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Journal

EARTH AND SPACE SCIENCE, 6(7)

ISSN

2333-5084

Author

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Publication Date

2019

DOI

10.1029/2019EA000599

Peer reviewed

A century of nonlinearity in the geosciences

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Key Points:

- Nonlinear concepts and methods have greatly expanded the range of problems we can address
- There is still only a small but increasing number of nonlinear methodologies
- Prediction is a great test of our mathematical and physical understanding

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11 **Abstract**

12 This paper provides a thumbnail sketch of the evolution of nonlinear ideas in the math-
 13 ematics and physics of the geosciences, broadly construed, over the last hundred or so
 14 years. It emphasizes the mathematical concepts and methods, and outlines simple ex-
 15 amples of how they were, are, and maybe will be applied to the solid Earth — i.e., the
 16 crust, mantle, and core — and its fluid envelopes — i.e., the atmosphere and oceans.

17 **Plain-Language Summary.** Nonlinearity has become a buzzword, along with chaos,
 18 complexity, fractals, networks, tipping points, turbulence, and other concepts associated
 19 with modern science. We outline here what it all means and how it has affected the progress
 20 of the geosciences over the past century, mostly over the last six decades or so.

21 **1 Introduction and Motivation**

22 As we are celebrating 100 years since the founding of the American Geophysical
 23 Union (AGU), it appears of interest to consider the way that nonlinear concepts and meth-
 24 ods have modified the way that we are practicing the geosciences today and may prac-
 25 tice them over the next century.

26 While nonlinear approaches have rapidly expanded over the last half century, it is
 27 clear that their roots go back much further. One of the oldest nonlinear problems in the
 28 geosciences is certainly drawing a right angle on the face of the Earth, e.g., between a
 29 meridian and a parallel: this problem is equivalent to solving the Diophantine equation
 30 $a^2+b^2 = c^2$. It is conjectured that the ancient Egyptians applied this equivalence, com-
 31 monly called Pythagoras’s theorem, to build their pharaonic projects, from the basis up;
 32 specifically, that they used the simplest solution — namely $(a, b, c) = (3, 4, 5)$ — by ty-
 33 ing $12 = 3+4+5$ equidistant knots into a rope, and used it in order to build the great
 34 pyramids of Gizeh, and many other temples, palaces and tombs (e.g., Cooke, 2011).

35 But that is, of course, not what we all have in mind when discussing nonlinearity
 36 in the sciences in general and in the geosciences in particular. Linear approaches dom-
 37 inated the physical sciences in the 19th century; the explosion of a variety of method-
 38 ologies that deviate from them is well illustrated by the saying, often attributed to Stanis-
 39 law Ulam (Gleick, 1987), that linear dynamics is akin to elephant zoology, or words to
 40 that effect. What we mean by tracing back the rapid rise of nonlinear dynamics, non-
 41 linear sciences or what not to some time after World War II, is the following fact: ac-
 42 cording to the well-known story of the lamppost, and of attempts to find the forlorn keys
 43 in its circle of light, a superb development of methods for solving linear algebraic and
 44 differential equations in the 19th century led to great emphasis on solving problems for-
 45 mulated in terms of such equations in the first half of the 20th century.

46 Basically, linear problems are easily separable, and hence solvable, due to the su-
 47 perposition principle, projection onto orthonormal bases, and so on. Thus, many such
 48 problems were solved over the last 200 years, most often analytically, i.e., with pencil and
 49 paper or with very rudimentary computational devices. And these methods are still of
 50 great use to us, in deriving and determining the properties of tangent linear equations,
 51 adjoint operators, and many other mathematical approximations of real-world problems.

52 It is the rise of more-and-more powerful computational devices after World War
 53 II that changed our way of thinking about what the solution to the mathematical for-
 54 mulation of a physical problem really is, i.e., not necessarily an analytical expression but
 55 an algorithm for obtaining information about such a solution with prescribed accuracy.
 56 The improvement in observational methods — in the geosciences and elsewhere, whether
 57 *in vitro*, i.e., in the lab, or *in vivo*, i.e. outdoors— has also contributed greatly to our
 58 appetite for going beyond linear approximation to model, simulate, understand, and pre-
 59 dict the complexity of the phenomena under study.

60 The nonlinear way of thinking about problems, in the geosciences and many other
 61 sciences — physical sciences in general, biosciences, socio-economic sciences — still needs
 62 to operate within the circles of light projected into the night of our ignorance by a cer-
 63 tain number of lampposts. These lampposts include the theory of dynamical systems,
 64 statistical mechanics, scale invariances, the theory of localized coherent structures, and
 65 several others. Some lampposts that have been added or whose light circle has expanded
 66 in the last decade or so are network theory and the theory of non-autonomous and ran-
 67 dom dynamical systems.

68 The remainder of this paper will examine some of these lampposts and their re-
 69 spective circles of light, following Ghil, Kimoto, and Neelin (1991) and Ghil (2001). In
 70 the next section, we outline with a broad brush how linear results provided first insights
 71 into the behavior of fluid motions, around the turn of the 19th into the 20th century, and
 72 how nonlinear ones completed our knowledge after World War II.

73 Sections 3 and 4 examine in somewhat greater detail the dynamical systems and
 74 the scale invariance lamppost, respectively. Each section starts with a sketch of the ba-
 75 sic concepts and methods, in Secs. 3.1 and 4.1, respectively; each then follows up with
 76 some key applications. Thus, in Secs. 3.2 we discuss the mechanics of vacillation, mul-
 77 tiple weather regimes in the atmosphere, and multiple flow regimes in the oceans, while
 78 in Sec. 4.2 we cover succinctly fractals in dynamical systems, as well as scale invariance
 79 in general three-dimensional (3-D), two-dimensional (2-D) and geostrophic turbulence.

80 A few additional lampposts are examined in Sec. 5, each subsection starting again
 81 with theoretical foundations, followed by selected applications. Section 5.1 covers net-
 82 work theory, including both topology and dynamics, in particular that associated with
 83 Boolean delay equations; the applications illustrated are to earthquake and climate net-
 84 works. In Section 5.2 we discuss fluctuation–dissipation theory, outlining both the clas-
 85 sical theory for thermodynamic equilibrium and the more recent out-of-equilibrium gen-
 86 eralizations, and emphasizing applications to climate response.

87 In Sec. 5.3, we cover the extension of dynamical systems theory to nonautonomous
 88 and random dynamical systems; the applications are the stochastically perturbed Lorenz
 89 (1963a) model and the oceans’ wind-driven circulation subject to time-varying wind stress.
 90 This subsection ends with an introduction to climate sensitivity and the use of Wasser-
 91 stein distance to generalize the traditional concept of equilibrium sensitivity.

92 Section 6 presents two meanings of prediction as touchstones of progress in the non-
 93 linear geosciences: (i) forecasting, i.e. prediction in time of the quantitative realization
 94 of known phenomena; and (ii) theoretical prediction of qualitatively new phenomena. The
 95 former meaning is illustrated by forecasting atmospheric and oceanic phenomena on longer
 96 and longer time scales, from days through seasons and on to several decades. The lat-
 97 ter one is presented in the context of predicting an ice-covered Earth by simple energy
 98 balance models and leading to the current arguments about a snowball Earth. Section 7
 99 concludes this review paper with a brief coda.

100 **2 From Linear to Nonlinear Thinking: A Quick Review**

101 A paradigmatic success of linear concepts and methods at the beginning of the 20th
 102 century is the explanation by Lord Rayleigh (1916) of the striking patterns found in the
 103 thermal convection experiments of James Thomson (1882) and of Henri Bénard (1900).
 104 The word “paradigm” is used here advisedly in the sense of Thomas Kuhn (1962): it is
 105 easy to see how the transition from a linear — and for quite a while very successful —
 106 mode of thinking to a nonlinear one is not just an evolutive generalization but a gen-
 107 uine revolution.

108 In the next section, we will consider a few key traits of the nonlinear mode of think-
 109 ing. In many applications to the physical sciences, like fluid dynamics, the linear mode
 110 involves linearizing the equation of motion about a suitably symmetric steady state, most
 111 often a state of rest (Rayleigh, 1916, p. 534). The stability of the resulting linear oper-
 112 ator is examined and the spatial pattern of the most rapidly growing unstable mode can
 113 then be compared to observations. While Lord Rayleigh only examined a rectangular
 114 domain, subsequent work led to the study of convective rolls and hexagons as the most
 115 often occurring spatial patterns near equilibrium (e.g., Busse, 1978; Krishnamurti, 1973).
 116 It is interesting, though, that Rayleigh (1916, pp. 529–530) does describe the irregular
 117 transitions between two types of flow regimes. Pursuing an explanation thereof was clearly
 118 beyond the reach of the linear methodology available to him.

119 Be that as it may, linear methodology led to many other successes during the first
 120 half of the 20th century, in explaining flow patterns observed in the laboratory, in indus-
 121 try, and in nature. Thus, when we see parallel cloud streaks in the sky, we know that
 122 they are the result of either Rayleigh-Bénard or Kelvin-Helmholtz instability. Possibly
 123 the crowning success of this approach was the discovery of a truly 3-D instability of great
 124 importance for atmospheric and oceanic flows, namely baroclinic instability, by Jule G.
 125 Charney (1947) and, independently, by Eric T. Eady (1949).

126 Charney’s and Eady’s results on baroclinic instability and variations thereupon man-
 127 aged to explain various features of the initial stages of development of mid-latitude storms
 128 in the atmosphere and of mesoscale meanders in the oceans. But they could not explain
 129 the finite-amplitude interactions between separate storms nor help very much in predict-
 130 ing weather beyond 1–2 days. In fact, Eady (1949, pp. 51–52) already had a pretty clear
 131 vision of the difference between theoretically identifying recognizable initial patterns in
 132 a storm’s development and “the formidable task facing theoretical meteorology — that
 133 of discovering the nature of and determining quantitatively [sic] all the forecastable reg-
 134 ularities of a permanently unstable (i.e., permanently turbulent) system.” It is here that
 135 the paradigmatic jump from linear to nonlinear concepts and methods has to occur.

136 **3 The dynamical systems lamppost**

137 **3.1 The theory**

138 The mathematical theory of dynamical systems deals with modeling the behavior
 139 of systems that evolve on long time scales. Sufficiently long, that is, for assuming that
 140 solutions of the models exist for all times, from $-\infty$ to $+\infty$. This theory does not dis-
 141 tinguish, in principle, linear from nonlinear systems but has much to say about the lat-
 142 ter; it does not distinguish either between natural systems — whether physical, biolog-
 143 ical or socio-economical — and human-made systems but we will be interested here in
 144 the natural ones. Some basic facts of nonlinear life are outlined below, from the dynam-
 145 ical systems perspective, following Ghil et al. (1991).

- 146 1. The equations of continuum mechanics are nonlinear. Surprisingly many phenomena
 147 can be explained by linearization about a particular fixed basic state. Many more can-
 148 not; see Sec. 2 above.
- 149 2. Behavior of solutions to the nonlinear equations changes qualitatively only at isolated
 150 points in phase-parameter space, called bifurcation points. Behavior along a single branch
 151 of solutions, between such points, is modified only quantitatively and can be explored
 152 by linearization about the basic state, which changes as the parameters change. That
 153 is, nonlinear dynamics is much like linear dynamics, only more so (Ghil & Childress, 1987;
 154 Lorenz, 1963a, 1963b).
- 155 3. Bifurcation trees lead from the simplest, most symmetric states, to highly complex
 156 and realistic ones, with much lower symmetry in either space or time or both. These trees
 157 can be explored partially by analytic methods (Jin & Ghil, 1990; Jordan & Smith, 2007)

158 and more fully by numerical ones, such as pseudo-arclength continuation (Dijkstra, 2005;
159 Legras & Ghil, 1985).

160 4. The truly nonlinear behavior near bifurcation points involves robust transitions, of
161 great generality, between single and multiple fixed points (saddle-node, pitchfork and trans-
162 verse bifurcations), fixed points and limit cycles (Hopf bifurcation), limit cycles and strange
163 attractors (“routes to chaos”: Eckmann, 1981; Guckenheimer & Holmes, 1983). As the
164 complexity of the behavior increases, its predictability decreases (Ghil, 2001).

165 5. Behavior in the most realistic, chaotic regime can be described by the ergodic the-
166 ory of dynamical systems. In this regime, statistical information similar to, but more de-
167 tailed than for truly random behavior, can be extracted and used for predictive purposes
168 (Eckmann & Ruelle, 1985; Ghil & Robertson, 2000; Mo & Ghil, 1987).

169 6. Chaos and strange attractors are not restricted to low-order systems. They can be
170 shown to exist for the full equations governing continuum mechanics (Constantin, Foias,
171 Nicolaenko, & Temam, 1989; Temam, 2000). The detailed exploration of finite- but high-
172 dimensional attractors is in full swing (Dijkstra, 2005; Ghil, 2017; Legras & Ghil, 1985).

173 7. Single time series (Takens, 1981) and single numbers derived from them (e.g., Grass-
174 berger, 1983) have been used to describe chaotic behavior. This very simple and straight-
175 forward use of a nonlinear concept has attracted considerable attention to determinis-
176 tically chaotic dynamics, including in the geosciences (Nicolis & Nicolis, 1984; Tsonis &
177 Elsner, 1988). The use of single time series, while exciting in theory, is not very promis-
178 ing when the series are short and noisy (Ruelle, 1990; Smith, 1988). The increasing avail-
179 ability of a large number of similar series at different points in space, combined with phys-
180 ical insight, is compensating more and more for the shortcomings of each individual time
181 series in describing the complexity of many phenomena in the geosciences, as well as ad-
182 vancing their prediction (Ghil et al., 2002).

183 3.2 Some results

184 *The mechanics of vacillation.* Two steps beyond linear theory, in the direction already
185 outlined by Eady (1949), were taken by Edward N. Lorenz (1963a, 1963b). The first was
186 stimulated by the work on convection mentioned in Sec. 2 above, and revisited by Barry Saltz-
187 man (1962). This step yielded the paradigmatic strange attractor of Lorenz (1963a), too
188 well known to be reviewed here yet another time; see Sparrow (1982), Guckenheimer and
189 Holmes (1983), Ghil and Childress (1987, Sec. 5.4), and McWilliams (2019) in this is-
190 sue. It showed the road to understanding deterministic chaos in a low-dimensional case.

191 The second step was going beyond the linear theory of baroclinic instability and
192 was stimulated by the rotating-annulus experiments with differential heating of David Fultz
193 (e.g., Fultz et al., 1959) and Raymond Hide (Hide & Mason, 1975, and references therein);
194 see also Ghil, Read, and Smith (2010). In this step, Lorenz (1963b) showed how to pro-
195 ceed from the initial baroclinic instability of Charney (1947), via successive bifurcations,
196 to the so-called index cycle of atmospheric mid-latitude variability. Namias (1950) de-
197 scribed this cycle of the zonal index as a recurrence of changes in intensity of the pre-
198 vailing westerlies, with a rough periodicity of 4–6 weeks.

199 Lorenz (1963b, Fig. 3) reproduced key features of this phenomenon — such as the
200 changes in strength, latitude and meandering of the westerly jet — by associating them
201 with the tilted-trough vacillation in the rotating annulus experiments. The correspond-
202 ing bifurcation tree appears as Fig. 5.8 in Ghil and Childress (1987).

203 *Multiple weather regimes.* Charney (1947) and Eady (1949) followed the linear approach
204 outlined in Secs. 1 and 2 and assumed small perturbations about a stationary mid-latitude
205 state of zonally symmetric flow. But observational meteorologists knew already that pre-
206 dominantly zonal flow is only one of the mid-latitudes’ persistent states, and that episodes
207 of so-called blocked flow — with large deviations from zonality — can persist for fairly
208 long times (e.g., Baur, 1947; Namias, 1968). ‘Long’ here is defined as longer than the life

209 cycle of a typical mid-latitude storm, which is 5–7 days, while blocking events can last
 210 for up to a month (e.g., Dole & Gordon, 1983); see also Ghil and Childress (1987, Fig. 6.1).

