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# A Certain Class of *t*-Intuitionistic Fuzzy Subgroups

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**ABSTRACT** In this study, the $t$-intuitionistic fuzzy normalizer and centralizer of $t$ intuitionistic fuzzy subgroup are proposed. The $t$-intuitionistic fuzzy centralizer is normal subgroup of $t$-intuitionistic fuzzy normalizer and investigate various algebraic properties of this phenomena. We also introduce the concept of $t$-intuitionistic fuzzy Abelian and cyclic subgroups and prove that every $t$-intuitionistic fuzzy subgroup of Abelian (cyclic) group is $t$-intuitionistic fuzzy Abelian (cyclic) subgroup. We show that the image and pre-image of $t$-intuitionistic fuzzy Abelian (cyclic) subgroup are $t$-intuitionistic fuzzy Abelian (cyclic) subgroup under group homomorphism.

**INDEX TERMS** $t$-intuitionistic fuzzy set, $t$-intuitionistic fuzzy subgroup, $t$-intuitionistic fuzzy Abelian subgroup, $t$-intuitionistic fuzzy cyclic subgroup.

## I. INTRODUCTION

Camille Jordan named Abelian group due to the pioneer work of Norwegian mathematician Niels Henrik Abel, because Abel established the commutativity of the groups interprets that the roots of the polynomial could be evaluated by using radicals. The Abelian group theory is widely used in present mathematical theories. Therefore, the duality theory about qualities of finite Abelian groups has expanded to the duality theory of locally compact Abelian groups. The homological algebra made it easier to address a whole series of problems in Abelian groups such as classifying the set of all generalizations of one group to another group. The module theory is narrowly connected with Abelian groups such as module over the ring of integers. Furthermore, many results in the Abelian group theory can be used to module case on principal ideal ring. There are numerous studies and availability of multiple objects in which Abelian groups have now extremely improved. The Abelian group has become a standard and powerful tool to study universal genetic code and codon sequences behavior. The more development about Abelian group may be viewed in [1].

Ambiguity is a pervasive part of human life. This world is not pertaining to accurate calculations or hypothesis. This error in assessment is really problematic for human intelligence. A variety of mathematical notions have been formulated as handy approaches to tackle this difficulty wherein fuzzy sets are included. The fuzzy logic has been formed on a system of group having ambiguous knowledge. Owing to elastic quality of intuitionistic fuzzy sets to handle unreliability, this event is regarded wonderfully great for humanistic logic underlying inaccurate reality and limitless knowledge. This doctrine is doubtlessly a core point of classical fuzzy sets as it provides further opportunity to put forth incorrect information, leading to more suitable solution for numerous challenges. These certain sets developed favorable models in situation wherein we are to deal with highly limited choices like yes or no. The other significant property of this knowledge as it empowers man to analyze negative and positive aspects of inaccurate concepts. The intuitionistic fuzzy set theory is used in modern mathematics at broad spectrum as it infers with fuzzy set. This specific theory is greatly at work in galaxy of disciplines covering venerability of gas pipeline, medical diagnostic approaches, neural networks models and travel time history. The most striking feature for discoursing the intuitionistic fuzzy theory is that it incorporates vagueness

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and uncertainty of physical difficulties better than ancient fuzzy set does like in the field of psychological examination, decision making and financial services.

The fuzzy sets and their elementary consequences were introduced by Zadeh [2] in 1965. Rosenfeld [3] commenced the fuzzification algebraic structure in 1971. He initiated the concept of fuzzy subgroups and explored a variety of key features relating to concept. Liu [4] developed a link between ring theory and fuzzy sets and innovated the theory of fuzzy subring. Atanassov [5] presented the theory of intuitionistic fuzzy sets and thrived fundamental algebraic characteristics of this situation. This concept became more efficient in scientific field because it deals with degree of membership and non-membership in unit interval. This doctrine is doubtlessly a core point of conventional fuzzy sets as it provides further opportunity to put forth incorrect information to cope with more suitable solution for numerous challenges. The other significant property of this knowledge as it empowers man to analyze negative and positive aspects of inaccurate concepts. Biswas [6] started the conception of intuitionistic fuzzy subgroups. De Gang and Jun [7] described fuzzy factor ring in 1998. A new concept of complex fuzzy sets was presented by Ramot *et al.* [8]. The extension of fuzzy sets to complex fuzzy sets is comparable to the extension of real numbers to complex numbers. Banerjee and Basnet [9] studied intuitionistic fuzzy subrings in 2003. Yan and Wang [10] introduced the notion of intuitionistic $I$-fuzzy topological spaces. He developed the connection between the category of intuitionistic $I$-fuzzy quasi-coincident neighborhood spaces and the category of intuitionistic $I$-fuzzy topological spaces. Gunduz and Bayramov [11] described a connection between intuitionistic fuzzy soft set and module theory and commenced the idea of intuitionistic fuzzy soft module. Yetkin and Olgun [12] studied the direct product of fuzzy subgroup and fuzzy subring in 2011. Isaac and John [13] discussed the fundamental algebraic properties of intuitionistic fuzzy module in 2011. They also developed the intuitionistic fuzzy quotient module and investigated intuitionistic fuzzy module under module homomorphism. The fundamental algebraic attribution of intuitionistic fuzzy subgroups and $t$-intuitionistic fuzzy quotient module were inquired in [14], [15]. Singh and Srivaslava [16] analyzed the separation axioms in intuitionistic fuzzy topological spaces. The idea of intuitionistic fuzzy module over intuitionistic fuzzy ring was presented in [17]. Sharma [18] innovated the concept of t-intuitionistic fuzzy subgroups in 2012. Alkouri and Salleh [19] commenced the idea of complex intuitionistic fuzzy subsets and enlarge the basic characteristic of this phenomena. They also initiated the concept of complex intuitionist fuzzy relation and developed fundamental operation of complex intuitionistic fuzzy sets in [20], [21]. Ersoy and Davvaz [22] explored the structure of intuitionistic fuzzy set in $\Gamma$-semihyper-ring. Azam *et al.* [23] defined some properties of anti fuzzy ideal in 2013. Broumi *et al.* [24] developed some new results of intuitionistic fuzzy soft sets. Davvaz and Sadrabadi [25] studied the sequences of join spaces and intuitionistic fuzzy set connected with the direct product of two hyper-groupoids with particular characterization. In 2014, Mandal and Ranadive [26] discussed rough semiprime intuitionistic fuzzy ideal and rough intuitionistic fuzzy ideal of intuitionistic fuzzy subring. The idea of intuitionistic fuzzy graph with categorical properties was investigated in [27]. Intuitionistic fuzzy set theory plays a significant role in vast range of medical fields. It works wonder in diagnosing and selecting suitable treatments in medicine. Beg and Rashid [28] extended the study of intuitionistic fuzzy theory to solve medical diagnosis decision making problems. Bakhadach *et al.* [29] explored some features of intuitionistic fuzzy prime ideal in 2016. Anandh and Giri [30] examined the notion of intuitionistic fuzzy subfield with respect to $(T, S)$ norm in 2017. Anandh and Giri [30] commenced a new concept of complex intuitionistic fuzzy subrings and depicted the elementary properties of this fact. Yamin and Sharma [32] debated the intuitionistic fuzzy ring with operators in 2018. Lakshmi and Priyaa [33] discussed a novel approach on ambiguity using intuitionistic fuzzy based rule generation system that addresses the problem of deduction by including the degree of belief, disbelief and the hesitation margin in wireless sensor networks. Solairaju and Mahalakshmi [34] investigated some characterization of hesitant intuitionistic fuzzy soft set and presented the idea hesitant intuitionistic fuzzy soft group. Latif *et al.* [35] made remarkable effort to extend the notion of $t$-intuitionistic fuzzy sets in group theory and proved fundamental theorem of $t$-intuitionistic fuzzy isomorphism. Many interesting results about complex intuitionistic fuzzy graph and cellular network provider companies were presented in [36]. Aloaiyan *et al.* [37] described a novel framework of $t$-intuitionistic fuzzification of Lagrange's theorem. Alcantud *et al.* [38] commenced the method for aggregating infinite sequences of intuitionistic fuzzy sets. They also solved the decision making issues where data come in the shape of intuitionistic fuzzy set along with an indefinitely long number of periods. Liu *et al.* [39] investigated the existing transformation technique between fuzzy set and soft set. Pan *et al.* [40] presented a novel singularity-free fixed-time fuzzy control technique for ambiguous constrained n-link robotic system. Liang *et al.* [41] studied the stability analysis, controller design and design of the distributed reduced-order dynamic gain observer.

