$M(\mu_{mj}) = 0$$

(24) $$\lim_{m\to\infty} D_1(\mu_m) = t$$

(25) $$\lim_{m\to\infty} d^2(\mu_m) = u$$

(26) $$\lim_{m\to\infty} \sum_{j=1}^{k_m} \int \frac{\|H\|^3}{1+\|H\|^2} d\mu_{mj} = 0$$

(27) $$\lim_{m\to\infty} \sum_{j=1}^{k_m} \int_{\{\|H\|>1\}} (\mathrm{Tr}H)^2 d\mu_{mj} = 0.$$

Then the measures μ_m converge weakly to the Gaussian measure $\nu_t * \eta_u$.

Proof. — First observe that for any $\varepsilon > 0$

$$\int \|H\| d\mu_{mj} = \int_{\{\|H\|\leq\varepsilon\}} \|H\| d\mu_{mj} + \int_{\{\|H\|>\varepsilon\}} \|H\| d\mu_{mj}$$
$$\leq \varepsilon + \frac{1+\varepsilon^2}{\varepsilon^2} \int \frac{\|H\|^3}{1+\|H\|^2} d\mu_{mj}.$$

Then Lemma 2 and (26) imply that $\lim_m D_1(\mu_{mj}) = 0$ uniformly in j. Similarly,

$$\int (\sum h_i)^2 d\mu_{mj} \leq \int \|H\| d\mu_{mj} + \int_{\{\|H\|>1\}} (\mathrm{Tr} H)^2 d\mu_{mj}$$

and by (27) $\lim_m d^2(\mu_{mj}) = 0$ uniformly in j.

Fix $\mathbf{s}$ such that $\Phi_{\mathbf{s}}$ is bounded. By Corollary 4 and (23) the conditions (26) and (27) imply

$$\lim_m \sum_{j=1}^{k_m} \int |R_{1,\mathbf{s}}(H)| d\mu_{mj} = 0. \tag{28}$$

In particular $\int |R_{1,\mathbf{s}}(H)| d\mu_{mj}$ tends to 0 uniformly with respect to j. Then (21) implies

$$\lim_m \sup_{1\leq j\leq k_m} |\hat{\mu}_{mj}(\mathbf{s}) - 1| = 0. \tag{29}$$

By (21)

$$\sum_j [1-\hat{\mu}_{mj}(\mathbf{s})] = -\left(\gamma_2 - \frac{1}{n}\gamma_1^2\right) D_1(\mu_m) - \frac{1}{2}\gamma_1^2 d^2(\mu_m) - \sum_j \int R_{1,\mathbf{s}}(H) d\mu_{mj}.$$

By (24),(25),(28) and (29) we have then

$$\lim_m \sum_j [1-\hat{\mu}_{mj}(\mathbf{s})] = -\left(\gamma_2 - \frac{1}{n}\gamma_1^2\right) t - \frac{1}{2}\gamma_1^2 u;$$
$$\lim_m \sum_j [1-\hat{\mu}_{mj}(\mathbf{s})]^2 = 0.$$

Using again (29) we get

$$\lim_m \hat{\mu}_m(\mathbf{s}) = \exp\left[\lim_m \sum_j \log \hat{\mu}_{mj}(\mathbf{s})\right]$$
$$= \exp\left[\left(\gamma_2 - \frac{1}{n}\gamma_1^2\right) t + \frac{1}{2}\gamma_1^2 u\right] = \widehat{\nu_t * \eta_u}(\mathbf{s}).$$

By the Lévy continuity theorem on $\mathcal{P}_n$ (look [5]Thm.4.2 in the case of G semisimple; the proof of Gangolli works on $\mathcal{P}_n$) we get $\mu_m \Rightarrow \nu_t * \eta_u$. □

Remark. — Under the hypotheses of Theorem 4 $\lim_m D_l(\mu_m) = t$ for $l = 2, \ldots, n-1$ (see Corollary 3).

COROLLARY 5. — *Let $(\kappa_t)_{t>0}$ be the heat semigroup on $\mathcal{P}_n$. Let the family of measures $\{\mu_{mj}\}$ be centralised and verify (26) and (27). If*

$$\lim_m D_1(\mu_m) = D_1(\kappa_t) = t$$
$$\lim_m d^2(\mu_m) = d^2(\kappa_t) = 2tn^{-1}$$

then

$$\mu_m \Rightarrow \kappa_t.$$ □

BIBLIOGRAPHY

[1] J. FARAUT, Dispersion d'une mesure de probabilité sur $SL(2, \mathbb{R})$ biinvariante par $SO(2, \mathbb{R})$ et théorème de la limite centrale, exposé Oberwolfach, 1975.

[2] J. FARAUT, A. KORANYI, Jordan Algebras, Symmetric Cones and Symmetric Domains, to appear.

[3] R. GANGOLLI, Isotropic infinitely divisible measures on symmetric spaces, Acta Math., 111 (1964), 213-246.

[4] S. HELGASON, Groups and Geometric Analysis, Academic Press, New York, 1984.

[5] G.A. HUNT, Semi-groups of measures on Lie groups, Trans. Amer. Math. Soc., 81 (1956), 264-293.

[6] F.I. KARPELEVICH, V.N. TUTUBALIN, M.G. SHUR, Limit theorems for the compositions of distributions in the Lobachevsky plane and space, Theory Prob. Appl., 4 (1959), 399-402.

[7] B. KOSTANT, On convexity, the Weyl group and the Iwasawa decomposition, Ann. Sci. Ecole Norm. Sup., 6 (1973), 413-455.

[8] D.St.P. RICHARDS, The Central Limit Theorem on Spaces of Positive Definite Matrices, J. Multivariate Anal., 29 (1989), 326-332.

[9] A. TERRAS, Noneuclidean Harmonic Analysis, the Central Limit Theorem and Long Transmission Lines with Random Inhomogeneities, J. Multivariate Anal., 15 (1984), 261-276.

[10] A. TERRAS, Asymptotics of Special Functions and the Central Limit Theorem on the Space $\mathcal{P}_n$ of Positive $n \times n$ Matrices, J. Multivariate Anal., 23 (1987), 13-36.

[11] A. TERRAS, Harmonic Analysis on Symmetric Spaces and Applications II, Springer-Verlag, New York, 1988.

Manuscrit reçu le 27 juin 1991.

Piotr GRACZYK,
Département de Mathématiques
Université Pierre et Marie Curie
Paris VI
4 Place Jussieu
75230 Paris Cedex 05 (France)
&
Institute of Mathematics
Wroclaw Technical University
W. Wyspianskiego 27
50-370 Wroclaw (Poland).