211 Charney and DeVore (1979) studied a low-order barotropic model with merely three
 212 modes in a β -channel — i.e., in a rectangular domain on a tangent plane to the sphere
 213 (e.g., Gill, 1982; Pedlosky, 1987) — that had two stable stationary solutions: one with
 214 features similar to zonal flow, the other resembling blocked flow; see the bifurcation di-
 215 agram in Ghil and Childress (1987, Fig. 6.5). Charney, Shukla, and Mo (1981) and Benzi,
 216 Malguzzi, Speranza, and Sutera (1986) provided observational evidence for the existence
 217 of blocked-vs.-zonal bimodality in the Northern Hemisphere extratropics, while Mo and
 218 Ghil (1987) also found bimodality in the Southern Hemisphere extratropics. The latter
 219 bistability involved different amplitudes and phases of a dominant wavenumber-three,
 220 quasi-stationary wave; a third quasi-stationary pattern, of regional rather than hemispheric
 221 extent, was called by Mo and Ghil (1987) the Pacific–South-American (PSA) pattern.

222 Legras and Ghil (1985) showed that, using just 25 modes of a barotropic model on
 223 the sphere, one could go well beyond two stable fixed points, to obtain not only more
 224 realistic zonal and blocked flow, but also stable limit cycles and deterministically chaotic
 225 behavior. In the latter regime, depending on the Rossby number Ro that determines the
 226 relative importance of the planet’s rotation (see Sec. 4.2 for further details), it is either
 227 a zonal, a blocked or an intermittent regime that dominates. In the presence of inter-
 228 mittency, the relative time spent in zonal and blocked episodes changes smoothly as Ro
 229 increases (Ghil & Childress, 1987, Fig. 6.14). Weeks et al. (1997, Fig. 5B) used a barotropic
 230 rotating annulus with topography and found that the dependence of persistence times
 231 of zonal vs. blocked flow on the experiment’s Rossby number exhibited marked similar-
 232 ities to the numerical results of Legras and Ghil (1985).

233 The existence of several weather regimes in the Northern Hemisphere’s atmosphere
 234 is statistically pretty well established now by a number of distinct clustering methods
 235 and their application to several data sets; see, for instance, Table 1 in Ghil, Groth, Kon-
 236 drashov, and Robertson (2018) and references therein. Even so, the exact number of such
 237 regimes supported by the data, as well as their description and dynamical explanation,
 238 remains a matter of debate. Moreover, high-resolution numerical weather prediction (NWP)
 239 models — which are otherwise quite skillful at predicting weather a few days in advance
 240 — still have difficulties in predicting the onset of blocking and transitions between it and
 241 zonal flow (Dawson & Palmer, 2014).

242 *Multiple flow regimes in the oceans.* The horizontal extent of storms in the atmosphere
 243 and of eddies in the oceans is given by the Rossby radius of deformation R (Ghil & Chil-
 244 dress, 1987; Gill, 1982; Pedlosky, 1987) that determines the so-called synoptic scale. Be-
 245 cause of the differences in stratification between the two fluid media, $R_{oc} \simeq 100 \text{ km} \ll$
 246 $R_{atm} \simeq 1000 \text{ km}$. Thus, when first discovered, oceanic eddies have been erroneously
 247 called “mesoscale eddies,” since 100 km is termed the mesoscale in the atmosphere. Be-
 248 that as it may, the name has stuck (e.g., McWilliams, 2011).

249 While the diameter of oceanic eddies is much smaller than that of the atmospheric
 250 ones, their life cycle is much longer: months rather than days. Thus, low-frequency vari-
 251 ability (LFV) in the oceans is on the scale of years-to-decades, while in the atmosphere
 252 it is subseasonal-to-seasonal, 10–100 days. In the oceans, one tends to distinguish be-
 253 tween two types of causes of LFV: the wind-driven circulation and the thermohaline or
 254 meridional-overturning circulation (THC or MOC). The former is predominantly in a
 255 horizontal plane, driven by atmospheric momentum fluxes, and contributes to the inter-
 256 annual LFV of the oceans, while the latter is predominantly in a meridional plane, driven
 257 by buoyancy fluxes and contributes to the interdecadal LFV (Dijkstra & Ghil, 2005).

258 Important contributions to the nonlinear understanding of oceanic LFV are roughly
 259 contemporaneous to, or even earlier than, the pioneering contributions of Lorenz (1963a,

1963b) for the atmosphere. Henry Stommel (1961) obtained two stable stationary solutions in a simple two-box model of the THC. He was originally interested in the seasonal reversal of local THC's, such as in the Red Sea or the Eastern Mediterranean (Stommel, 1961, p. 225) but did note on p. 228 that “One wonders whether other quite different states of flow are permissible in the ocean [...] and if such a system might jump into one of these with a sufficient perturbation. If so, the system is inherently fraught with possibilities for speculation about climatic change.” Speculations on this matter continue apace, and some of the relevant research is reviewed in Dijkstra and Ghil (2005, Sec. 3).

George Veronis (1963) considered the wind-driven ocean circulation in a rectangular basin on the β -plane, subject to time-independent wind stress, and truncating the expansion of the barotropic, single-layer streamfunction at four sine modes. He obtained two stable steady states, as well as a limit cycle for various parameter values.

Jiang, Jin, and Ghil (1995) introduced a different expansion of the shallow-water equations in the same geometry, with an exponential multiplier in the zonal, x -direction to allow for a western boundary current, as well as carrying out numerical integrations on an eddy-permitting grid with $\Delta x = \Delta y = 20$ km. They obtained exact steady states, as well as exactly periodic solutions (Fig. 1) for the numerical integrations that used 15 000 grid variables. These authors also showed that the generation of the nearly mirror-symmetric steady states in the numerical integrations was well captured by the perturbed pitchfork bifurcation of their highly idealized, intermediate-order model.

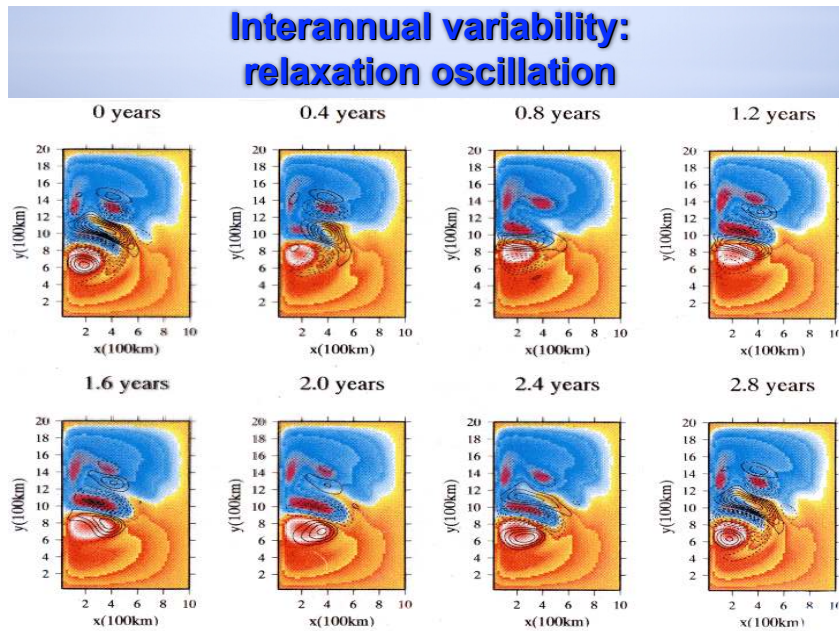


Figure 1. Snapshots from an exactly periodic relaxation oscillation of the Jiang et al. (1995) model; see also Fig. 7 (black and white) there. Color indicates contours of the model’s upper-layer thickness, with warm colors for the subtropical gyre and cold ones for the subpolar one; black lines indicate contours of potential vorticity, with a modified Rossby wave propagating across the basin. Courtesy of Shi Jiang.

280 The periodic solutions became more and more anharmonic and sawtooth-shaped
 281 as the time-constant wind stress intensity was increased, and finally led to aperiodic, in-
 282 termittent solutions. This transition to chaos can be followed in Fig. 2 via a homoclinic
 283 bifurcation for a quasi-geostrophic (QG) model with a resolution of $\Delta x = \Delta y = 10$ km.
 284 Dijkstra and Ghil (2005, Sec. 2) provide further details on this particular model, as part
 285 of an entire hierarchy of increasingly detailed and realistic models that confirm its re-
 286 sults, and many additional references. Concerning geostrophy and its effect on turbu-
 287 lent fluid behavior, see Sec. 4.2 below.

288 Overall, the line of work outlined in the preceding paragraphs has provided fairly
 289 convincing evidence that intrinsic oceanic LFV, even in the absence of variable atmo-
 290 spheric forcing, is an important source of interannual climate variability. Detailed con-
 291 frontation of model results with recent reanalysis data for both atmosphere and oceans
 292 supports these ideas, at least in the case of the North Atlantic basin (Groth, Feliks, Kon-
 293 drashov, & Ghil, 2017), where this mechanism also provides a possible explanation of
 294 the North Atlantic Oscillation (NAO) and of its approximate 7-8-year periodicity. The
 295 situation for time-dependent wind forcing will be discussed in Sec. 5.3.

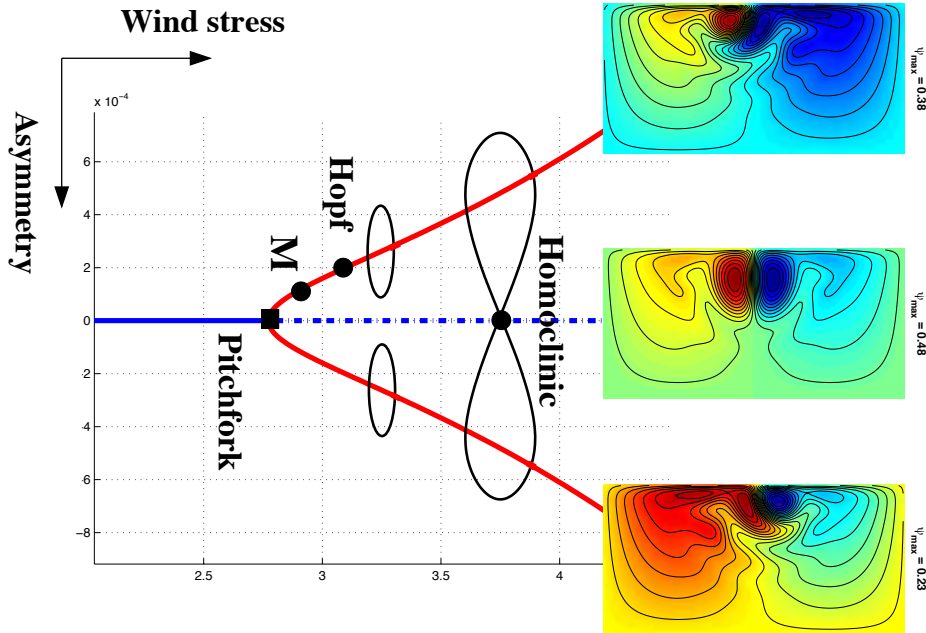


Figure 2. Generic bifurcation diagram for the barotropic QG model of the double-gyre prob-
 lem: the asymmetry of the solution is plotted versus the intensity of the wind stress τ . The
 streamfunction field $\psi = \psi(x, y)$ is plotted for a steady-state solution associated with each of the
 three branches; positive values in red and negative ones in blue. After Simonnet et al. (2005).

296 *Bifurcations and tipping points.* In the applications covered herein, we have limited our-
 297 selves to classical bifurcations (e.g., Arnol'd, 2012; Guckenheimer & Holmes, 1983), which
 298 go back to the work of Leonhard Euler (1757) on buckling of a beam (e.g., Timoshenko
 299 & Gere, 1961). Recently, the interest in bifurcations in the geosciences has greatly in-
 300 creased due to the introduction of the concept of tipping points from the social sciences
 301 (Gladwell, 2000; Lenton et al., 2008). Clearly, a tipping point sounds a lot more threat-
 302 ening than a bifurcation point, especially when dealing with an earthquake or a dramatic
 303 and irreversible climate change.

304 Aside from their rhetorical impact, though, tipping points generalize classical bi-
 305 furcations when considering open, rather than closed systems. In fact, there are three
 306 kinds of tipping points under consideration recently for open systems (Ashwin, Wic-
 307 zorek, Vitolo, & Cox, 2012):

- 308 (i) B-Tipping or Bifurcation-due tipping — slow change in a parameter leads to the sys-
 309 tem’s passage through a classical bifurcation;
- 310 (ii) N-Tipping or Noise-induced tipping — random fluctuations lead to the system’s cross-
 311 ing an attractor basin boundary;
- 312 (iii) R-Tipping or Rate-induced tipping — rapid changes lead to the system’s losing track
 313 of a slow change in its attractors.

314 We will have more to say about open systems and their attractors in Sec. 5.3.

315 4 The Scale Invariance Lamppost

316 The light of this lamppost has to do with insights about patterns that appear to
 317 keep their spatial structure at increasing magnification. Such spatial patterns — like the
 318 Cantor set on the line and the Peano curve in the plane — were well known by the late
 319 19th century (e.g., Sagan, 2012) but their pervasiveness in nature and connection to a
 320 system’s evolution in time only became evident in the second half of the 20th century.

321 4.1 The theory

322 Probably the best-known set with strange properties that arise by an iterative con-
 323 struction is the Cantor ternary set. Consider the closed unit interval $\mathcal{C}_0 = [0, 1]$ of length
 324 $\ell_0 = 1$ on the real line \mathbb{R} and remove the open middle third $(1/3, 2/3)$, which leaves the
 325 set $\mathcal{C}_1 = (\mathcal{C}_0/3) \cup (2/3 + \mathcal{C}_0/3) = [0, 1/3] \cup [2/3, 1]$ of length $\ell_1 = 2/3$. Removing induc-
 326 tively the open middle third of the two closed intervals left, then of the four ones left at
 327 the next stage and so on, one gets

$$328 \quad \mathcal{C}_n = \frac{\mathcal{C}_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{\mathcal{C}_{n-1}}{3} \right), \quad (1)$$

329 of length $\ell_n = (2/3)\ell_{n-1} = (2/3)^n$. This construction is perfectly self-similar and scale
 330 invariant.

331 Clearly $\ell_n \rightarrow 0$, so that the limit set $\mathcal{C}_\infty = \mathcal{C}$ has length $\ell_\infty = 0$. But the deep
 332 result is that there is a one-to-one correspondence between the points in the set \mathcal{C} of zero
 333 Lebesgue measure and those in the unit interval \mathcal{C}_0 , i.e., the two sets have the same un-
 334 countable cardinality $|\mathcal{C}| = |\mathcal{C}_0|$, which equals also the transfinite cardinality \aleph_1 of the
 335 real line itself. The former result was stated by Georg Cantor (1887) without proof; the
 336 modern proof is based on what became known as the Cantor-Schröder-Bernstein the-
 337 orem, with Felix Bernstein and Ernst Schröder having almost simultaneously given two
 338 different proofs in 1897, as did Felix Dedekind. The 2-D generalization of the Cantor set
 339 in the plane \mathbb{R}^2 is called the Sierpiński (1916) carpet; see Fig. 3.

340 Many mathematicians at the time were not comfortable with transfinite numbers
 341 nor with statements like the inequality $\aleph_0 < \aleph_1$, where \aleph_0 is the cardinality of natu-
 342 ral, integer and rational numbers, among other countable sets. Nor did physicists in the
 343 late 19th century appreciate functions that were not continuously differentiable every-
 344 where. This inequality and the absence of any cardinals between \aleph_0 and \aleph_1 depended
 345 on difficult issues raised by the axiomatization of mathematics (e.g., Suppes, 1972) that
 346 were not that palatable for most mathematicians and almost all physicists. This fact tran-
 347 spires even in Cantor’s choice of journal for his 1887 paper, namely a philosophical rather
 348 than a standard mathematical one.