Keeping the advantage of $t$-intuitionistic fuzzy sets and taking the importance of group theory, this article presents the theory of $t$-intuitionistic fuzzy Abelian (cyclic) subgroups of group. We propose t-intuitionistic fuzzy normalizer (centralizer) of $t$-intuitionistic fuzzy subgroup analog to normalizer (centralizer) of classical group. The image (pre-image) of $t$-intuitionistic fuzzy Abelian (cyclic) subgroup is also $t$-intuitionistic fuzzy Abelian (cyclic) subgroup under natural group homomorphism.

An outline of this manuscript is shaped as: In section 2, We define some basic concept of $t$-intuitionistic fuzzy sets and $t$-intuitionistic fuzzy subgroups which are compulsory for our further discussion. In section 3, we introduce the concept $t$-intuitionistic fuzzy normalizer and centralizer of $t$-intuitionistic fuzzy subgroup and also prove that $t$-intuitionistic fuzzy centralizer is normal subgroup of $t$-intuitionistic fuzzy normalizer. We also introduce the idea of $t$-intuitionistic fuzzy Abelian subgroups and $t$-intuitionistic fuzzy cyclic subgroups and prove that every $t$-intuitionistic fuzzy subgroup of Abelian (cyclic) group is $t$-intuitionistic fuzzy Abelian (cyclic) subgroup. We establish that direct product of two $t$-intuitionistic fuzzy Abelian subgroups is $t$-intuitionistic fuzzy Abelian subgroup. In section 4, We show that the image and inverse image of $t$-intuitionistic fuzzy Abelian (cyclic) subgroup are $t$-intuitionistic fuzzy Abelian (cyclic) subgroup under group homomorphism.

## II. PROPERTIES OF $t$-INTUITIONISTIC FUZZY SUBGROUPS

In this section, we define some basic definition and results of $t$-intuitionistic fuzzy subgroups, which play a key role our further discussion.

*Definition 2.1 [5]:* Let $P$ be a universe of discourse. Then the intuitionistic fuzzy set (IFS) $A$ of $P$ is an object of the form $A = \{< w, \mu_A(w), \nu_A(w) >: w \in P\}$, where $\mu_A : P \to [0, 1]$ and $\nu_A : P \to [0, 1]$ define the degree of belief and degree of disbelief of the element $w \in P$, respectively, these functions must be satisfied the condition $0 \leqslant \mu_A(w) + \nu_A(w) \leqslant 1$.

*Definition 2.2 [18]:* Let $A$ be a IFS of a universe of discourse $P$ and $t \in [0, 1]$. Then $t$-intuitionistic fuzzy set ($t$-IFS) $A^t$ of $P$ is an object triplet of the form $A^t = \{< x, \mu_{A^t}(w), \nu_{A^t}(w) >: w \in P\}$, where $\mu_{A^t} : P \to [0, 1]$ and $\nu_{A^t} : P \to [0, 1]$ define the degree of membership and degree of non-membership of the element $w \in P$, respectively, these functions must be satisfied the condition $0 \leqslant \mu_{A^t}(w) + \nu_{A^t}(w) \leqslant 1$.

*Definition 2.3 [18]:* Let $G$ be the group, for any $w, z \in G$. A $t$-IFS $A^t = (\mu_{A^t}, \nu_{A^t})$ of $G$ is called a $t$-intuitionistic fuzzy subgroup ($t$-IFSG) of $G$ if

1) $\mu_{A^t}(zw) \geq \min\{\mu_{A^t}(w), \mu_{A^t}(z)\}$
2) $\mu_{A^t}(w^{-1}) = \mu_{A^t}(w)$
3) $\nu_{A^t}(zw) \leq \max\{\nu_{A^t}(w), \nu_{A^t}(z)\}$
4) $\nu_{A^t}(w^{-1}) = \nu_{A^t}(w)$

or equivalently, A $t$-IFS $A^t$ of $G$ is said to be $t$-IFSG of $G$ if $\mu_{A^t}(wz^{-1}) \geq \min\{\mu_{A^t}(w), \mu_{A^t}(z)\}$ and $\nu_{A^t}(wz^{-1}) \leq \max\{\nu_{A^t}(w), \nu_{A^t}(z)\}$.