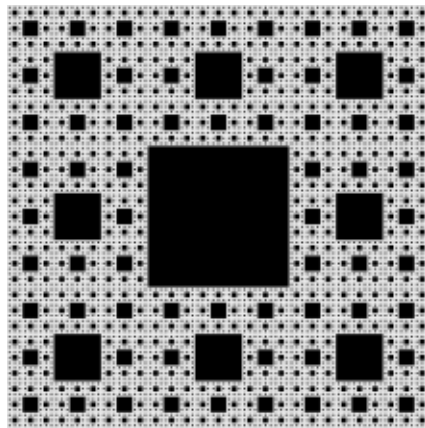


Figure 3. The 6th-level iteration for obtaining the Sierpiński carpet on the unit square $[0, 1] \times [0, 1]$. The carpet has topological dimension $d_T = 1$ but Hausdorff dimension $d_H \simeq 1.893 < 2$. From Wikimedia, public domain.

349 The situation was as bad or worse with respect to functions that were not contin-
 350 uously differentiable anywhere. Bernard Bolzano and Augustin-Louis Cauchy had given
 351 early definitions of continuity in 1817 and 1823, respectively, and Karl Weierstrass had
 352 given the better-known $(\epsilon-\delta)$ definition a few decades later. As discussed in Sec. 2, physi-
 353 cists were extensively using ordinary and partial differential equations (ODEs and PDEs)
 354 around the turn of the 19th into the 20th century and lack of continuous differentiability
 355 was considered a mathematical oddity of little use in studying natural phenomena.

356 Benoît Mandelbrot (1967) had an important role in stressing that this was not so.
 357 Hugo Steinhaus (1954) had already discussed what we now call fractional dimension, when
 358 Lewis Fry Richardson (1961) pointed out the “coastline paradox” and provided the polyg-
 359 onal method for correctly overcoming this paradox; see also Hunt (1998). Essentially,
 360 the length L of a coastline, river (e.g., Steinhaus’s Vistula) or geographic border depends
 361 on the scale G used to approximate it by a polygon. Based on several examples avail-
 362 able at the time, Richardson (1961, Fig. 17) proposed the approximation $L(G) = \kappa G^{1-D}$,
 363 where κ is a constant and $D \geq 1$ is the fractional dimension; the latter equals unity if
 364 the curve is smooth. Quite recently, Losa, Ristanović, Ristanović, Zaletel, and Beltraminelli
 365 (2016) found “[...] that among many fractal analysis techniques, only Richardson’s method
 366 enables correct calculation of the length of an object’s border or irregular line.”

367 The basic ingredients of Mandelbrot’s development of fractal concepts and meth-
 368 ods became available in the early 20th century. First, Felix Hausdorff (1918) provided
 369 a generalization of dimension that allowed one to evaluate it for the kinds of odd sets
 370 we discussed above, cf. Fig. 3; it is now called the Hausdorff dimension and it can take
 371 on noninteger values. Second, the same year, Gaston Julia (1918) considered a class of
 372 iteratively defined sets in the complex plane \mathbb{C} that have the right oddity.

373 Speaking loosely, for a given holomorphic (i.e., complex analytic) function $f(z)$, with
 374 $z = x+iy$, the Julia set $\mathcal{J}(f)$ and the Fatou (1919) set $\mathcal{F}(f)$ are complements of each
 375 other, with $\mathcal{J}(f)$ being the set of points on which repeated iterations of $z \rightarrow f(z)$ di-
 376 verge, while on $\mathcal{F}(f)$ these iterations behave similarly. In other words, f is regular on
 377 $\mathcal{F}(f)$ and chaotic on $\mathcal{J}(f)$. As for the Cantor set \mathcal{C} above, we only outline here the sim-
 378 plest case, namely that of quadratic polynomials, written as $f_c(z) = z^2 + c$, with $c \in$
 379 \mathbb{C} . It is this case that Benoît Mandelbrot (2013, and references therein) made famous
 380 in the late 20th century.

For $c = 0$, the Julia set is simply the unit circle $|z| = 1$, and the two Fatou sets are its interior and exterior, with iterations that converge to 0 and ∞ , respectively. In general, though, the Julia set $\mathcal{J}(f_c)$ is much more complicated and Mandelbrot (1977) introduced the term “fractals” for such complicated sets. A beautiful illustration of the self-similarity that characterizes many fractals is given by the Mandelbrot set $\mathcal{M}(f)$, defined as the set of points c in the complex plane for which the iterates

$$\{z_{n+1}(c) = f(z_n; c); n = 0, \dots\}$$

381 stay bounded as n increases, when starting at $z_0 = 0$. The most often studied and cited
382 case is that of $f(z; c) = f_c(z) = z^2 + c$.

383 While there is no definitive consensus on how best to define a fractal, there are two
384 key ingredients: (i) a degree of self-similarity and (ii) a Hausdorff dimension d_H that ex-
385 ceeds the classical, topological dimension d_T . The rigorous mathematical definition of
386 the latter is also laborious, but its integer values are obvious for the usual Euclidean spaces,
387 namely $d_T = n$ for \mathbb{R}^n ; the former is often an irrational number, although a “fractal
388 dimension,” while often used, is an obvious misnomer: it is the set that is a fractal, while
389 the dimension is a simple scalar in all cases, and a fraction in many.

390 Both Julia sets, defined for a fixed c as z varies, and Mandelbrot sets, defined for
391 a fixed $z_0 = 0$ as c varies, have fascinating properties and there are interesting connec-
392 tions between the two. Peitgen and Richter (2013) provide both mathematical substance
393 and beautiful illustrations on these topics. Figure 4 illustrates just one such case, but
394 for this set, the scale invariance is more qualitative: things look roughly the same rather
395 than exactly the same at different scales.

396 4.2 Some results

397 *Fractals in dynamical systems.* A number of factors concurred in the second half of the
398 20th century to greatly increase the circle of light of this lamppost, as well as the inter-
399 est in it. First, there was the increase of interest in dynamical systems and their appli-
400 cations, as reviewed in Secs. 2 and 3 herein. Next, like in the case of dynamical systems,
401 it was the great progress in computing power and storage capacity.

402 It’s great fun computing Julia or Mandelbrot sets on your laptop, as it is comput-
403 ing the strange attractor of the Lorenz (1963a) model. Moreover, this attractor is a frac-
404 tal for a broad range of parameter values, i.e. when you drill through it perpendicular
405 to the tangent manifold, anywhere except at the origin, you get a Cantor-like set.

406 The dimensions of the attractor, for the standard nondimensional parameter val-
407 ues illustrated in the original Lorenz (1963a) paper — namely the Rayleigh number $\rho =$
408 28, the Prandtl number $\sigma = 10$ and the wavenumber $\beta = 8/3$ — are $d_H = 2.06 \pm$
409 $0.01 > 2 = d_T$ and its volume is zero, as for the Cantor set. Please see, again, Spar-
410 row (1982), Guckenheimer and Holmes (1983), Ghil and Childress (1987, Sec. 5.4), and
411 McWilliams (2019) in this issue for further details.

412 While several metric dimensions have been defined for dynamical systems (e.g., Farmer,
413 Ott, & Yorke, 1983), a particularly useful one is the Lyapunov dimension. It is given by
414 the Lyapunov spectrum of the underlying system and is also called the Kaplan-Yorke di-
415 mension (Kaplan & Yorke, 1979):

$$416 \quad d_{KY} \equiv k + \sum_{j=1}^k \frac{\lambda_j}{\lambda_{k+1}}; \quad (2)$$

417 here k is the maximum integer such that the sum of the k largest exponents is still non-
418 negative. We shall return to the Lyapunov spectrum in Sec. 6 below. Leonov, Kuznetsov,
419 Korzhemanova, and Kusakin (2016) obtained the following remarkable formula for the

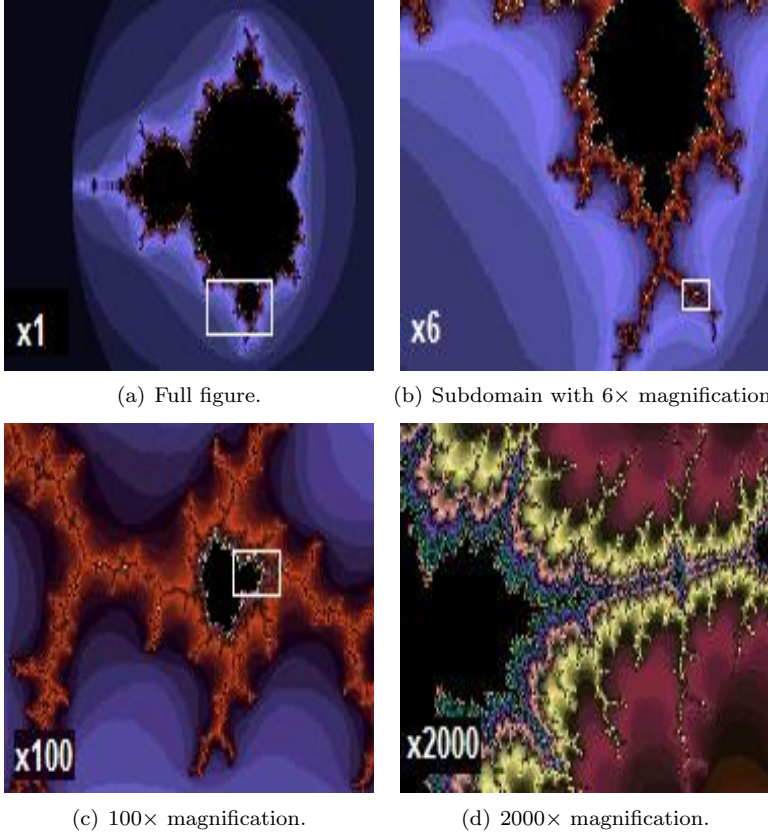


Figure 4. Mandelbrot set, with c_r on the abscissa and c_i on the ordinate. The white rectangles indicate the domain of the zoom in the next panel. From Nadim Ghaznavi, under the Creative Commons Attribution-Share Alike licence <https://creativecommons.org/licenses/by-sa/3.0/deed.en>.

420 Lyapunov dimension d_{KY} of the global attractor of the Lorenz (1963a) model:

$$421 \quad d_{KY} = 3 - \frac{2(\sigma + \beta + 1)}{\sigma + 1 + ((\sigma - 1)^2 + 4\rho\sigma)^{1/2}} < 3. \quad (3)$$

422 *Fractals in turbulence.* Of course, it is one thing to describe numerically and study mathematically fractals in dynamical systems and quite another thing to do so in natural phenomena. As already indicated in Sec. 1, the improvement in making and in analyzing observations, with their rapidly increasing number and accuracy, has also greatly accelerated the uses of scale invariance in the natural environment.

427 A particularly stimulating example is given by turbulence in general and by geophysical turbulence more specifically. Turbulent flow arises in many areas of engineering, as well as in nature, from blood flow to galactic evolution. Its presence and intensity is characterized by the Reynolds number $R \equiv UL/\nu$, where U , L and ν are a characteristic velocity, length and kinematic viscosity of the flow and the fluid: the higher U or L and the smaller ν , the more turbulent the flow.

433 Understanding and predicting turbulent behavior is probably the hardest problem in continuum physics. Compared to huge progress throughout the 20th century in quantum and relativistic physics, progress in turbulence studies has been more moderate.

436 In fact, in the opening article of the *Annual Reviews of Fluid Mechanics*, Sydney Goldstein (1969) attributes to Sir Horace Lamb the following statement “I am an old
437

438 man now, and when I die and go to Heaven there are two matters on which I hope for
 439 enlightenment. One is quantum electrodynamics, and the other is the turbulent motion
 440 of fluids. And about the former I am really rather optimistic.” The “old man,” of course,
 441 was a leader in fluid dynamics at the turn of the 19th into the 20th century and the au-
 442 thor of the Lamb (1932) book on which the generation of S. Goldstein, Ludwig Prandtl
 443 and Theodor von Kármán had grown up. Goldstein’s comment on this quote is, “Lamb
 444 was correct on two scores. All who knew him agreed that it was Heaven that he would
 445 go to, and he was right to be more optimistic about quantum electrodynamics than tur-
 446 bulence.” Goldstein’s prediction still holds exactly 50 years later.

447 Rapid progress of technology still obliged engineers and other practitioners to find
 448 empirical results even in the absence of deeper understanding of the causes of turbulence
 449 and the behavior of turbulent flows. In particular, once the crucial role of boundary lay-
 450 ers in mediating the transition between the fairly frictionless flow far from a wall and
 451 the necessity of a viscous fluid to be at rest at the boundary was understood, several em-
 452 pirical formulas were developed. Schlichting and Gersten (2016, and earlier editions) are
 453 a good source for this important subfield of turbulent fluid dynamics.

454 Thus, assumptions about the phenomena at play that appear at first sight rather
 455 strong, along with dimensional analysis (e.g., Barenblatt, 1996), lead to the well-known
 456 “law of the wall.” Let U be the (nearly constant) velocity outside the boundary layer,
 457 τ_w the shear stress at the solid surface, y the distance perpendicular to the surface, $u_\tau =$
 458 $(\tau_w/\rho)^{1/2}$, with ρ the density of the fluid, μ its molecular viscosity, and $\nu = \mu/\rho$ its kine-
 459 matic viscosity. The law is then given by $U/u_\tau = f((u_\tau y)/\nu)$ and it holds for the “in-
 460 ner layer” $y \leq 0.2\delta$, where δ is the total thickness of the boundary layer.

461 Based on work variously attributed to Lev Landau in the former Soviet Union and
 462 to L. Prandtl and Th. von Kármán in the western literature, the form of the function
 463 f above is logarithmic, resulting in the log-law

$$464 \quad \frac{U}{u_\tau} = \frac{1}{\kappa} \ln \frac{u_\tau y}{\nu} + C; \quad (4)$$

465 see Bradshaw and Huang (1995, and references therein). Extensive experimental work
 466 shows that Eq. (4) holds for $\kappa \simeq 0.41$ and $C \simeq 5.0$, provided the pressure gradient par-
 467 allel to the wall is not too large, within the region $30\nu/u_\tau \leq y \leq 0.1\delta$. The goodness
 468 of fit of the log-law above decreases as the pressure gradient increases and one approaches
 469 separation of the boundary layer.

470 Such semi-empirical relations, based on physical approximations and dimensional
 471 analysis, served practitioners well. Still, there was an increasing need for fundamental
 472 understanding of the complexities involved in turbulent flows.

473 A truly major step forward was due to the development of the concept of energy
 474 cascade and the statistical theory of turbulence. In his pioneering study of numerical weather
 475 prediction, L. F. Richardson (1922, p. 66) formulated the key idea of a turbulent cas-
 476 cade via the verse “Big whirls have little whirls that feed on their velocity, and little whirls
 477 have lesser whirls and so on to viscosity—in the molecular sense.”

478 This idea was refined first by distinguishing between the largest scales in a fluid
 479 that are most energetic and are affected by the geometry of the domain, and the small-
 480 est ones, at which energy input from nonlinear interactions and the energy drain from
 481 viscous dissipation are in exact balance. The latter have high frequency and are locally
 482 isotropic and homogeneous (e.g., Batchelor, 1953). In between these two scales, geomet-
 483 ric and directional information is lost in the A. N. Kolmogorov (1941) inertial cascade,
 484 between the large scales L and the Kolmogorov scale ℓ_K , provided the Reynolds num-
 485 ber R is sufficiently high.

The value of ℓ_K is given again merely by dimensional arguments and the physical assumption that the statistics of the small scales are universally and uniquely determined by the rate of energy dissipation ϵ and the kinematic viscosity ν , as $R \rightarrow \infty$,

$$\ell_K = (\nu^3/\epsilon)^{1/4}.$$

486 Between L and ℓ_K , instabilities break up the larger eddies into smaller ones that inter-
 487 act nonlinearly, while viscous effects are negligible. Once more, these assumptions and
 488 dimensional analysis lead — for scalar wavenumbers $k = 2\pi/r$ and $L > r > \ell_K$, where
 489 $r = |\mathbf{r}|$ and \mathbf{r} is the distance in the physical space \mathbb{R}^3 — to the kinetic-energy spectrum
 490 $E = E(k)$, namely

$$491 \quad E(k) = C\epsilon^{2/3}k^p, \tag{5}$$

492 with $p = -5/3$ and C a presumably universal constant.

493 Frisch (1995) presents this statistical theory of 3-D turbulence elegantly and reviews
 494 the experimental evidence, which confirms broadly the theory. This so-called direct en-
 495 ergy cascade appears in Fig. 5(a). There are two related difficulties, though. First, to
 496 cite again Goldstein (1969), “[...] distinguished mathematical statisticians, some of whom
 497 had hopes of contributing to the theory of turbulence, [when] they saw the physical, rather
 498 than mathematical, nature of Kolmogorov’s contribution [...] decided that such research
 499 was not for them.” Indeed, to this day — and in spite of considerable progress in the
 500 mathematical theory of the Navier-Stokes equations that govern fluid dynamics (e.g., Temam,
 2001) — there is no rigorous derivation of the Kolmogorov $(-5/3)$ law.

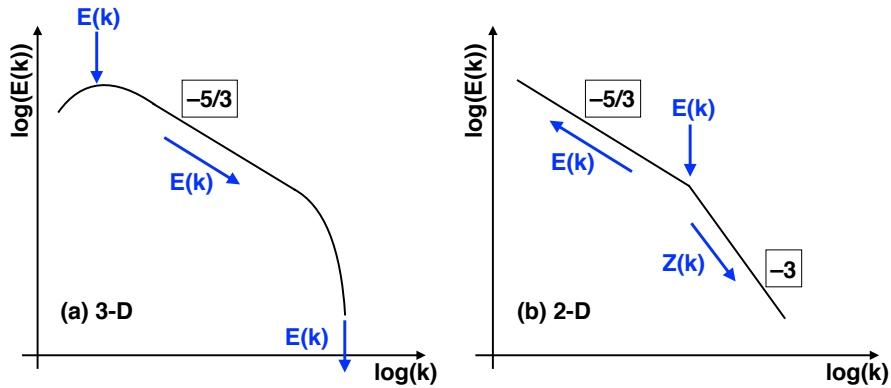


Figure 5. Energy and enstrophy cascades in (a) three-dimensional (3-D) and (b) two-dimensional (2-D) turbulence. The latter panel also characterizes the dual cascades in QG turbulence. Courtesy of Niklas Boers.

502 Second, the Kolmogorov (1941) theory implicitly assumes that 3-D turbulence is
 503 statistically self-similar at different scales in the inertial range. Thus the flow velocity
 504 increments $\delta\mathbf{u}(r) = \mathbf{u}(\mathbf{x} + \mathbf{r}) - \mathbf{u}(\mathbf{x})$, when scaled by $\lambda > 0$, should behave as $\delta\mathbf{u}(r) \simeq$
 505 $\lambda^\beta \delta\mathbf{u}(r)$, with β independent of r , where \simeq stands for equality in distribution. It follows
 506 that the structure functions of order n , i.e. the n^{th} -order statistical moments of the flow
 507 velocity increments $\delta\mathbf{u}$, should scale as

$$508 \quad \left\langle (\delta\mathbf{u}(r))^n \right\rangle = C_n (\epsilon r)^{n/3}, \quad (6)$$

509 where the brackets denote the statistical average, and the C_n are universal constants.

510 More generally, given $1 < |p| < 3$ in Eq. (5), one can show that the second-order
 511 structure function, i.e. $n = 2$ in Eq. (6) behaves like r^{p-1} . Since the latter is easier to
 512 measure accurately, $\langle (\delta\mathbf{u}(r))^2 \rangle \propto r^{2/3}$ implies that $p = 5/3$, confirming Kolmogorov
 513 (1941) theory. In fact, experimental differences are of the order of 2 % (Mathieu & Scott,
 514 2000). So far, so good.

515 Higher-order structure functions, though, deviate more and more from the scaling
 516 predicted by Eq. (6), as they become a sublinear function of n , and the constants C_n are
 517 far from universal, according to both laboratory and numerical experiments. The main
 518 reason for the observed deviations is the lack of homogeneity in the turbulent flow field,
 519 in either time or space; this feature of turbulence is referred to as intermittency and Man-
 520 delbrot (1969) highlighted its role: he conjectured that, as $R \rightarrow \infty$, the dissipation of
 521 the energy, far from being uniform, tends to concentrate on a fractal set with $d_H < 3$.
 522 Lagrangian coherent structures play an important role in reducing dissipation, produc-
 523 ing intermittency in turbulent flows, and increasing their predictability (e.g., Haller, 2015).

524 *Geophysical turbulence.* Large-scale atmospheric and oceanic flows are characterized by
 525 the key role of rotation and shallowness (e.g., Ghil & Childress, 1987; Gill, 1982; Ped-
 526 losky, 1987). The theoretical study of such flows is referred to as geophysical fluid dy-
 527 namics (GFD) and an important tool in this study is the QG approximation; see Ghil
 528 and Childress (1987, Ch. 4) for a succinct introduction.

529 Shallowness is due to the small aspect ratio $\delta \equiv H/L \ll 1$, where H is the char-
 530 acteristic height — with $H \simeq 10$ km in the atmosphere and even smaller in the oceans
 531 — and L the characteristic horizontal extent, with $L \simeq 10^3$ km in the atmosphere and
 532 $L \simeq 10^2$ km in the oceans. The dominant role of planetary rotation is due to the small-
 533 ness of the Rossby number $Ro \equiv U/fL \ll 1$, where U is a characteristic horizontal
 534 velocity, $f = 2\Omega \sin \phi$ is the Coriolis parameter that measures the local angular veloc-
 535 ity, while Ω is the planet’s angular velocity of rotation around its axis and ϕ the latitude.

536 QG flows are hydrostatic, i.e., vertical accelerations are negligible due to the flows’
 537 shallowness, and they are dominated by geostrophic balance between the Coriolis force
 538 and the pressure gradient. These two features result in QG flows being 2-D to a good
 539 first approximation, which suggests that geostrophic turbulence should also have 2-D fea-
 540 tures (e.g., Cushman-Roisin & Beckers, 2011; McWilliams, 2011; Salmon, 1998). We start
 541 by rapidly reviewing the differences between 2-D and 3-D turbulence.

542 The key difference is the existence of two quadratic invariants, enstrophy and ki-
 543 netic energy, rather than energy alone; see the references in Charney (1971, pp. 1087-
 544 1088), with enstrophy Z being the mean-squared vorticity. In 3-D turbulence, the con-
 545 servation of the kinetic energy $E(k)$ leads to the direct cascade from large to small scales,
 546 cf. Eq. (5), as illustrated in Fig. 5; the slope of the $E(k)$ spectrum over the inertial range
 547 $L \leq k \leq \ell_K$ equals approximately $-5/3$.

548 In 2-D turbulence, the existence of the two separate, positive-definite quadratic in-
 549 variants, kinetic energy E and enstrophy $Z(k)$, leads to two cascades. Indeed, Fjørtoft
 550 (1953) showed that an energy transfer from k to $k+\Delta k$ must, to conserve $Z(k)$, be ac-
 551 companied by a larger transfer of energy to $k-\Delta k$; this follows, essentially, from $Z(k) \propto$

552 $k^2 E(k)$. Based on this crucial fact, Kraichnan (1967) showed that, in 2-D turbulent flows,
 553 there are two inertial ranges: one with a reverse energy cascade and zero enstrophy flux,
 554 between L and L_* , the other with a direct enstrophy cascade and zero energy flux, be-
 555 tween L_* and ℓ_K . The slope of the energy spectrum in the former is $(-5/3)$, and it is
 556 (-3) in the latter, as illustrated in Fig. 5(b).

557 Charney (1971) noted atmospheric observations (e.g., Wiin-Nielsen, 1967) and numeri-
 558 cal simulations (e.g., Manabe, Smagorinsky, Holloway, & Stone, 1970) of a k_z^{-3} en-
 559 ergy spectrum, where $7 \leq k_z \leq 20$ is the zonal wavenumber, with a corresponding range
 560 of linear scales from 1 500 to 4 000 km. He emphasized, though, that the previously ac-
 561 cepted analogies between 2-D and QG flows are not really sufficient to argue for a simi-
 562 larity of the turbulent physics, given the fact that the baroclinic instability that injects
 563 energy at $L_* \simeq 10^3$ km is highly 3-D.

564 Charney (1971) argued that a deeper reason for the k_z^{-3} spectrum is the possibil-
 565 ity, in geostrophic turbulence, to combine its two quadratic invariants into a single one,
 566 which he termed “pseudo-potential vorticity,” following previous work of his own. In 1971,
 567 no sufficiently accurate observations or simulations were available for distinguishing among
 568 several hypotheses for the atmospheric spectrum beyond $k_z = 20$. As such observations
 569 did become available, Nastrom and Gage (1985) showed that (i) all the way down to 2.6 km,
 570 there are no spectral gaps; and (ii) in fact, the k_z^{-3} spectrum associated with the k_z^{-3}
 571 enstrophy cascade is followed by yet another $(-5/3)$ slope, as the flow becomes 3-D at
 the smallest scales; see Fig. 6.

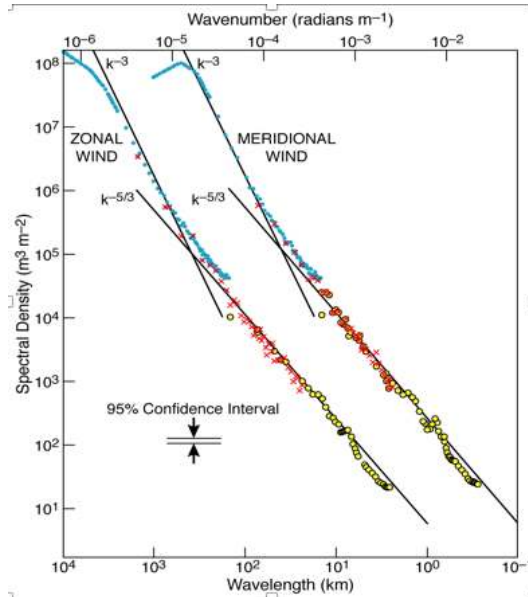


Figure 6. Wavenumber spectra of zonal and meridional velocity composited from three groups of flight segments of different lengths; these groups were selected from over 6 000 commercial aircraft flights. The three types of symbols (blue, red, and yellow) show results from each group. The least-square-fitted straight lines indicate slopes of (-3) and $(-5/3)$. The meridional wind spectra are shifted one decade to the right for greater legibility. The actual observational results show the typical deviations from straight lines in log-log coordinates. After Nastrom and Gage (1985).

573 Subsequent work, reviewed by Rhines (1979), Salmon (1998), and McWilliams (2011),
 574 among others, has greatly refined understanding of both atmospheric and oceanic tur-
 575 bulence, including the role of intermittency in deviating from simple $-5/3$ and -3 laws.
 576 The interest of GFD practitioners for 2-D turbulence, combined with the computationally
 577 much easier task of carrying out high-resolution, high- R calculations in 2-D led to
 578 an important discovery linking localized coherent structures with intermittency and in-
 579 creased predictability (Legras, Santangelo, & Benzi, 1988; McWilliams, 1984).

580 These structures were shown to be stable nonlinear solutions of the 2-D Euler equa-
 581 tions. They represent, therewith, a depletion of nonlinearity in the turbulent flow field,
 582 locally inhibit the direct enstrophy cascade, and can survive for long times. As a result,
 583 the predictability time of large-scale dynamics increases, being no longer limited as much
 584 by the small-scale fluctuations; see the recent review of Haller (2015).

585 Sakuma and Ghil (1991) also reviewed some of the pertinent GFD literature, as
 586 well as proving stability for such localized coherent structures in the shallow-water equa-
 587 tions, and emphasizing the analogies with magnetohydrodynamics (MHD). These analogies
 588 arise from the similarity between the role of the magnetic field vector \mathbf{B} in the lat-
 589 ter and the angular rotation vector $\mathbf{\Omega}$ in GFD (e.g., Ghil & Childress, 1987; Hide, 1989).

590 Helicity \mathbf{H} is an additional quadratic invariant in both 3-D and 2-D turbulence (Chorin,
 591 2013, and references therein) but it is not sign-definite, and hence does not have the same
 592 effect as enstrophy on balancing energy transfers. Still, it does give rise to both inverse
 593 and dual cascades, which are important in GFD as well as in MHD. Helicity dynamics
 594 and bidirectional cascades are discussed in this issue by Pouquet, Stawarz, Rosenberg,
 595 and Marino (2019). A particularly important application is to astrophysics in general
 596 and to the solar wind in particular (e.g., Pouquet, Marino, Mininni, & Rosenberg, 2017).

597 **5 A Few More Lampposts**

598 So far we have covered, to the extent allowed by the constraints of this special is-
 599 sue, some fundamental concepts, methods and results of dynamical systems theory in
 600 Sec. 3 and of scale invariance in Sec. 4. We will sketch now, even more briefly, the skele-
 601 ton of three additional lampposts that increasingly are helping shed some light on non-
 602 linear effects in the geosciences.

603 **5.1 The network lamppost**

604 We live in a world that is more and more dependent on networks of computing de-
 605 vices, as well as of people. Network theory thus is playing a bigger role in both under-
 606 standing and modifying this world. Its applications extend to a rapidly growing num-
 607 ber of areas, which include of course the geosciences.

608 Arguably, it is the Burridge and Knopoff (1967) model of friction along a fault that
 609 is the first and still one of the most important models of this kind in the geosciences. The
 610 model consists of a string of blocks connected by springs and can also be thought of as
 611 a modification of the Fermi, Pasta, Ulam, and Tsingou (1955) model with differing non-
 612 linear spring laws and the addition of nonlinear friction forces. It provided an understand-
 613 ing of the gradual accumulation and sudden release of potential energy associated with
 614 slow pre-seismic build-up and rapid displacement along an earthquake fault. The Burridge-
 615 Knopoff model, by its simple-model explanation of a baffling phenomenon, played a role
 616 in nonlinear solid-Earth studies that resembles that of the Lorenz (1963a) model in non-
 617 linear atmospheric studies.

618 *Network theory.* More generally, network theory is a field of graph theory. A graph is
 619 an object with nodes that are connected by edges. The nodes and edges have certain at-
 620 tributes, e.g. the physics at each node may be described by an ODE, while the link be-

621 tween two nodes may correspond to couplings between their ODEs. Such a network could
 622 then correspond to the method of lines being applied to a PDE (e.g., Schiesser, 2012).

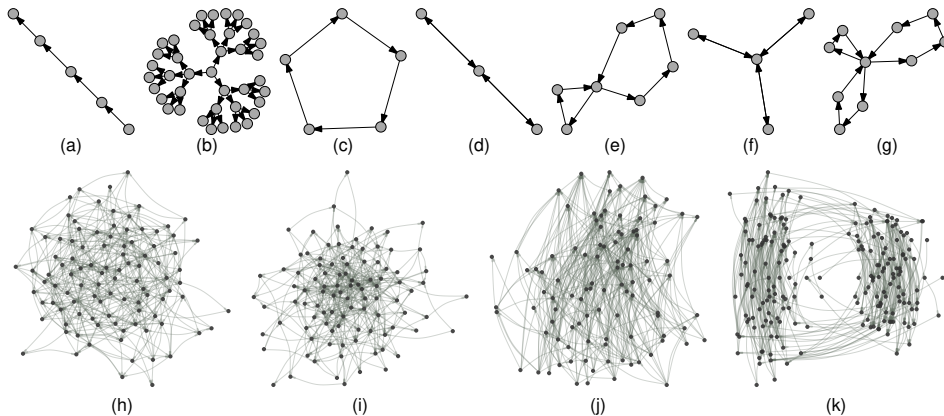


Figure 7. Schematic diagram of the network classes studied by Colon and Ghil (2017). (a)–(g) Simple network motifs: (a) linear networks; (b) trees; (c) isolated loops; (d) and (e) two interacting loops, connected through a pivotal node; (f) and (g) three interacting loops. (h)–(k) More complex classes of directed graphs, with $n = 100$ nodes and connectivity $c = 4$: (h) directed Erdős–Rényi (ER) networks; (i) scale-free networks with the specific, production-network distribution of in- and out-degree based on the Fujiwara and Aoyama (2010) dataset; (j) random acyclic (RA) networks in which production moves upward; and (k) a network of two interdependent RA networks — in the network at left, production moves upward, while it moves downward in the one at right. From Colon and Ghil (2017).