*Definition 2.4 [18]:* A $t$-IFSG $A = (\mu_{A^t}, \nu_{A^t})$ of a group $G$ is said to be $t$-intuitionistic fuzzy normal subgroup ($t$-IFNSG) of $G$ if

1) $\mu_{A^t}(zw) = \mu_{A^t}(wz)$
2) $\nu_{A^t}(zw) = \nu_{A^t}(wz)$, for all $w, z \in G$

or equivalently, $A^t$ is a $t$-IFNSG of a group $G$ is *normal* if and only if $\mu_{A^t}(z^{-1}wz) = \mu_{A^t}(w)$ and $\nu_{A^t}(z^{-1}wz) = \nu_{A^t}(w)$, for all $w, z \in G$

*Definition 2.5:* Let $A^t$ be a $t$-IFS of a universal set $P$. Then $(\alpha, \beta)$-cut of $A^t$ is a crisp subset $C_{\alpha,\beta}(A^t)$ of the $t$-IFS $A^t$ is given by $C_{\alpha,\beta}(A^t) = \{w : w \in P$ such that $\mu_{A^t}(w) \geq \alpha$, $\nu_{A^t}(w) \leq \beta\}$, where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$.

*Theorem 2.1:* Let $A^t$ be a $t$-IFS of a group $G$, then $A^t$ is a $t$-IFSG of $G$ if and only if $C_{\alpha,\beta}(A^t)$ is a subgroup of group $G$, for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$.

*Proof:* Given that $A^t$ is a $t$-IFSG of $G$. Clearly $C_{\alpha,\beta}(A^t) \neq \phi$. Let $w, z \in C_{\alpha,\beta}(A^t)$. Then $\mu_{A^t}(w), \mu_{A^t}(z) \geq \alpha$ and $\nu_{A^t}(w), \nu_{A^t}(z) \leq \beta$.

This implies that $\mu_{A^t}(wz) \geq \min\{\mu_{A^t}(w), \mu_{A^t}(z)\} \geq \alpha$ and $\nu_{A^t}(wz) \leq \max\{\nu_{A^t}(w), \nu_{A^t}(z)\} \leq \beta$.

Thus $wz \in C_{\alpha,\beta}(A^t)$. Also, $\mu_{A^t}(w^{-1}) = \mu_{A^t}(w) \geq \alpha$ and $\nu_{A^t}(w^{-1}) = \nu_{A^t}(w) \leq \beta$. Therefore $w^{-1} \in G$. Hence $C_{\alpha,\beta}(A^t)$ is a subgroup of $G$. ■

*Definition 2.6:* Let $A = (\mu_{A^t}, \nu_{A^t})$ and $B = (\mu_{B^t}, \nu_{B^t})$ be two $t$-IFS's of $P$ and $Q$, respectively. Then the *Cartesian product* of $A^t$ and $B^t$ is denoted by $A^t \times B^t$ and is defined as: $A^t \times B^t = \{\langle (w, z), \mu_{A^t \times B^t}(w, z), \nu_{A^t \times B^t}(w, z)\rangle : w \in P$ and $z \in Q\}$. where, $\mu_{A^t \times B^t}(w, z) = \min\{\mu_{A^t}(w), \mu_{B^t}(z)\}$ and $\nu_{A^t \times B^t}(w, z) = \max\{\nu_{A^t}(w), \nu_{B^t}(z)\}$.

*Proposition 2.1:* Let $A^t$ and $B^t$ be two $t$-IFSG of $G_1$ and $G_2$, respectively, then $C_{\alpha,\beta}(A^t \times B^t) = C_{\alpha,\beta}(A^t) \times C_{\alpha,\beta}(B^t)$ for all $\alpha, \beta \in [0, 1]$ with $0 \leq \alpha + \beta \leq 1$.

*Proof:* Let $(w, z) \in C_{\alpha,\beta}(A^t \times B^t)$ be any element

$$
\begin{aligned}
&\Leftrightarrow \mu_{A^t \times B^t}(w, z) \geq \alpha \text{ and } \nu_{A^t \times B^t}(w, z) \leq \beta \\
&\Leftrightarrow \min\{\mu_{A^t}(w), \mu_{B^t}(z)\} \geq \alpha \\
&\qquad \text{and } \max\{\nu_{A^t}(w), \nu_{B^t}(z)\} \leq \beta \\
&\Leftrightarrow \mu_{A^t}(w) \geq \alpha, \quad \mu_{B^t}(z) \geq \alpha \\
&\qquad \text{and } \nu_{A^t}(w) \leq \beta, \quad \nu_{B^t}(z) \leq \beta \\
&\Leftrightarrow \mu_{A^t}(w) \geq \alpha, \quad \nu_{B^t}(z) \leq \beta \\
&\qquad \text{and } \mu_{B^t}(z) \geq \alpha, \quad \nu_{A^t}(w) \leq \beta \\
&\Leftrightarrow w \in C_{\alpha,\beta}(A^t) \text{ and } z \in C_{\alpha,\beta}(B^t) \\
&\Leftrightarrow (w, z) \in C_{\alpha,\beta}(A^t \times B^t).
\end{aligned}
$$

Hence $C_{\alpha,\beta}(A^t \times B^t) = C_{\alpha,\beta}(A^t) \times C_{\alpha,\beta}(B^t)$. ■

*Theorem 2.2:* Let $A^t$ and $B^t$ be a $t$-IFSG ($t$-IFNSG) of groups $G_1$ and $G_2$, respectively. Then $A^t \times B^t$ is also a $t$-IFSG ($t$-IFNSG) of group $G_1 \times G_2$.

## III. FUNDAMENTAL PROPERTIES OF $t$-INTUITIONISTIC FUZZY ABELIAN SUBGROUPS

We start this section with $t$-intuitionistic fuzzy normalizer and $t$-intuitionistic fuzzy centralizer and show that $t$-intuitionistic fuzzy normalizer (centralizer) is subgroup of group. We also prove that this newly defined centralizer is normal subgroup of $t$-intuitionistic fuzzy normalizer and investigate some fundamental algebraic properties of these situations. We also introduce the notion of $t$-intuitionistic fuzzy Abelian (cyclic) group and prove that every $t$-intuitionistic fuzzy subgroup is $t$-intuitionistic fuzzy Abelian group analog to classical group theory and also discuss their properties.

*Definition 3.1:* Let $G$ be a group and $A^t$ be a $t$-IFSG of $G$. Then the $t$-intuitionistic fuzzy normalizer of $A^t$ is described by $N(A^t) = \{a \in G : \mu_{A^t}(a^{-1}wa) = \mu_{A^t}(w)$ and $\nu_{A^t}(a^{-1}wa) = \nu_{A^t}(w)$, for all $w \in G\}$.

The following results explain the essential attribute about $t$-intuitionistic fuzzy normalizer of $t$-IFSG of $G$.

*Theorem 3.1:* Let $A^t$ be a $t$-IFSG of a group $G$. Then

1) $N(A^t)$ is a subgroup of $G$.
2) $A^t$ is a $t$-IFNSG of $G$ if and only if $N(A^t) = G$.
3) $A^t$ is a $t$-IFNSG of group $N(A^t)$.

*Proof:*

1) Let $a, b \in N(A^t)$ be any two elements. Then we have $\mu_{A^t}(a^{-1}wa) = \mu_{A^t}(w)$ and $\nu_{A^t}(a^{-1}wa) = \nu_{A^t}(w)$, for all $w \in G$, (i)
$\mu_{A^t}(b^{-1}zb) = \mu_{A^t}(z)$ and $\nu_{A^t}(b^{-1}zb) = \nu_{A^t}(z)$, for all $z \in G$, (ii)
Put $z = a^{-1}wa$ in (ii) and using (i), we get
$$\mu_{A^t}(b^{-1}a^{-1}wab) = \mu_{A^t}(a^{-1}wa) = \mu_{A^t}(w)$$
$$\text{and } \nu_{A^t}(b^{-1}a^{-1}wab) = \nu_{A^t}(a^{-1}wa) = \nu_{A^t}(w).$$
Implies that $\mu_{A^t}((ab)^{-1}w(ab) = \mu_{A^t}(w)$
and $\nu_{A^t}((ab)^{-1}w(ab)) = \nu_{A^t}(w)$.
Thus $ab \in N(A^t)$.
Further, we change $w$ into $w^{-1}$ in (i), we get
$$\mu_{A^t}(a^{-1}w^{-1}a) = \mu_{A^t}(w^{-1}) = \mu_{A^t}(w)$$
$$\text{and } \nu_{A^t}(a^{-1}w^{-1}a) = \nu_{A^t}(w^{-1}) = \nu_{A^t}(w)$$
$$\Rightarrow \mu_{A^t}((awa^{-1})^{-1}) = \mu_{A^t}(awa^{-1}) = \mu_{A^t}(w)$$
$$\text{and } \nu_{A^t}((awa^{-1})^{-1}) = \nu_{A^t}(awa^{-1}) = \nu_{A^t}(w),$$
$$\Rightarrow \mu_{A^t}((a^{-1})^{-1}w(a^{-1})) = \mu_{A^t}(w)$$
$$\text{and } \nu_{A^t}((a^{-1})^{-1}w(a^{-1})) = \nu_{A^t}(w). \Rightarrow a^{-1} \in N(A^t).$$
Hence $N(A^t)$ is subgroup of $G$.
2) Suppose that $N(A^t) = G$, then $\mu_{A^t}(a^{-1}wa) = \mu_{A^t}(w)$ and $\nu_{A^t}(a^{-1}wa) = \nu_{A^t}(w)$, for all $w, a \in G$. Hence $A^t$ is a $t$-IFNSG of group $G$.
Conversely, assume that $A^t$ is a $t$-IFNSG of group $G$. Then $\mu_{A^t}(a^{-1}wa) = \mu_{A^t}(w)$ and $\nu_{A^t}(a^{-1}wa) = \nu_{A^t}(w)$, for all $w, a \in G$. Implies that, the set $\{a \in G : \mu_{A^t}(a^{-1}wa) = \mu_{A^t}(w)$ and $\nu_{A^t}(a^{-1}wa) = \nu_{A^t}(w)$, for all $w \in G\} = G$. Consequently, $N(A^t) = G$.
3) Let $a, b \in N(A^t)$ be any elements. Then $\mu_{A^t}(a^{-1}wa) = \mu_{A^t}(w)$ and $\nu_{A^t}(a^{-1}wa) = \nu_{A^t}(w)$, for all $w \in G$. Putting $w = ab$, we get $\mu_{A^t}(ab) = \mu_{A^t}(a^{-1}aba) = \mu_{A^t}(ba)$ and $\nu_{A^t}(ab) = \nu_{A^t}(a^{-1}aba) = \nu_{A^t}(ba)$. Hence $A^t$ is a $t$-IFNSG of $N(A^t)$.

*Definition 3.2:* Let $A^t$ be a $t$-IFSG of group $G$. Then the set $C(A^t) = \{a \in G : \mu_{A^t}([a, w])\} = \mu_{A^t}(e)$ and $\nu_{A^t}([a, w]) = \nu_{A^t}(e)$, for all $w \in G\}$ is called *$t$-intuitionistic fuzzy centralizer* of $A^t$ in $G$, where $[a, w]$ is the commutator of the two elements $a$ and $w$ in $G$, i.e., $[a, w] = a^{-1}w^{-1}aw$.

The following result illustrates that $t$-intuitionistic fuzzy centralizer is subgroup of group $G$ and also define the relation between $t$-intuitionistic fuzzy centralizer and $t$-intuitionistic fuzzy normalizer.

*Theorem 3.2:* Let $A^t$ be a $t$-IFSG of a group $G$. Then

1) $C(A^t)$ is a subgroup of $G$.
2) $C(A^t)$ is a normal subgroup of $N(A^t)$.