623

624 A much simpler network could be a geometrically linear one, each of its nodes hav-
 625 ing an identical Boolean expression attached to it, while being instantaneously connected
 626 to neighboring nodes. Such a network is called a cellular automaton (e.g., Von Neumann,
 627 1951; Wolfram, 1983). For illustration purposes, Fig. 7 shows a number of network classes
 628 recently studied by Colon and Ghil (2017).

629 A graph may be undirected, meaning that there is no distinction between the two
 630 nodes associated with each edge, or its edges may be directed from one node to another.
 631 The latter can be the case of river networks (Zaliapin, Foufoula-Georgiou, & Ghil, 2010,
 632 and references therein), supplier–producer networks (e.g., Colon & Ghil, 2017; Fujiwara
 633 & Aoyama, 2010), and many others (e.g., Albert & Barabási, 2002; Newman, 2010). A
 634 good example of the former is an Ising model on a 2-D lattice in statistical mechanics
 635 (e.g., Onsager, 1944) or a forest fire model of lesser (Malamud, Morein, & Turcotte, 1998)
 636 or greater (Spyratos, Bourgeron, & Ghil, 2007) complexity.

637 The topology of a network can be described by its adjacency matrix $\mathbf{A} = (a_{ij})$,
 638 where the entry a_{ij} equals 1 or 0 depending on whether an edge does exist between the
 639 nodes i and j or not. Much of network theory concentrates on various topological fea-
 640 tures, and on measures of centrality (Albert & Barabási, 2002; Newman, 2010, and ref-
 641 erences therein). Each of these measures aims to rank nodes by their importance, and
 642 they differ in how this importance is defined.

643 The simplest measure of centrality is the number of edges that it participates in,
 644 which is called the degree k . For directed graphs, one also distinguishes between the in-
 645 and out-degree. The distribution of degrees can be uniform, e.g., $k \equiv 1$ for either a lin-

ear graph or a simple cycle and $k \equiv 2$ for a braid (e.g., Coluzzi, Ghil, Hallegatte, & Weisbuch, 2011); it can be fully connected, $k \equiv N - 1$, where N is the number of nodes; it can be fully random, in which case the mean degree is $z = \bar{k} > N/2$; or it can be scale-free, i.e. it obeys a power law, with $p(k) \simeq k^{-\alpha}$, with $\alpha > 0$.

The dynamics on a network depends on the mathematical description of the state of each node, its set of linked neighbors, and on the nature of the links, i.e., on the coupling between the nodes. The state of each node can be described by a time series of real- or Boolean-valued variables; such time series, in turn, can either be provided by observations or be the result of evolution equations, be they systems of ODEs, PDEs or of Boolean equations. The links, as previously mentioned, can be directed or not; they can also change in time in an evolving network.

Network applications, I: Boolean delay equations (BDEs). We will give here an application to earthquake modeling and prediction. First, we introduce the framework of Boolean delay equations (BDEs) to describe the state of the nodes and the nature of the links.

A system of BDEs is a semi-discrete dynamical model with Boolean-valued variables that evolve in continuous time (Dee & Ghil, 1984; Ghil & Mullhaupt, 1985). The place occupied by BDEs in the world of dynamical systems is illustrated in Fig. 8.

Systems of BDEs can be classified into conservative or dissipative, in a manner that parallels the classification of ODEs or PDEs. Solutions to certain conservative BDEs exhibit growth of complexity in time; such BDEs can be seen therefore as metaphors for biological evolution or human history. Dissipative BDEs are structurally stable and exhibit multiple equilibria and limit cycles, as well as more complex, fractal solution sets, such as Devil’s staircases and “fractal sunbursts” (Ghil, Zaliapin, & Coluzzi, 2008, and references therein).

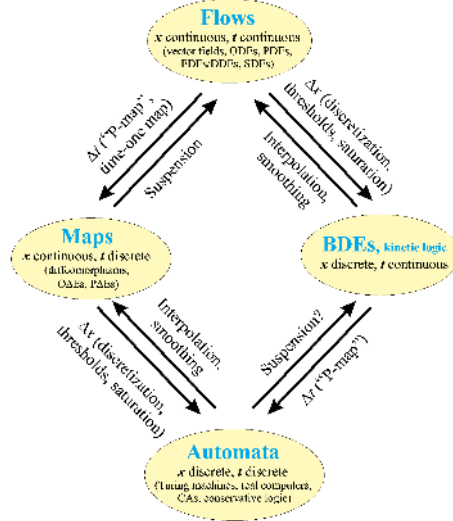


Figure 8. Schematic diagram of the distinct classes of dynamical systems, in terms of the state x and time t . Note the links: the discretization of time t can be achieved by the Poincaré map (P-map) or a time-one map, leading from **Flows** to **Maps**. The opposite connection is achieved by suspension. To go from **Maps** to **Automata** one has to discretize the state and smoothing lead in the opposite direction. Similar connections lead from **BDEs** to **Automata** and to **Flows**, respectively. Please see the glossary in Table A.1 for acronyms. Modified after Mullhaupt (1984).

669 More generally, Fig. 8 raises the question of which one of the various types of dy-
 670 namical systems therein apprehends best the complexities of the world surrounding us?
 671 Clearly, the amount of detail provided by each increases as we move from the **Automata**
 672 at the bottom to the **Flows** at the top of the rhomboid in the figure.

673 Thus, one level at which one can read the figure is as an illustration of the hier-
 674 archy of models discussed further in Sec. 6. But there is also another way of reading it.
 675 In fact, each one of the downward-pointing arrows between a class of models and an ad-
 676 jacent one below it represents a perfectly self-consistent simplification, obtained as one
 677 discretizes either time t or space x . We all know how to obtain an ordinary or partial
 678 difference equation (O Δ E or P Δ E) from an ODE or PDE respectively, by discretizing
 679 time. The extent to which the solutions of the O Δ E so obtained converge to those of the
 680 corresponding ODE depend on certain stability and consistency properties of the ODE's
 681 right-hand side (e.g., Isaacson & Keller, 2012).

682 In the case of a P-map, topological properties are preserved as one goes from a **Flow**
 683 to a **Map**, and maps are easier to study. Under certain technical assumptions dealing
 684 with smoothness and one-to-oneness, one gets most of what one wants from studying the
 685 map, since the suspension that goes back from the **Map** one has studied to the **Flow**
 686 can be proven to have the right properties. For instance, a periodic solution of the **Flow**
 687 will appear as a point in the **Map** and vice-versa.

688 Can similar equivalence results be proven for other pairs of arrows in Fig. 8? There
 689 exists numerical evidence, at least, to suggest that it might be true under suitable cir-
 690 cumstances. Two such examples of, at least partial, equivalences are given below.

691 Saunders and Ghil (2001) provided a thorough BDE treatment of the El-Niño/Southern-
 692 Oscillation (ENSO) mechanism postulated by J. J. Bjerknes (1969). Their Fig. 7 of the
 693 “Devil’s bleachers” shows the dependence of the model ENSO’s periodicity on two model
 694 parameters that characterize the wave propagation along the equator and the local ocean-
 695 atmosphere heat exchanges, respectively; see also Ghil, Zaliapin, and Coluzzi (2008, Fig. 6).
 696 The projection of the latter 3-D axionometric plot on its 2-D parameter plane is strik-
 697 ingly similar to Fig. 9 herein.

698 This similarity is the first example of good numerical correspondence between two
 699 adjacent vertices of the rhomboid in Fig. 8, since the “Devil’s terrace” in Fig. 9 is based
 700 on the intermediate model of Jin, Neelin, and Ghil (1994, 1996). The latter model is gov-
 701 erned by a system of nonlinear PDEs in one space dimension, namely longitude along
 702 the equator, with the parameters μ and δ_s that appear in Fig. 9 here; the two play a roughly
 703 similar role in the PDE model to that of the two parameters, local and global, in Ghil,
 704 Zaliapin, and Coluzzi (2008, Fig. 6).

705 The early applications of BDEs to the climate sciences only used small systems of
 706 a few variables (e.g., Darby & Mysak, 1993). The first BDE application on a network
 707 was to a very simple model of seismic activity. The model consists of a ternary tree with
 708 a direct cascade of loading from a top node that represents a major plate, down to smaller
 709 and smaller plates. This direct cascade collides with an inverse cascade of failures that
 710 starts with the bottom nodes and travels up to larger and larger plates, possibly all the
 711 way to the top, depending on the delayed effects of healing (Zaliapin, Keilis-Borok, &
 712 Ghil, 2003a, 2003b, and references therein).

713 Clearly, to analyze extensively and systematically systems of 3^L ODEs would be
 714 fairly prohibitive, even for a tree depth L as small as 6 or 7. Fairly surprisingly, though,
 715 the BDE model could be easily analyzed as a function of the loading and healing param-
 716 eters, yielding the three well-known seismic regimes of high (**H**), low (**L**) and intermit-
 717 tent (**I**) seismicity, as shown in Fig. 10.

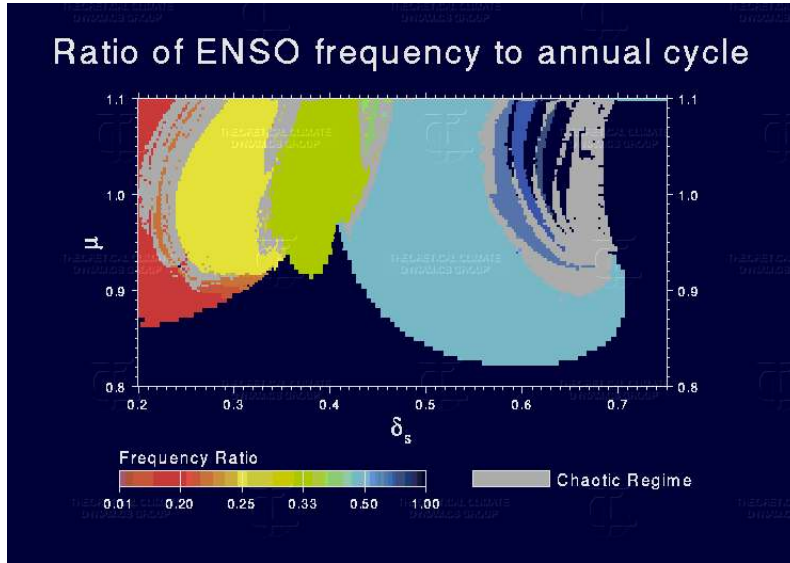


Figure 9. Regimes of subharmonic, frequency-locked and chaotic solutions in the (μ, δ_s) parameter plane; here μ is the local ocean–atmosphere coupling parameter and δ_s is an ocean mixed layer parameter that determines the model’s intrinsic periodicity, in the absence of the annual cycle. Black areas represent regions where no interannual signal is present. Color scale represents the frequency ratio of the interannual oscillation to the annual cycle in regimes that are frequency locked; e.g., 0.25 indicates one ENSO cycle every four years, 0.222 indicates two ENSO cycles repeating every nine years. Chaotic regimes are plotted in grey. Courtesy of Fei-Fei Jin.

718 The three regimes are characterized, respectively, by the following key features:

- 719 **H:** A cyclo-stationary behavior, with the maximum earthquake intensity reached on
- 720 every cycle;
- 721 **I:** A highly intermittent behavior, with irregular intervals between major earthquakes
- 722 and high, but not necessarily maximum intensity of the latter; and
- 723 **L:** A fairly low and nearly constant level of white-noise-like seismic activity overall.

724 These features are present in observations (e.g., Press & Allen, 1995; Romanow-
 725 icz, 1993), as well as in much more detailed and sophisticated models (Ben-Zion, 2008,
 726 and references therein). On the whole, it is the intermittent behavior that is most widespread,
 727 but a particular region can also change regime over time, as parameter values that af-
 728 fect the collective behavior of earthquakes and faults change. This is the second numer-
 729 ical example of at least partial equivalence between a **BDE** model and a **Flow**.

730 *Network applications, II: Teleconnections and centrality.* A very different network-theoretical
 731 setting was applied to climatic variability, and we discuss it now very succinctly herein,
 732 following Tsonis and Swanson (2008) and Donges, Zou, Marwan, and Kurths (2009a).
 733 The idea that meteorological, oceanographic or coupled climatic variability might involve
 734 “centers of action” that are widely separated in space goes back to Hildebrandsson and
 735 Teisserenc de Bort (1898) and to G. Walker’s “teleconnections” between them (Walker
 736 & Bliss, 1932). The statistical and dynamical study of such teleconnections engaged many
 737 important figures in the history of these disciplines over the last century (J. Bjerknes,
 738 1969; Hoskins & Karoly, 1981; Wallace & Gutzler, 1981).

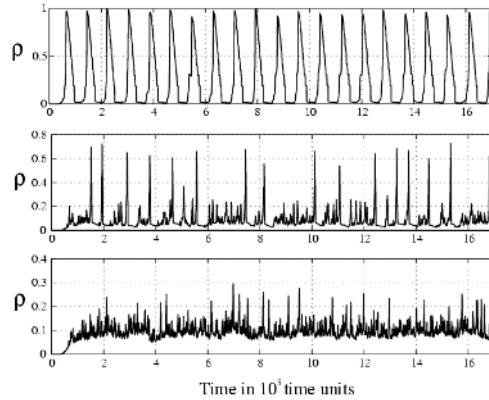


Figure 10. Three seismic regimes in the internal dynamics of the BDE model, for a tree depth of $L = 6$, i.e. for $n = 1093$ nodes. The panels show the density $\rho = \rho(t)$ of broken elements in the system. See Figs. 7 and 8 in Zaliapin et al. (2003a) for loading and healing parameter values and other details. (a) Regime **H**; (b) Regime **I**; and (c) Regime **L**. Note the difference in vertical scale for the three panels. Reproduced from Zaliapin et al. (2003a) with kind permission of Springer Science and Business Media.

739 One of the main approaches used by A. A. Tsonis and colleagues (e.g., Tsonis, Swan-
 740 son, & Kravtsov, 2007), as well as by the groups around J. Kurths (e.g., Donges, Zou,
 741 Marwan, & Kurths, 2009b) and around S. Havlin (e.g., Gozolchiani, Havlin, & Yamasaki,
 742 2011), was labeled complex networks (CNs) and essentially consists in identifying the
 743 strongest correlations among time series at different locations. Boers, Bookhagen, Mar-
 744 wan, Kurths, and Marengo (2013) review relevant climate network literature and provide
 745 an application to the South American Monsoon System and to the spatial patterns
 746 associated with synchronization of extreme rainfall events; see also Boers et al. (2019)
 747 for a global analysis of extreme-rainfall teleconnections.

748 Many of the dynamical studies of the atmosphere’s low-frequency variability that
 749 involve teleconnections have used the highly simplified geometry of a so-called β -channel
 750 with periodicity in longitude and solid walls along parallels to the north and south of
 751 the channel, away from both the North Pole and the Equator (Ghil & Childress, 1987;
 752 Pedlosky, 1987); see also Sec. 3.2 herein. Colon and Ghil (2017, and references therein)
 753 showed that signal propagation in networks with distinct topologies in the plane can have
 754 very different properties; these properties are quite likely to be entirely different, in turn,
 755 from those of networks on the sphere. It is the latter that are most relevant to dynam-
 756 ical studies on a spherical domain, whether linear (e.g., Hoskins & Karoly, 1981) or non-
 757 linear (e.g., Legras & Ghil, 1985). Thus BDE models in such geometrically different set-
 758 tings as shown in Fig. 7 here, on the plane and on the sphere, might complement or even
 759 guide further network-based investigations of teleconnections and climate variability.

760 5.2 The fluctuation–dissipation lamppost

761 Fluctuation–dissipation theory (FDT) has its roots in the classical theory of statis-
 762 tical mechanics of many-particle systems in thermodynamic equilibrium. The idea is
 763 very simple: the system’s return to equilibrium will be the same whether the perturba-
 764 tion that modified its state is due to a small external force or to an internal, random fluctu-
 765 ation (e.g., Kubo, 1966, and references therein). We outline below the simplest cases,

766 and point to the generalization to systems out of equilibrium, such as the climate sys-
767 tem or a network of seismic faults.

768 *Fluctuation–dissipation theory (FDT)*. Like so many other ideas in the physical sciences,
769 FDT goes back to Einstein and his *Annus mirabilis*, 1905. Einstein (1905) formulated
770 the problem of the Brownian motion of a large particle immersed in a fluid formed of
771 many small ones as follows. The presentation here follows Ghil and Childress (1987, Sec. 10.3),
772 where further details can be found. Consider the large particle as moving on a straight
773 line with velocity $u = u(t)$, subject to a random force $\eta(t)$ and to linear friction $-\lambda u$,
774 with coefficient λ . The equation of motion is

$$775 \quad du = -\lambda u dt + \eta(t), \quad (7)$$

776 The random force $\eta(t)$ is assumed to be a “white noise,” i.e., it has mean zero $\mathcal{E}[\eta(t; \omega)] =$
777 0 and autocorrelation $\mathcal{E}[\eta(t; \omega)\eta(t+s; \omega)] = \sigma^2 \delta(s)$, where $\delta(s)$ is a Dirac function, σ^2
778 is the variance of the white-noise process, ω labels the realization of the random process,
779 and \mathcal{E} is the expectation operator, which averages over the realizations ω . Alternative
780 notations for the latter are the overbar, in climate sciences, and the angle brackets, in
781 quantum mechanics, $\mathcal{E}[F] := \bar{F} := \langle F \rangle$.

782 Equation (7), with $\eta = \sigma dW$, is a linear stochastic differential equation (SDE)
783 of a form that is now referred to as a Langevin equation, where $W(t)$ is a normalized Brow-
784 nian motion or Wiener process. The necessary stochastic concepts are explained at a com-
785 fortable level in Dijkstra (2013, Ch. 3). Einstein’s main results are that

$$786 \quad \mathcal{E}[u^2] = \frac{\tau^*}{2\lambda}, \quad \mathcal{E}[x^2] = \frac{\tau^*}{\lambda^2} t, \quad (8)$$

787 with $x(t) = x_0 + \int_0^t u(s) ds$ the displacement of the particle and $\tau^* = \int_{-\infty}^{+\infty} \sigma^2 \delta(s) ds$.
788 There are two remarkable features in Eqs. (8) above. First, the fact that the variance
789 $\mathcal{E}[x^2]$ of the displacement is proportional to time. This leads to the mathematical theo-
790 ry of SDEs distinguishing between the time differential dt and the stochastic differen-
791 tial dW , since $\int_0^t ds = t$, while $\int_0^t dW(s) = t^{1/2}$; in other words, $dW \propto (dt)^{1/2}$.

792 Second, the friction coefficient λ characterizes in this simple case a dissipation of
793 the fluctuations, since $\mathcal{E}[u(t)] = \mathcal{E}[u_0] \exp(-\lambda t)$. More generally, as Kubo (1966, Se. 2)
794 points out, the dissipation constant is $D = \lim_{t \rightarrow \infty} \mathcal{E}[(x(t) - x(0))^2]$, and one gets

$$795 \quad \mu = \frac{D}{kT} = \frac{1}{kT} \int_0^\infty \mathcal{E}[u(t_0)u(t_0 + t)] dt; \quad (9)$$

796 here $\mu = 1/\lambda$ is the mobility of the particles, T is the temperature of the thermal bath,
797 and k is the Boltzmann constant. And *voilà*, you have the original and simplest version
798 of FDT, where the acronym also stands for the fluctuation–dissipation theorem.

799 FDT in general can thus be used either to infer the statistics of thermal fluctua-
800 tions from the drag law (e.g., Nyquist, 1928) with known λ or the reverse (e.g., Onsager,
801 1931). The former is more practical in laboratory or industrial situations, like an elec-
802 tric circuit, where it is relatively easy to measure the admittance or impedance of the
803 system and we are not that interested in details of what happens at such-and-such a lo-
804 cation in an individual wire. It is the latter, though, that is more useful for natural sys-
805 tems, like the climate system, where we have many observations localized in time and
806 space, and wish to estimate future response to as-yet-unknown forcings.

807 All of the above apply, however, to systems in thermodynamic equilibrium, and most
808 natural systems — including, of course, the climate system — are not. As Kubo (1966)
809 notes, it is precisely for this reason that FDT has attracted much greater attention “re-
810 cently” — i.e., in the middle of the 20th century — as it has been extended to “nonequi-
811 librium states [and to] irreversible processes in general.”

812 *FDT applications.* Cecil E. (“Chuck”) Leith (1975) showed that FDT applies to a 2-D
 813 or QG turbulent flow with two integral invariants, kinetic energy E and enstrophy Z ,
 814 under additional assumptions of normal distribution of the realizations and stationar-
 815 ity. Such flows were reviewed in Sec. 4.2 herein. Subject to the above assumptions (Leith,
 816 1975), the unperturbed covariance matrix \mathbf{U} and the average response matrix \mathbf{G} are then
 817 related by the FDT relation

$$818 \quad \mathbf{U}(\tau) = \mathbf{G}(\tau)\mathbf{U}(0), \quad (10)$$

819 where τ is the interval over which we wish to estimate the response of the system to an
 820 arbitrary external forcing. Noting that the regression matrix \mathbf{R} for linear prediction of
 821 the stationary multivariate time series with lagged covariance matrix \mathbf{U} equals \mathbf{G} , one
 822 then gets that

$$823 \quad \mathbf{R}(\tau) = \mathbf{U}(\tau)\mathbf{U}^{-1}(0). \quad (11)$$

824 Since the problem of estimating the response of the climate system to both natu-
 825 ral and anthropogenic forcing on multidecadal time scales is becoming scientifically, as
 826 well as socio-economically, more and more important, Eqs. (10, 11) present a huge ad-
 827 vantage over conventional methods of attacking this problem. Indeed, successive assess-
 828 ment reports of the Intergovernmental Panel on Climate Change (IPCC: e.g., Houghton,
 829 Jenkins, & Ephraums, 1990; IPCC, 2007) carried out ensembles of high-end global cli-
 830 mate model simulations with a number of prescribed scenarios of such forcings, but were
 831 limited by the enormous computational expense of such simulations.

832 In comparison, the linear response of Eq. (11) can be computed, at least in a re-
 833 duced subspace of leading eigenvectors of the covariance matrix \mathbf{U} — the so-called em-
 834 pirical orthogonal functions ((EOFs: Jolliffe & Cadima, 2016; Preisendorfer, 1988) —
 835 relatively easily. And, once that is done, changes in any prescribed scalar or vector observ-
 836 able, say in the globally averaged surface air temperatures or in the entire sea surface
 837 temperature field $\{T_{ij}(t)\}$, can be evaluated in turn for arbitrary small forcings $\delta\mathbf{f}(t)$.

838 Let $\hat{\mathbf{U}}$ and $\hat{\mathbf{R}}$ be the reduced versions of \mathbf{U} and \mathbf{R} , respectively, with $\{\hat{\mathbf{u}}_\alpha\}$ the EOFs
 839 of $\hat{\mathbf{U}}$, and $\{T_\alpha(t)\}$ the projection of said temperature field onto the corresponding EOFs.
 840 Component-wise, we can write, following Leith (1975), that

$$841 \quad \delta T_\alpha(t) = \int_{-\infty}^t \sum_{\beta} \hat{R}_{\alpha\beta} \delta f_\beta(s) ds. \quad (12)$$

842 Once more, this is all very helpful for systems in thermodynamic equilibrium and
 843 normally distributed stochastic processes, which turbulent fluids and other subsystems
 844 of the climate system are not. Given a normal distribution of an initial state, nonlinear-
 845 ity will break that happy state of affairs to a greater or lesser degree.

846 Generalizations to systems out of equilibrium have been developed since the early
 847 1950s (e.g., Callen & Welton, 1951) and many references appear in Kubo (1966). But
 848 a particularly fruitful change in point of view was provided by D. Ruelle (1998, 2009),
 849 who considered the problem in the setting of dynamical systems theory, rather than that
 850 of statistical mechanics. The former point of view is justified in this context by the so-
 851 called chaotic hypothesis (e.g., Gallavotti & Cohen, 1995), which states, in rough terms,
 852 that chaotic systems with many degrees of freedom possess a physically relevant invari-
 853 ant measure ν such that averaging with respect to this measure is equivalent to averag-
 854 ing in time over the system’s attractor. This property suffices for using the measure ν
 855 in evaluating changes in any observable of the system with respect to any small pertur-
 856 bation, and we return to this point in Sec. 5.3 below.

857 In the footsteps of Leith (1975), several applications of FDT to climate (e.g., Abramov
 858 & Majda, 2008; Gritsun & Branstator, 2007) and ocean (Wirth, 2018) models have been
 859 carried out. It is V. Lucarini and colleagues, though, who have systematically applied

Ruelle’s linear response theory to generalize both equilibrium and transient climate sensitivity (Lucarini, Ragone, & Lunkeit, 2016; Ragone, Lucarini, & Lunkeit, 2015); they also obtained the resonant response and its spatial patterns in one or more frequency bands for time-dependent forcing (Lucarini et al., 2014, and references therein).

The study of resonant response is made possible by the study of the susceptibility operator $\tilde{\mathbf{S}}$, which is given by the Fourier transform of the linear response operator $\tilde{\mathbf{G}}$. The latter operator requires a generalization of the response matrix \mathbf{G} defined in Eq. (10) to the non-equilibrium setting, for which we refer to the work of Ruelle (1998, 2009) and of Lucarini et al. (2014).

5.3 The random dynamical systems (RDS) lamppost

In Sec. 3, we have considered mainly the deterministically nonlinear approach to apprehend the complexities of geosciences in general and climate variability in particular. In the previous subsection, we have also hinted, via the Langevin equation (7), at the complementary approach of stochastically linear dynamics to climate variability and change, due largely to K. Hasselmann (1976). Imkeller and Von Storch (2001, and references therein) give a broader view of this approach. In the present subsection, we briefly outline a promising unification of the two complementary approaches to climate variability and change, via the theory of random dynamical systems (RDS).

The theory of nonautonomous (NDS) and random (RDS) dynamical systems. As a result of sensitive dependence on initial data and on parameters, numerical weather forecasts, as well as climate projections, are both expressed these days in probabilistic terms. It is, in fact, more convenient — and becoming more and more necessary — to rely on a model’s (or set of models’) probability density function (PDF) rather than on its individual, pointwise simulations or predictions.

We summarize here results on the surprisingly complex statistical structure that characterizes stochastic nonlinear systems. This complex structure does provide meaningful physical information that is not described by the PDF alone; it lives on a random attractor, which extends the concepts of a strange attractor and of the invariant measure that is supported by it, from the deterministic to the stochastic framework. It is this extension that we describe, in the simplest possible terms, forthwith.

On the road to including random effects, one needs to realize first that the climate system, as well as any of its subsystems, is not closed: it exchanges energy, mass and momentum with its surroundings, whether other subsystems or the interplanetary space and the solid earth. Typical applications of dynamical systems theory to climate variability so far have only taken into account exchanges that are constant in time, thus keeping the model — whether governed by ODEs, PDEs or other differential equations — autonomous; i.e., the models had coefficients and forcings that were constant in time.

Succinctly, one can write such a system as

$$\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X}; \boldsymbol{\mu}), \quad (13)$$

where \mathbf{X} now may stand for any climate or other geophysical field, while \mathbf{f} is a smooth function of \mathbf{X} and of the vector of parameters $\boldsymbol{\mu}$, but does not depend explicitly on time. Being autonomous greatly facilitated the analysis of a model’s solutions. For instance, two distinct trajectories, $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$, of a well-behaved, smooth autonomous system cannot pass through the same point in phase space, which helps describe the system’s phase portrait. So does the fact that we only need to consider the behavior of solutions $\mathbf{X}(t)$ as we let time t tend to $+\infty$: the resulting sets of points are — possibly multiple — stationary solutions, periodic solutions, and chaotic sets.

We know only too well, however, that the seasonal cycle plays a key role in climate variability on many time scales, while orbital forcing is crucial on the Quaternary time

scales of many millennia, and now anthropogenic forcing is of utmost importance on interdecadal time scales. How can one take into account such time-dependent forcings, and analyze the nonautonomous systems, written succinctly as

$$\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X}, t; \boldsymbol{\mu}), \quad (14)$$

907 to which they give rise? In Eq. (14), the dependence of \mathbf{f} on t may be periodic, $\mathbf{f}(\mathbf{X}, t + P) = \mathbf{f}(\mathbf{X}, t)$, as in various ENSO models, with $P = 12$ months, or monotone, $\mathbf{f}(\mathbf{X}, t + \tau) \geq \mathbf{f}(\mathbf{X}, t)$ for $\tau \geq 0$, as in studying scenarios of anthropogenic climate forcing.

910 To illustrate the fundamental character of the distinction between (13) and (14),
911 consider the simple scalar version of these two equations:

$$\dot{X} = -\beta X, \quad (15a)$$

$$\dot{X} = -\beta X + \gamma t, \quad (15b)$$

912 respectively. We assume that both systems are dissipative, i.e. $\beta > 0$, and that the forc-
913 ing is monotone increasing, $\gamma \geq 0$, as would be the case for anthropogenic forcing in
914 the industrial era. Lorenz (1963a) pointed out the key role of dissipativity in giving rise
915 to strange, but attracting solution behavior, while Ghil and Childress (1987, Sec. 5.4)
916 emphasized its importance and pervasive character in climate dynamics. Clearly the only
917 attractor for the solutions of Eq. (15a), given any initial point $X(0) = X_0$, is the fixed
918 point $X = 0$, attained as $t \rightarrow +\infty$.

919 For the nonautonomous case of Eq. (15b), though, this forward-in-time approach
920 yields blow-up as $t \rightarrow +\infty$, for any initial point. To make sense of what happens in the
921 case of time-dependent forcing, one introduces instead the pullback approach, in which
922 solutions are allowed to still depend on the time t at which we observe them, but also
923 on a time s from which the solution is started, $X(s) = X_0$; presumably $s \ll t$. With
924 this little change of approach, one can easily verify that
925
926
927

$$|X(s, t; X_0) - \mathcal{A}(t)| \rightarrow 0 \quad \text{as } s \rightarrow -\infty, \quad (16)$$

928 for all t and X_0 , where the pullback attractor (PBA) $\mathcal{A}(t)$ is given explicitly by

$$\mathcal{A}(t) = \frac{\gamma(t - 1/\beta)}{\beta}. \quad (17)$$

929 We thus obtain, in this pullback sense, the intuitively obvious result that the solutions,
930 if started far enough in the past, all approach the time-dependent attractor set $\mathcal{A}(t)$, which
931 grows linearly in time and thus follows the linear forcing.