*Proof:*

1) Note that, $C(A^t) \neq \phi$, because $e \in C(A^t)$. Let $a, b \in C(A^t)$. Then $\mu_{A^t}([a, w]) = \mu_{A^t}(e)$ and $\nu_{A^t}([a, w]) = \nu_{A^t}(e)$ and $\mu_{A^t}([b, z]) = \mu_{A^t}(e)$ and $\nu_{A^t}([b, z]) = \nu_{A^t}(e)$, hold for all $w, z \in G$.
$$\text{Implies that } \mu_{A^t}(a^{-1}w^{-1}aw) = \mu_{A^t}(e)$$
$$\text{and } \nu_{A^t}(a^{-1}w^{-1}aw) = \nu_{A^t}(e), \quad (1)$$
$$\mu_{A^t}(b^{-1}z^{-1}bz) = \mu_{A^t}(e)$$
$$\text{and } \nu_{A^t}(b^{-1}z^{-1}bz) = \nu_{A^t}(e), \quad (2)$$
Putting $z = a^{-1}pa$ in (2), we get
$$\mu_{A^t}(b^{-1}a^{-1}p^{-1}aba^{-1}pa) = \mu_{A^t}(e)$$
$$\text{and } \nu_{A^t}(b^{-1}a^{-1}p^{-1}aba^{-1}pa) = \nu_{A^t}(e).$$
$$\Rightarrow \mu_{A^t}(((ab)^{-1}p^{-1}(ab)p)(p^{-1}a^{-1}pa)) = \mu_{A^t}(e)$$
$$\text{and } \nu_{A^t}(((ab)^{-1}p^{-1}(ab)p)(p^{-1}a^{-1}pa)) = \nu_{A^t}(e).$$
$$\Rightarrow \mu_{A^t}((ab)^{-1}p^{-1}(ab)p) = \mu_{A^t}(e)$$
$$\text{and } \nu_{A^t}((ab)^{-1}p^{-1}(ab)p) = \nu_{A^t}(e).$$
Thus, $ab \in C(A^t)$.
Also, from equation (1), we have $\mu_{A^t}(e) = \mu_{A^t}(a^{-1}w^{-1}aw) = \mu_{A^t}((a^{-1}w^{-1}aw))^{-1} = \mu_{A^t}((w^{-1}a^{-1}wa))$.
Implies that $\mu_{A^t}((w^{-1}a^{-1}wa)) = \mu_{A^t}(e)$.
Similarly, we have $\nu_{A^t}((w^{-1}a^{-1}wa)) = \nu_{A^t}(e)$.
Putting $w = ta^{-1}$, we get $\mu_{A^t}((at^{-1}a^{-1}ta^{-1}a)) = \mu_{A^t}((at^{-1}a^{-1}t)) = \mu_{A^t}(e)$ and $\nu_{A^t}((at^{-1}a^{-1}ta^{-1}a)) = \nu_{A^t}((at^{-1}a^{-1}t)) = \nu_{A^t}(e)$. Therefore, $a^{-1} \in C(A^t)$. Hence $C(A^t)$ is subgroup of $G$.
2) Let $a \in C(A^t)$ and $b \in N(A^t)$ be any elements. We show that $b^{-1}ab \in C(A^t)$.
Consider
$$\mu_{A^t}(a^{-1}w^{-1}aw) = \mu_{A^t}(e)$$
and
$$\nu_{A^t}(a^{-1}w^{-1}aw) = \nu_{A^t}(e) \quad \forall\, w \in G, \quad (3)$$
$$\mu_{A^t}((b^{-1}zb)) = \mu_{A^t}(z)$$
and
$$\nu_{A^t}((b^{-1}zb)) = \nu_{A^t}(z) \quad \forall\, z \in G, \quad (4)$$
Put $z = a^{-1}w^{-1}aw$ in (4) and using (3), we get
$$\mu_{A^t}((b^{-1}a^{-1}w^{-1}awb)) = \mu_{A^t}(a^{-1}w^{-1}aw) = \mu_{A^t}(e)$$
$$\text{and } \nu_{A^t}((b^{-1}a^{-1}w^{-1}awb)) = \nu_{A^t}(a^{-1}w^{-1}aw) = \nu_{A^t}(e)$$
Again putting $w = bpb^{-1}$ above, we get
$$\mu_{A^t}((b^{-1}a^{-1}bp^{-1}b^{-1}abpb^{-1}b)) = \mu_{A^t}(e)$$
$$\text{and } \nu_{A^t}((b^{-1}a^{-1}bp^{-1}b^{-1}abpb^{-1}b)) = \nu_{A^t}(e).$$
$$\text{Implies that } \mu_{A^t}((b^{-1}a^{-1}bp^{-1}b^{-1}abp) = \mu_{A^t}(e)$$
$$\text{and } \nu_{A^t}(b^{-1}a^{-1}bp^{-1}b^{-1}abp) = \nu_{A^t}(e).$$
$$\text{Implies that } \mu_{A^t}((b^{-1}ab)^{-1}p^{-1}(b^{-1}ab)p) = \mu_{A^t}(e)$$
$$\text{and} \nu_{A^t}((b^{-1}ab)^{-1}p^{-1}(b^{-1}ab)p) = \nu_{A^t}(e).$$

Implies that $b^{-1}ab \in C(A^t)$. Hence $C(A^t)$ is a normal subgroup of $N(A^t)$.

*Proposition 3.1:* Let $A^t$ be a $t$-IFNSG of group $G$ and $M = \{a \in G : \mu_{A^t}(a) = \mu_{A^t}(e) \text{ and } \nu_{A^t}(a) = \nu_{A^t}(e)\}$. Then $M \subseteq C(A^t)$.

*Proof:* Given that $A^t$ is a $t$-IFNSG of group $G$. Therefore, $\mu_{A^t}(z^{-1}wz) = \mu_{A^t}(w)$ and $\nu_{A^t}(z^{-1}wz) = \nu_{A^t}(w)$, for all $w, z \in G$. Let $a \in M$. Then $\mu_{A^t}(a) = \mu_{A^t}(e)$ and $\nu_{A^t}(a) = \nu_{A^t}(e)$.