932 For the more complicated case of RDSs, where the random attractor \mathcal{A} depends
933 on the particular realization ω of the driving noise, $\mathcal{A} = \mathcal{A}(t; \omega)$, we refer to Chekroun,
934 Simonnet, and Ghil (2011); Ghil, Chekroun, and Simonnet (2008) and Dijkstra (2013,
935 Ch. 4). The beauty and complexity of the results is illustrated herein by four snapshots
936 at successive times $\{t_1, \dots, t_4\}$ for the Lorenz (1963a) model perturbed by multiplicative
937 noise; see Fig. 11. Note that the support of the invariant measure $\nu(t; \omega)$ may change
938 quite abruptly, from time t to time $t + \Delta t$; see the related short video given as Supple-
939 mentary Information in Chekroun et al. (2011), as well as at <https://vimeo.com/240039610>.
940 This video shows more clearly than a simple sequence of snapshots the interaction be-
941 tween the nonlinearly deterministic dynamics and the stochastic perturbations.
942
943

944 *NDS and RDS applications.* We outline here briefly an application of the theory of nonau-
945 tonomous dynamical systems (NDSs) to the so-called double-gyre problem of the wind-
946 driven ocean circulation, following Pierini, Ghil, and Chekroun (2016) and Ghil (2017).
947 The large-scale, near-surface flow of the mid-latitude oceans is dominated by the pres-
948 ence of a larger, anticyclonic and a smaller, cyclonic gyre. The two gyres share the east-
949 ward extension of western boundary currents, such as the Gulf Stream or Kuroshio, and

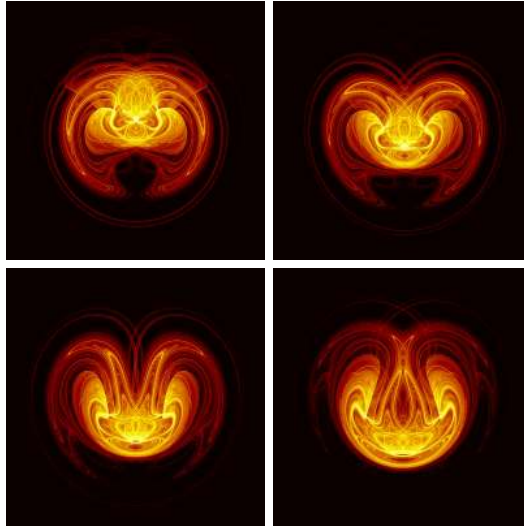


Figure 11. Four snapshots of the stochastically perturbed Lorenz (1963a) model's random attractor $\mathcal{A}(\omega)$ and the invariant measure $\nu(\omega)$ supported on it. The model can be written componentwise as $dX_i = f(\mathbf{X}; \boldsymbol{\mu})dt + \sigma X_i dW$, $i = 1, 2, 3$, with $\mathbf{X} \equiv (X_1, X_2, X_3) \equiv (X, Y, Z)$ and the parameter values $\boldsymbol{\mu}$ equal to the classical ones — normalized Rayleigh number $r = 28$, Prandtl number $Pr = 10$, and normalized wave number $b = 8/3$ — while the noise intensity is $\sigma = 0.5$ and the time step is $\delta t = 5 \cdot 10^{-3}$. The color bar used is on a log-scale and quantifies the probability to end up in a particular region of phase space; shown is a projection of the 3-D phase space (X, Y, Z) onto the (X, Z) -plane. Notice the complex, interlaced filament structures between highly (yellow) and moderately (red) populated regions. The time interval Δt between two successive snapshots — moving from left to right and from top to bottom — is $\Delta t = 0.0875$. Weakly populated regions cover an important part of the random attractor and are, in turn, entangled with regions that have near-zero probability (black). [After Chekroun et al. (2011) with permission from Elsevier.]

950 are induced by the shear in the winds that cross the respective ocean basins. Results for
 951 this problem in the presence of a surface wind stress that is constant in time were re-
 952 viewed briefly in Sec. 3.2; see, in particular, Figs. 1 and 2 there.

953 The model domain used by Pierini et al. (2016) is rectangular, like those in Sec. 3.2,
 954 and the model equations are based on the equivalent barotropic QG vorticity equation
 955 of Simonnet et al. (2005). This PDE is projected here onto four modes that take into
 956 account the presence of a western boundary current by including an exponentially de-
 957 caying factor for the streamfunction field, as suggested by Jiang et al. (1995). The forc-
 958 ing is deterministic, aperiodic, and dominated by interdecadal variability.

959 The autonomous system exhibits a global bifurcation associated with a homoclinic
 960 orbit, like the one illustrated in Fig. 2 herein; it occurs at the value $\gamma = 1.0$ for the pa-
 961 rameter γ that scales the intensity of the forcing. Pierini et al. (2016, Appendix) have
 962 rigorously demonstrated the existence of a global PBA for the time-dependent forcing
 963 case in the weakly dissipative, nonlinear model under discussion, based on general re-
 964 sults for nonautonomous dynamical systems (Carvalho, Langa, & Robinson, 2012; Kloeden
 965 & Rasmussen, 2011).

966 Numerically, though, this unique global attractor seems to possess two separate lo-
 967 cal PBAs, as apparent from Fig. 12. Panels (a) and (b) in the figure refer to parame-

968 ter values that correspond to subcritical vs. supercritical values of γ in the autonomous
 969 model, respectively. While formula (16) seems to require an infinite pullback time, it turns
 970 out that convergence to the PBAs in this model only takes about 15 yr.

971 The mean normalized distance Δ plotted in the figure is defined as $\Delta = \langle \delta_n \rangle_{\tilde{T}}$.
 972 Here δ_t is the distance, at time t , between two trajectories of the model that were a dis-
 973 tance δ_0 apart at time $t = t_0$, and the normalized distance $\delta_n = \delta_t / \delta_0$ is averaged over
 the whole forward time integration \tilde{T} of the available trajectories, with $\tilde{T} = 400$ yr.

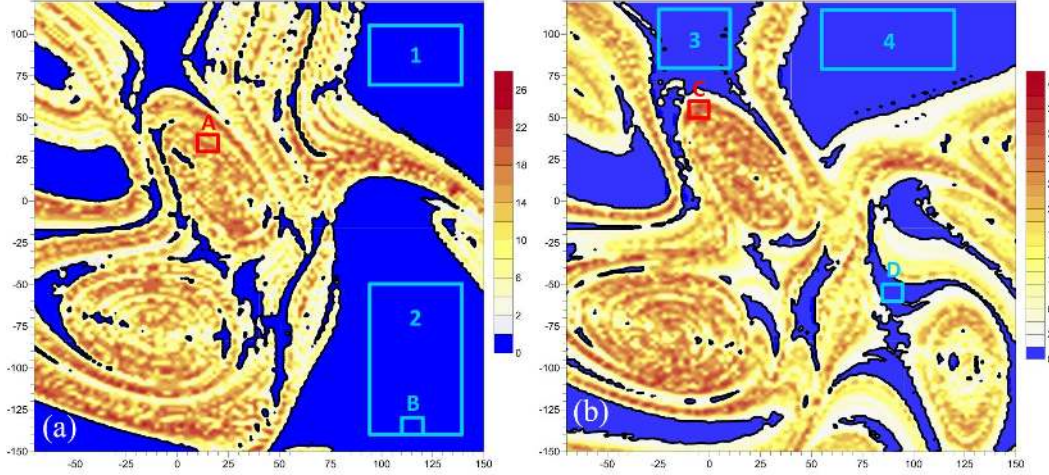


Figure 12. Mean normalized distance Δ for 15 000 trajectories of the double-gyre ocean model: (a) $\gamma = 0.96$, and (b) $\gamma = 1.1$. Reproduced from Pierini et al. (2016), with the permission of the American Meteorological Society.

974

975 The maps of Δ in Fig. 12 reveal large chaotic regions where $\delta_n \gg 1$ on average
 976 (warm colors) but also non-chaotic regions, in which $\sigma \leq 1$ (blue) and thus initially close
 977 trajectories do remain close on average. The rectangular regions in the two panels that
 978 are labeled by letters A and B and by numbers 1–4 correspond to subdomains of the
 979 initial set Γ ; see Pierini et al. (2016, Sec. 5). The numerical evidence in Fig. 12 suggests
 980 that the boundary between the two types of local attractors has fractal properties.

981 In the autonomous context, the coexistence of topologically distinct local attrac-
 982 tors is well known in the climate sciences (Dijkstra, 2013; Dijkstra & Ghil, 2005; Ghil
 983 & Childress, 1987; Simonnet et al., 2005, and references therein). The coexistence of lo-
 984 cal PBAs with chaotic vs. non-chaotic characteristics, within a unique global PBA, as
 985 illustrated by Fig. 12 here, seems to be novel, at least in the geosciences literature.

986 *Climate sensitivity and Wasserstein distance.* Tamás Tél and associates (Bódai, Károlyi,
 987 & Tél, 2011; Bódai & Tél, 2012; Drótos, Bódai, & Tél, 2015) have applied NDS and RDS
 988 concepts and methods to climate modeling, while emphasizing the distinctions and ad-
 989 vantages of the pullback point of view with respect to the much more common one of
 990 ensemble simulations (Houghton et al., 1990; IPCC, 2007, and references therein). The-
 991 oretically speaking, the latter practice merely approximates the PDF that would be ob-
 992 tained by the forward-in-time solution of the Fokker-Planck equation associated with a
 993 given model, a solution that is impossible to obtain for high-dimensional climate mod-
 994 els (Leith, 1974). An important point raised by the work of these authors is that, aside
 995 from the computational difficulties with ensemble size and the PDF approximation, the

996 finite-time averages obtained by the ensemble method do not reflect correctly the changes
 997 in time of the climate system’s statistics in a transient world.

998 Following up on the work of Lucarini and colleagues (e.g., Lucarini et al., 2014) in
 999 applying linear response theory to climate change and on that of Tél and associates above,
 1000 Ghil (2015, 2017) proposed using the Wasserstein or “earth mover’s” distance Δ_W to gen-
 1001 eralize the concept of equilibrium climate sensitivity; $\Delta_W\nu$ is the distance between two
 1002 invariant measures of equal mass, ν_1 and ν_2 , on a metric space, like an n -dimensional
 1003 Euclidean space (Dobrushin, 1970; Kantorovich, 2006; Monge, 1781; Wasserstein, 1969).
 1004 Roughly speaking, and dropping the subscript ‘W’, $\Delta\nu$ represents the total work needed
 1005 to move the “dirt” (i.e., the measure) from a trench you are digging to another one you
 are filling, over the distance between the two trenches.

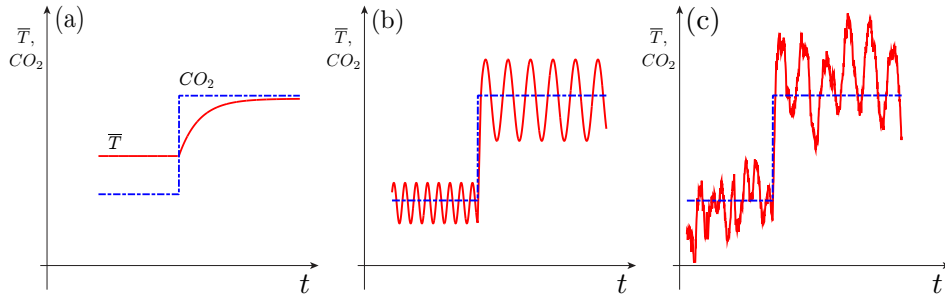


Figure 13. Climate sensitivity (a) for an equilibrium model; (b) for a nonequilibrium, oscillatory model; and (c) for a nonequilibrium, chaotic model, including possibly random perturbations. As a forcing (atmospheric CO₂ concentration, say, dash-dotted line) changes suddenly, global temperature (light solid) undergoes a transition: in panel (a) only the mean temperature changes; in panel (b) the mean adjusts, as it does in panel (a), but the period, amplitude and phase of the oscillation can also decrease, increase or stay the same, while in panel (c) the entire intrinsic variability changes as well. From Ghil (2017), with permission from the American Institute of Mathematical Sciences.

1006

1007 Equilibrium climate sensitivity γ_e is usually defined as $\gamma_e = \partial\bar{T}/\partial\mu$, where \bar{T} is
 1008 the globally and seasonally averaged surface air temperature and μ is a parameter, such
 1009 as the suitably normalized incoming net radiation. It was introduced by Charney et al.
 1010 (1979) and used extensively by the IPCC’s first three assessment reports (e.g., Houghton
 1011 et al., 1990). The associated evolution of $\bar{T}(t)$ for a jump in CO₂ concentration in a scalar
 1012 linear model is illustrated in Fig. 13(a).

1013 This picture is clearly oversimplified, given the complex evolution of temperatures
 1014 in the historical record. Figure 13(b) illustrates $\bar{T}(t)$ in a world in which ENSO would
 1015 be purely periodic, and Fig. 13(c) illustrates schematically the even more realistic case
 1016 of temperature evolution in a deterministically chaotic, turbulent and stochastically per-
 1017 turbed system. For such a system, a better definition of climate sensitivity would be

$$1018 \quad \gamma_{cs} = \frac{\Delta\nu}{\Delta\mu}; \quad (18)$$

1019 here $\{\nu_i = \nu_i(\mu_i) : i = 1, 2\}$ can be the invariant measures on a system’s strange at-
 1020 tractor, in the autonomous case, or its PBA, whether deterministically nonautonomous
 1021 or random, and $\{\nu_i = \nu_i(\mu_i) : i = 1, 2\}$ are the corresponding values of a parameter,
 1022 such as the forcing parameter γ in the Pierini et al. (2016) model in Fig. 12. In this sense,
 1023 one can think of Eq. (18) as a generalization of the linear response in Eq. (12).

1024 The Wasserstein distance $\Delta(\nu_1, \nu_2)$ between two measures ν_1 and ν_2 on a metric
 1025 space X is defined as

$$1026 \quad \Delta(\nu_1, \nu_2) = \inf \mathcal{E}[m(\xi, \eta)]; \quad (19)$$

1027 here m is a metric, the infimum is taken over all possible pairs of random variables ξ and
 1028 η that have the distributions ν_1 and ν_2 , respectively, and \mathcal{E} is the corresponding expect-
 1029 ation. When $X = \mathbb{R}$ is just the real line and m the usual Euclidean metric, let Q_1 and
 1030 Q_2 be the PDFs of the absolutely continuous measures ν_1 and ν_2 . Then

$$1031 \quad \Delta_W(Q_1, Q_2) = \int |G_1(x) - G_2(x)| dx, \quad (20)$$

1032 where G_1 and G_2 are the cumulative distribution functions of the two PDFs Q_1 and Q_2 ,
 1033 respectively (Vallender, 1974).

1034 In general, though, the shape of the two trenches as well as the depth along the trench
 1035 — i.e., both the support of the measure and its density — can differ. Chekroun, Ghil,
 1036 and Neelin (2018) have described a so-called critical transition of this type — essentially
 1037 a generalized tipping point — for a simple ENSO model with a seasonal cycle. In such
 1038 a case, the actual distance calculations require some model reduction to a smaller phase
 1039 space (e.g., Kondrashov, Chekroun, & Ghil, 2015, and references therein) and they have
 1040 to rely on more advanced methodologies in the reduced space (e.g., Villani, 2009).

1041 Robin, Yiou, and Naveau (2017) have argued that the usual quadratic norms used
 1042 to judge distance in the phase space of climate models do not provide an easy interpre-
 1043 tation of the dynamics on the attractor. They calculated the Wasserstein distance be-
 1044 tween the PBAs of the Lorenz (1984) model subject to summer vs. winter forcings and
 1045 showed how this metric does provide a more intuitive discrimination between the two.

1046 Vissio and Lucarini (2018) evaluated the performance of a stochastic parametriza-
 1047 tion by using the Wasserstein distance to measure the difference between the behavior
 1048 of a full fast–slow system and that of a reduced system in which the parametrization had
 1049 replaced the fast subsystem. In their setting, the Lorenz (1984) model governed the slow
 1050 behavior and the Lorenz (1963a) model the fast one. Applying the Wouters and Lucarini
 1051 (2016) parametrization to the fast component, they showed that “Wasserstein distance
 1052 provides a robust tool for assessing the quality of the parametrization, and that mean-
 1053 ingful results can be obtained when considering [a very coarse-grained] representation
 1054 of the phase space.”

1055 **6 The Way Ahead: Prediction and Prediction**

1056 There are two important meanings of “prediction” in the physical sciences. First,
 1057 there is the relatively straightforward meaning of predicting in time. There are many other
 1058 areas of science in which one needs or, at least, wishes to predict: the evolution of an in-
 1059 dividual illness or of an epidemic, that of human population numbers, the outcomes of
 1060 national, ethnic or class conflicts.

1061 In the geosciences, this kind of prediction is clearly of paramount importance: pre-
 1062 dicting routine weather progress, as well as extreme weather events, like a hurricane land-
 1063 fall or a flash flood; earthquakes, volcanic eruptions; global and regional temperatures
 1064 and precipitations many years from now. In all these cases, the usefulness of detailed,
 1065 physics-based models is largely predicated on the understanding of the phenomena and
 1066 processes involved. Thus, good predictions validate the knowledge that entered a spe-
 1067 cific model or class of models, while unsatisfactory ones give a sense of the distance still
 1068 ahead in the field of interest.

1069 Second, there is the sense in which a theoretical model predicts a phenomenon that
 1070 had not been observed at the time of the prediction. The paradigmatic example of this
 1071 kind of prediction is the observational confirmation (Dyson, Eddington, & Davidson, 1920)

1072 of the Einstein (1916) prediction of light rays' bending in the gravity field of the Sun.
 1073 More precisely, the 1919 solar eclipse confirmed that the bending of starlight passing near
 1074 the Sun was about twice as much as predicted by using Newtonian gravity alone. This
 1075 kind of prediction tends to be rare, and rather undervalued in the geosciences.

1076 *Real-time forecasting.* It is clear that numerical weather prediction (NWP) skill has steadily
 1077 improved over the years, since its post-World War II beginnings in the mid-1950s (Thomp-
 1078 son, 1961). Operational forecasts with good local accuracy in surface air temperatures
 1079 for up to 3–5 days are fairly routine, although precipitation forecasts, with their greater
 1080 dependence on more poorly resolved vertical velocities are typically less accurate.

1081 Global forecasts of atmospheric fields on larger scales are much of the time rather
 1082 accurate up to ten days, thanks to improvements in the physical parametrizations of subgrid-
 1083 scale phenomena and the assimilation of massive amounts of remote-sensing data, along
 1084 with the substantial increase of spatial resolution due to huge increases in computing power
 1085 and storage capacity.

1086 It appears that the NWP situation is well in hand (e.g., Kalnay, 2003), although
 1087 there is still room for improvement with respect to theoretical limits of predictability,
 1088 and substantial misses still occur. A better understanding of the mechanisms associated
 1089 with the onset, maintenance and termination of blocking, as discussed in Sec. 3.2 herein
 1090 could help. And so could a better understanding of the interaction between smaller and
 1091 larger scales, as reviewed in Palmer and Williams (2009) and in Sec. 4.2 here.

1092 John von Neumann's role in starting these modern developments in NWP is well
 1093 known, cf. Charney, Fjørtoft, and von Neumann (1950). What is a little less so is his
 1094 longer-range outlook on the three levels of difficulty in understanding and predicting at-
 1095 mospheric and climate phenomena (Von Neumann, 1955): (a) short-term NWP is the
 1096 easiest, since it represents a pure initial-value problem, as formulated by V. Bjerknes (1904)
 1097 and L. F. Richardson (1922); (b) long-term climate prediction is next easiest, since it cor-
 1098 responds to studying the system's asymptotic behavior, i.e., the possible attractors and
 1099 the statistical properties thereof (Dijkstra, 2013; Dijkstra & Ghil, 2005; Ghil & Childress,
 1100 1987); and (c) intermediate-term prediction is hardest, since both the initial data and
 1101 the parameter values are important.

1102 In fact, long-term climate prediction is a bit harder than Von Neumann envisaged
 1103 at the time, because the forcing changes in time, too, as discussed here in Sec. 5.3. Con-
 1104 cerning the intermediate term, matters tend to get more and more difficult as the pre-
 1105 diction horizon is extended further and further, because additional subsystems, with longer
 1106 time scales and additional evolution mechanisms have to be accounted for (Ghil, 2001).

1107 Thus, subseasonal-to-seasonal prediction is receiving increased attention and is mak-
 1108 ing good progress (Robertson & Vitart, 2018, and references therein). Interannual cli-
 1109 mate variability being dominated by ENSO, its prediction concentrates on the coupled
 1110 ocean–atmosphere system in the Tropical Pacific and the teleconnection therefrom to the
 1111 extratropics. ENSO prediction has made great strides, with the emphasis shifting from
 1112 statistical and stochastic-dynamic models in the 1990s to high-end climate models in the
 1113 last decade; compare, for instance, the assessments of real-time ENSO forecast skill in
 1114 Barnston, Glantz, and He (1999) vs. Barnston, Tippett, Heureux, Li, and DeWitt (2012).
 1115 And interdecadal climate prediction is becoming the hardest problem of the climate sci-
 1116 ences, and one of humanity's hardest ones as well.

1117 On the other hand, there are areas of the geosciences in which even the possibil-
 1118 ity of prediction is viewed with suspicion, e.g., earthquake prediction (Geller, Jackson,
 1119 Kagan, & Mulargia, 1997). In spite of the sustained scepticism, the approach outlined
 1120 by Zaliapin et al. (2003b) might deserve some attention. A key obstacle to prediction
 1121 is clearly the relative rarity of large earthquakes and of long and accurate earthquake
 1122 catalogs. One way to extend the record might be to use a model, albeit a more detailed

1123 and complete one than the ternary-tree model mentioned here in Sec. 5.1, to generate
 1124 additional, synthetic catalogs of arbitrary length, which agree in their statistics with ex-
 1125 isting catalogs of real sequences, as far as the latter go. And then proceed from there.

1126 The situation with respect to predicting volcanic eruptions is somewhat less con-
 1127 troversial than for earthquake prediction but still far from being as routine as in NWP.
 1128 Some volcanoes, like Mt. Etna in Sicily, seem to behave fairly periodically — like the
 1129 synthetic earthquakes in Fig. 10(a) of Sec. 5.1 — and their infrasound rumblings have
 1130 been used fairly successfully for automated, real-time forecasts (e.g., Hall, 2018). Oth-
 1131 ers behave more irregularly, like in Fig. 10(b), but still may exhibit characteristic relax-
 1132 ation oscillations of their magma chambers, which could lead to a certain degree of pre-
 1133 dictability (Walwer, Ghil, & Calais, 2019).

1134 *Predicting new phenomena.* The typical way that theory, observation in the field or in
 1135 the laboratory, and numerical simulation interact in the geosciences is: (i) observation
 1136 in the field, be it the atmosphere, ocean or solid Earth, in situ or remotely; (ii) analy-
 1137 sis and description of the observations; and (iii) attempts at explanation of the observed
 1138 phenomena via competing theories and numerical simulations. Moreover, with increas-
 1139 ing computer power and storage capacity, Ockham’s razor is neglected more and more,
 1140 preference being given to high-end models with massive details over the simpler and more
 1141 easily understandable models.

1142 In fact, philosophical objections do exist to the parsimony principle and it, too, is
 1143 not infallible. Still, it is simpler to put the Sun at, or near, the center of the solar sys-
 1144 tem than to keep adding epicycles to the geocentric system (e.g., Kuhn, 1962). The main
 1145 point of applying the principle is that simpler theories cover more observations and should
 1146 therefore be easier to falsify, in the terminology of Karl Popper (2005), i.e., as the dic-
 1147 tionary antonym of “verify” and synonym of “disprove.” Recall that, according to Pop-
 1148 per (2005), to be scientific, a statement has to be falsifiable.

1149 A way of using more systematically parsimonious models in the geosciences is that
 1150 of model hierarchies. Introduced into the climate sciences by Schneider and Dickinson
 1151 (1974), they extend from simple, low-order conceptual models, through intermediate ones
 1152 with one or more space dimensions, all the way to high-end ones that encompass many
 1153 processes and have high 3-D spatial resolution. Rather than hurling epithets of “toy”
 1154 models towards one end of the hierarchy and “overkill” towards the other, it is impor-
 1155 tant to recognize the role of the entire hierarchy in developing ideas, concepts and tools,
 1156 on the one hand, and testing them against observations, on the other.

1157 More specifically, Held (2005) has argued for the need to use simpler models in or-
 1158 der to understand many aspects of the simulations produced by the more detailed ones.
 1159 The author of this paper and his colleagues (e.g., Dijkstra & Ghil, 2005; Ghil, 2001; Ghil
 1160 & Robertson, 2000) have argued that successive bifurcations can play the role of Ari-
 1161 adne’s thread across the rungs of this hierarchy. An illustration of this role in the case
 1162 of the double-gyre problem for the wind-driven ocean circulation was given in Sec. 3.2.

1163 It is important also to remember that when a simpler model and a more detailed
 1164 disagree, it is not always the former that is wrong; i.e., adding details does not always
 1165 add realism. Ghil (2015) reviewed a situation of this type, based on the work of Dijk-
 1166 stra (2007). Inspired by the work of Stommel (1961), a series of papers using THC mod-
 1167 els from simple to intermediate and beyond, had obtained bistability of the MOC, es-
 1168 pecially in situations mimicking the Atlantic Ocean; see Dijkstra and Ghil (2005, Sec. 3)
 1169 and Dijkstra (2005, Ch. 6) for a review.

1170 High-end ocean models used in the third Coupled-Model Intercomparison Program
 1171 (CMIP3) — on which the conclusions of the IPCC’s Fourth Assessment Report (IPCC,
 1172 2007) were based — obtained, however, results that contradicted this bistability. As shown
 1173 by Dijkstra (2007), observations of the evaporation-minus-precipitation fluxes over the

1174 Atlantic, between the southern tips of Greenland and Africa, tend to agree better with
1175 the simpler models than with the CMIP3 ones; and it is this better agreement that sup-
1176 ports the bistability results of the former, simpler models.

1177 One more interesting story of bistability will shed further light on the correct use
1178 of a model hierarchy, as well as on that of nonlinearity in the geosciences in general. En-
1179 ergy balance models (EBMs) are fairly simple climate models that emphasize the role
1180 of incoming and outgoing radiative fluxes in determining the atmosphere's temperature
1181 field, while parameterizing the role of the velocity field in the energy fluxes (Budyko, 1969;
1182 Sellers, 1969). Studies of the number and stability of the stationary solutions of these
1183 models in the early and mid-1970s showed that — in spite of various differences in their
1184 physical formulation and mathematical details (e.g., Ghil & Childress, 1987, Table 10.1)
1185 — they exhibited two stable stationary solutions separated in phase space by an unsta-
1186 ble one (Ghil, 1976; Held & Suarez, 1974; North, Howard, Pollard, & Wielicki, 1979).

1187 The warmer of the two stable fixed points could be identified with something like
1188 the present climate or, more generally, an interglacial one. The colder one corresponds
1189 to an ice-covered planet and was labeled at the time a “deep freeze.” The unstable fixed
1190 point (e.g., Bódai, Lucarini, Lunkeit, & Boschi, 2015) has been explored more recently
1191 by using an edge tracking algorithm (Lucarini & Bódai, 2017).

1192 The presence of the saddle-node bifurcation between the interglacial climate and
1193 the unstable one was promptly confirmed by the results of a simple general circulation
1194 model (Wetherald & Manabe, 1975, Fig. 5). In fact, the authors of the latter study com-
1195 mented that “As stated in the Introduction, it is not, however, reasonable to conclude
1196 that the present results are more reliable than the results from the one-dimensional stud-
1197 ies mentioned above simply because our model treats the effect of transport explicitly
1198 rather than by parameterization. [...] Nevertheless, it seems to be significant that both
1199 the one-dimensional and three-dimensional models yields qualitatively similar results in
1200 many respects.”

1201 In spite of this encouraging confirmation, the fact that a sharp global temperature
1202 drop by tens of degrees Celsius could occur given very small insolation changes was not
1203 taken seriously for quite a while by many climate scientists. The thinking went that the
1204 Sun is a main sequence star and its radiative flux had thus been larger in the past and
1205 not smaller, as required by the models for a deep freeze to set in. More recently, though,
1206 considerable evidence has accumulated for Neoproterozoic (1 000–543 Myr ago) glacia-
1207 tions at low latitudes, which suggest a completely glaciated Earth, labeled “snowball Earth”
1208 (e.g., Hoffman, Kaufman, Halverson, & Schrag, 1998).

1209 Considerable disagreement persists as to whether the Neoproterozoic glaciation was
1210 total or partial, a slushball rather than a snowball; it seems, moreover, to have consisted
1211 of ups and downs in temperatures and ice cover, somewhat like the Quaternary glacia-
1212 tion cycles, only longer and stronger. Even so, the much greater difficulty in getting out
1213 of a glaciated Earth than into it (e.g., Crowley, Hyde, & Peltier, 2001; Pierrehumbert,
1214 2004) is in substantial agreement with early EBM results on the hysteresis cycle of tran-
1215 sition between the high- and low-temperature solution branches (e.g., Ghil, 2001, Fig. 1).
1216 Finally, atmospheric composition and life clearly played a role not accounted for in the
1217 early work on EBMs or Quaternary glaciations (Rothman, Hayes, & Summons, 2003;
1218 Tziperman, Halevy, Johnston, Knoll, & Schrag, 2011, and references therein).

1219 To summarize, simple models can offer predictive insights into phenomena only dis-
1220 covered after such a prediction. And nonlinear concepts and methods — applied con-
1221 sistent across a hierarchy of models — can help disentangle the additional complex-
1222 ities to be explained once the phenomena have been identified in observations and de-
1223 scribed in greater detail.

1224 **7 Coda**

1225 We have visited several lampposts that have shed a little light — over the last cen-
1226 tury, and especially its more recent decades — into the darkness of phenomena in the
1227 geosciences in general, and into Earth’s fluid envelopes and the climate sciences more
1228 specifically. In each case, we have tried to outline the basic ideas and methods that fuel
1229 and focus this light, and to give a few examples of successful application of the theoret-
1230 ical ingredients. It is time to conclude with the hope that more lampposts will spring
1231 up over the coming century, and that the overlaps between pairs and triplets of circles
1232 of light will provide even greater clarity.

1233 **Acknowledgments**

1234 It is a distinct pleasure to express my deepest gratitude to all my graduate students, post-
1235 docs and other co-authors, from and with whom I learned most of what I know about
1236 the material covered, however imperfectly, in this review paper. I am also grateful to An-
1237 nick Pouquet, who solicited the paper in the context of this Special Issue, and gave me
1238 therewith the opportunity to take a view that is both longer and broader than in one’s
1239 usual research papers. Niklas Boers, Valerio Lucarini, James C. McWilliams and Annick
1240 Pouquet carefully read the draft and provided detailed and very helpful input. Niklas
1241 Boers, Shi Jiang and Fei-Fei Jin kindly provided Figures 5, 1 and 9, respectively; Fig-
1242 ures 1 and 9 are based on the numerical results reported in Jiang et al. (1995) and Jin
1243 et al. (1994), respectively. This review relies on knowledge accumulated over four decades
1244 of support by the European Union’s New and Emerging Science and Technology (NEST)
1245 Programme; the French Agence Nationale de la Recherche and Centre National de la Recherche
1246 Scientifique; and the U.S. Department of Energy, National Air and Space Administra-
1247 tion, National Science Foundation, and Office of Naval Research’s Multidisciplinary Uni-
1248 versity Research Initiative (MURI).

1249 **A Acronyms****Table A.1.** Acronyms^a

Acronym	Meaning
<i>BDE</i>	Boolean delay equation
<i>CA</i>	Cellular automaton (sing.) or automata (pl.)
<i>CNs</i>	Complex networks
<i>DDE</i>	Delay differential equation
<i>ER</i>	Erdős–Rényi (network)
<i>FDE</i>	Functional differential equation
<i>GFD</i>	Geophysical fluid dynamics
<i>IPCC</i>	International Panel on Climate Change
<i>NAO</i>	North Atlantic Oscillation
<i>NDS</i>	Nonautonomous dynamical system
<i>MHD</i>	Magnetohydrodynamics
<i>OΔE</i>	Ordinary difference equation
<i>ODE</i>	Ordinary differential equation
<i>P</i> -map	Poincaré map
<i>PΔE</i>	Partial difference equation
<i>PDE</i>	Partial differential equation
<i>PSA</i>	Pacific South American (pattern)
<i>RA</i>	Random acyclic (network)
<i>RDS</i>	Random dynamical system

^aList of acronyms.

1250 **References**

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