Consider

$$\begin{aligned}\mu_{A^t}([a,w]) &= \mu_{A^t}(a^{-1}w^{-1}aw)\\ &\geq \min\{\mu_{A^t}(a^{-1}), \mu_{A^t}(w^{-1}aw)\}\\ &= \min\{\mu_{A^t}(a), \mu_{A^t}(a)\}\\ &= \min\{\mu_{A^t}(e)\mu_{A^t}(e)\}\\ &= \mu_{A^t}(e).\end{aligned}$$

Thus $\mu_{A^t}([a,w]) = \mu_{A^t}(e)$.

Moreover,

$$\begin{aligned}\nu_{A^t}([a,w]) &= \nu_{A^t}(a^{-1}w^{-1}aw)\\ &\leq \max\{\nu_{A^t}(a^{-1}), \nu_{A^t}(w^{-1}aw)\}\\ &= \max\{\nu_{A^t}(a), \nu_{A^t}(a)\\ &= \max\{\nu_{A^t}(e)\ \nu_{A^t}(e)\}\\ &= \nu_{A^t}(e).\end{aligned}$$

Thus $\nu_{A^t}([a,w]) = \nu_{A^t}(e)$. Therefore, $a \in C(A^t)$. Hence $M \subseteq C(A^t)$.

*Definition 3.3:* Let $A^t$ be a $t$-IFSG of group $G$. Then $A^t$ is called a $t$-intuitionisic fuzzy Abelian subgroup ($t$-IFASG) of $G$ if $C_{\alpha,\beta}(A^t)$ is an Abelian subgroup of $G$, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1$.

*Remark 3.1 [1]:* Every subgroup of Abelian group is Abelian.

Next, we show that every $t$-IFSG of Abelian group $G$ is $t$-IFASG of $G$. In this direction we prove the following result.

*Theorem 3.3:* If $G$ is an Abelian group, then every $t$-IFSG of $G$ is $t$-IFASG of $G$.

*Proof:* Given that $G$ is Abelian group. Then $xy = yx$, hold for all $x, y \in G$. Since $A^t$ is a $t$-IFSG of group $G$. By Theorem (2.1), we have $C_{\alpha,\beta}(A^t)$ is subgroup of $G$. In the view of Remark (3.1), we know that $C_{\alpha,\beta}(A^t)$ is Abelian subgroup of $G$. By using definition of $t$-IFASG, we conclude that $A^t$ is $t$-IFASG of group $G$.

The following example leads us to note that the converse of Theorem (3.3) may not be true.

*Example 3.1:* Consider $G = S_3 = \{i, (12), (13), (23), (123), (132)\}$ be the symmetric group. Consider the $t$-IFS $A^t$ of $G$ defined by

$$\mu_{(A^t)}(w) = \begin{cases} 1, & \text{if } w = i,\\ 0, & \text{if } w^2 = i,\\ 0.5, & \text{if } w^3 = i\end{cases}$$

and

$$\nu_{(A^t)}(w) = \begin{cases} 0, & \text{if } w = i,\\ 0.3, & \text{if } w^2 = i,\\ 0.4, & \text{if } w^3 = i.\end{cases}$$

where $w \in G$ and $i$ is the identity element of $G$.

Clearly, $A^t$ is $t$-IFSG of group $G$. Moreover, all $C_{\alpha,\beta}(A^t)$ are Abelian subgroups of $G$, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1$. Hence $A^t$ is a $t$-IFASG of $G$, but $G$ is non-Abelian group.

*Theorem 3.4:* Let $A^t$ is a $t$-IFASG of $G$. Then the set $H = \{a \in G : \mu_{A^t}(aw) = \mu_{A^t}(wa) \text{ and } \nu_{A^t}(aw) = \nu_{A^t}(wa), \text{ for all } w \in G\}$ is an Abelian subgroup of $G$.

*Proof:* Since $A^t$ be a $t$-IFASG of $G$, $C_{\alpha,\beta}(A^t)$ is an Abelian subgroup of $G$, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha+\beta \leq 1$. To show that the $H$ is an Abelian subgroup of $G$: obviously, $H \neq \phi$, because $e \in H$.

Let $a, b \in H$ be any two elements. Then $\mu_{A^t}(aw) = \mu_{A^t}(wa)$ and $\nu_{A^t}(aw) = \nu_{A^t}(wa)$, for all $w \in G$.

Consider, $\mu_{A^t}((ab)w) = \mu_{A^t}(a(bw)) = \mu_{A^t}((bw)a) = \mu_{A^t}(b(wa)) = \mu_{A^t}((wa)b) = \mu_{A^t}(w(ab))$ and $\nu_{A^t}((ab)w) = \nu_{A^t}(a(bw)) = \nu_{A^t}((bw)a) = \nu_{A^t}(b(wa)) = \nu_{A^t}((wa)b) = \nu_{A^t}(w(ab))$ hold, for all $w \in G$. Therefore, $ab \in H$. Further, let

$$\begin{aligned} a \in H \Rightarrow \mu_{A^t}(aw) &= \mu_{A^t}(wa)\\ \text{and } \nu_{A^t}(aw) &= \nu_{A^t}(wa) \text{ holds,}\quad \text{for all } w \in G. \end{aligned} \tag{5}$$

Putting $w = z^{-1}$ in (5), we get $\mu_{A^t}(az^{-1}) = \mu_{A^t}(z^{-1}a)$and$\nu_{A^t}(az^{-1}) = \nu_{A^t}(z^{-1}a)$. Further,

$$\begin{aligned}\mu_{A^t}(a^{-1}z) &= \mu_{A^t}((a^{-1}z)^{-1}) = \mu_{A^t}(z^{-1}a)\\ &= \mu_{A^t}(az^{-1}) = \mu_{A^t}((az^{-1})^{-1})\\ &= \mu_{A^t}(za^{-1})\\ \text{and }\ \nu_{A^t}(a^{-1}z) &= \nu_{A^t}((a^{-1}z)^{-1}) = \nu_{A^t}(z^{-1}a)\\ &= \nu_{A^t}(az^{-1}) = \nu_{A^t}((az^{-1})^{-1})\\ &= \nu_{A^t}(za^{-1}).\end{aligned}$$

Consequently, $H$ is a subgroup of $G$.

Moreover, we show that $H$ is an Abelian subgroup of $G$. Let $a, b \in H$. Without loss of generality, assume that $\mu_{A^t}(a) \geq \alpha$, $\nu_{A^t}(a) \leq 1-\alpha$ and $\mu_{A^t}(b) \geq \alpha_1$, $\nu_{A^t}(b) \leq 1-\alpha_1$. Then $a \in C_{\alpha,1-\alpha}(A^t)$, $b \in C_{\alpha_1,1-\alpha_1}(A^t)$. Let $\alpha < \alpha_1$. Then $\mu_{A^t}(b) \geq \alpha_1 > \alpha$ and $\nu_{A^t}(b) \leq 1-\alpha_1 < 1-\alpha$. $\Rightarrow b \in C_{\alpha,1-\alpha}(A^t)$. Thus $a, b \in C_{\alpha,1-\alpha}(A^t)$ and so $ab = ba$. Hence $H$ is an Abelian subgroup of $G$.

*Remark 3.2:* If $A^t$ is a $t$-IFASG of group $G$. Then $A^t$ is also a $t$-IFNSG of $G$.

*Theorem 3.5:* If $A^t$ is a $t$-IFASG of group $G$. Then the sets $H$ and $C(A^t)$ are same, i.e., $C(A^t) = H$.

*Proof:* $C(A^t) = \{a \in G : \mu_{A^t}([a,w]) = \mu_{A^t}(e)$ and $\nu_{A^t}([a,w]) = \nu_{A^t}(e)$, for all $w \in G\} = \{a \in G : \mu_{A^t}(a^{-1}w^{-1}aw) = \mu_{A^t}(e)$ and $\nu_{A^t}(a^{-1}w^{-1}aw) = \nu_{A^t}(e)$, for all $w \in G\} = \{a \in G : \mu_{A^t}((wa)^{-1}aw) = \mu_{A^t}(e)$ and $\nu_{A^t}((wa)^{-1}aw) = \nu_{A^t}(e)$, for all $w \in G\} = \{a \in G :$

$\mu_{A^t}(wa) = \mu_{A^t}(aw)$ and $\nu_{A^t}(wa) = \nu_{A^t}(aw)$, for all $w \in G\} = H$.

*Theorem 3.6:* Let $A^t$ and $B^t$ be two $t$-IFSG's of a group $G_1$ and $G_2$, respectively. Then $A^t \times B^t$ is a $t$-IFASG of $G_1 \times G_2$ if and only if both $A^t$ and $B^t$ are $t$-IFASG's of $G_1$ and $G_2$, respectively.

*Proof:* Suppose that $A^t$ and $B^t$ are $t$-IFASG's of $G_1$ and $G_2$, respectively. Then $C_{\alpha,\beta}(A^t)$ and $C_{\alpha,\beta}(B^t)$ are Abelian subgroups of $G_1$ and $G_2$, respectively, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1 \Rightarrow C_{\alpha,\beta}(A^t) \times C_{\alpha,\beta}(B^t)$ is an Abelian subgroup of $G_1 \times G_2$. In the view of Proposition 2.8, we have $C_{\alpha,\beta}(A^t \times B^t) = C_{\alpha,\beta}(A^t) \times C_{\alpha,\beta}(B^t)$. Therefore, $C_{\alpha,\beta}(A^t \times B^t)$ is an Abelian subgroup of $G_1 \times G_2$, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1 \Rightarrow A^t \times B^t$ is an IFASG of $G_1 \times G_2$.

Conversely, let $A^t \times B^t$ is an IFASG of $G_1 \times G_2$, Then $C_{\alpha,\beta}(A^t \times B^t)$ is an Abelian subgroup of $G_1 \times G_2$. Implies that $C_{\alpha,\beta}(A^t) \times C_{\alpha,\beta}(B^t)$ is an Abelian subgroup of $G_1 \times G_2 \Rightarrow C_{\alpha,\beta}(A^t)$ and $C_{\alpha,\beta}(B^t)$ are Abelian subgroups of $G_1$ and $G_2$, respectively. Thus, $A^t$ and $B^t$ are $t$-IFASG's of $G_1$ and $G_2$, respectively. Hence, proved our claim.

*Definition 3.4:* Let $A^t$ be a $t$-IFSG of a group $G$. Then $A^t$ is called $t$-intuitionistic fuzzy cyclic subgroup ($t$-IFCSG) of group $G$, if $C_{\alpha,\beta}(A^t)$ is a cyclic subgroup of $G$, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1$.

*Remark 3.3 [1]:* Every subgroup of cyclic group is cyclic.

*Theorem 3.7:* If $G$ is a cyclic group, then every $t$-IFSG of $G$ is $t$-IFCSG of $G$.

*Proof:* Given that $G$ is cyclic group. Then $G =< x >$ for some $x \in G$. Let $A^t$ be $t$-IFSG of group $G$. By Theorem (2.1), we have $C_{\alpha,\beta}(A^t)$ is subgroup of $G$. In the view of Remark (3.3), we know that $C_{\alpha,\beta}(A^t)$ is cyclic subgroup of $G$. By using definition of $t$-IFCSG, we conclude that $A^t$ is $t$-ICFSG of group $G$.

In following example we explain that the converse of the above stated result may not be true.

*Example 3.2:* Consider $G =< a, b | a^3 = b^2 = e,\ bab^{-1} = a^{-1} >$ be dihedral group of order six. Consider the $t$-IFS $A^t$ of $G$ is defined by:

$$\mu_{(A^t)}(w) = \begin{cases} 1, & \text{if } w = e, \\ 0, & \text{if } w^2 = e, \\ 0.4, & \text{if } w^3 = e \end{cases}$$

and

$$\nu_{(A^t)}(w) = \begin{cases} 0, & \text{if } w = e, \\ 0.1, & \text{if } w^2 = e, \\ 0.5, & \text{if } w^3 = e. \end{cases}$$

where $w \in G$ and $e$ is the identity element of $G$.

Clearly, $A^t$ is $t$-IFSG of group $G$. Moreover, all $C_{\alpha,\beta}(A^t)$ are cyclic subgroups of $G$. For all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1$. Hence $A^t$ is $t$-IFCSG of $G$ but $G$ is not cyclic group.

## IV. HOMOMORPHISM OF $t$-INTUITIONISTIC FUZZY ABELIAN SUBGROUPS

This section devoted to study of $t$-IFASG ($t$-IFCSG) subgroup under group homomorphism and show that homomorphic image and pre-image of $t$-IFASG ($t$-IFCSG) form a $t$-IFASG ($t$-IFCSG).

*Definition 4.1:* Let $A^t$ and $B^t$ be $t$-IFS's of sets $P$ and $Q$, respectively and $h : P \to Q$ be a mapping and $z \in Q,\ w \in P$. Then the image of $A^t$ under the map $h$ is denoted by $h(A^t)$ and is defined as $h(A^t)(z) = (\mu_{h(A^t)}(z), \nu_{h(A^t)}(z))$ , where

$$\mu_{h(A^t)}(z) = \begin{cases} \max\{\mu_{A^t}(w) : w \in h^{-1}(z)\}, \\ 0, \qquad \text{otherwise.} \end{cases}$$

and

$$\nu_{h(A^t)}(z) = \begin{cases} \min\{\nu_{A^t}(w) : w \in h^{-1}(z)\}, \\ 1, \qquad \text{otherwise.} \end{cases}$$

Also, the pre-image of $B^t$ under $h$ is denoted by $h^{-1}(B^t)$ and is defined as $h^{-1}(B^t)(w) = (\mu_{h^{-1}(B^t)}(w), \nu_{h^{-1}(B^t)}(w))$, where $\mu_{h^{-1}(B^t)}(w) = \mu_{B^t}(h(w))$ and $\nu_{h^{-1}(B^t)}(w) = \nu_{B^t}(h(w))$, i.e., $h^{-1}(B^t)(w) = (\mu_{B^t}(h(w)), \nu_{B^t}(h(w)))$.

*Theorem 4.1:* Let $h$ be a mapping from $P$ into $Q$. Let $A^t$ and $B^t$ be two $t$-IFS of $P$ and $Q$, respectively. Then the following holds

1) $h(C_{\alpha,\beta}(A^t)) \subseteq C_{\alpha,\beta}(h(A^t))$
2) $h^{-1}C_{\alpha,\beta}(B^t) = C_{\alpha,\beta}(h^{-1}(B^t))$

*Proof:* The proof of this theorem is straightforward.

*Theorem 4.2:* Let $B^t$ be a $t$-IFASG of group $G_2$. Let $h$ be group homomorphism from $G_1$ into $G_2$. Then $h^{-1}(B^t)$ is a $t$-IFASG of group $G_1$.

*Proof:* Given that $B^t$ is a $t$-IFASG of group $G_2$. Therefore, $C_{\alpha,\beta}(B^t)$ is an Abelian subgroup of $G_2$, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1$. From Theorem 4.1, we have $C_{\alpha,\beta}(h^{-1}(B^t)) = h^{-1}C_{\alpha,\beta}(B^t) = \{w \in G_1 : h(w) \in C_{\alpha,\beta}(B^t)\}$. Let $w_1, w_2 \in C_{\alpha,\beta}(h^{-1}(B^t))$ be any two elements. Then $h(w_1), h(w_2) \in C_{\alpha,\beta}(B^t)$. Let $w_1, w_2 \in C_{\alpha,\beta}(h^{-1}(B^t))$ be any two elements. Then we have

$$\begin{aligned} \mu_{h^{-1}(B^t)}(w_1) \geq \alpha, &\quad \nu_{h^{-1}(B^t)}(w_1) \leq \beta \\ \text{and } \mu_{h^{-1}(B^t)}(w_2) \geq \alpha, &\quad \nu_{h^{-1}(B^t)}(w_2) \leq \beta. \\ \text{Implies that } \mu_{B^t}(h(w_1)) \geq \alpha, &\quad \nu_{B^t}(h(w_1)) \leq \beta \\ \text{and } \mu_{B^t}(h(w_2)) \geq \alpha, &\quad \nu_{B^t}(h(w_2)) \leq \beta. \end{aligned}$$

Implies that $\min\{\mu_{B^t}(h(w_1)), \mu_{B^t}(h(w_2))\} \geq \alpha$ and $\max\{\nu_{B^t}(h(w_1)), \nu_{B^t}(h(w_2))\} \leq \beta$. As $B^t$ is $t$-IFSG of group $G_2$. Therefore,

$$\begin{aligned} \mu_{B^t}(h(w_1)h(w_2)^{-1}) &\geq \min\{\mu_{B^t}(h(w_1)), \mu_{B^t}(h(w_2))\} \\ &\geq \alpha, \text{ and} \\ \nu_{B^t}(h(w_1)h(w_2)^{-1}) &\leq \max\{\nu_{B^t}(h(w_1)), \nu_{B^t}(h(w_2)) \\ &\leq \beta. \\ \mu_{B^t}(h(w_1)h(w_2)^{-1}) &\geq \alpha, \text{ and } \nu_{B^t}(h(w_1)h(w_2)^{-1}) \leq \beta. \\ &\Rightarrow h(w_1)h(w_2)^{-1} \in C_{\alpha,\beta}(A^t) \\ &\Rightarrow h(w_1 w_2^{-1}) \in (C_{\alpha,\beta}(B^t)). \end{aligned}$$

$$\begin{aligned}
&\Rightarrow (w_1 w_2^{-1}) \in h^{-1}(C_{\alpha,\beta}(A^t)) \\
&= C_{\alpha,\beta}(h^{-1}(B^t)) \text{ [By theorem4.1(2)]} \\
&\Rightarrow \; w_1 w_2^{-1} \in C_{\alpha,\beta}(h^{-1}(B^t)).
\end{aligned}$$

Hence $C_{\alpha,\beta}(h^{-1}(B^t))$ is subgroup of $G_1$. As $C_{\alpha,\beta}(B^t)$ is an Abelian subgroup of $G_2$. Therefore, we have $h(w_1)h(w_2) = h(w_2)h(w_1)$. Implies that $h(w_1w_2) = h(w_2w_1)$ and so $\mu_{B^t}(h(w_1w_2)) = \mu_{B^t}(h(w_2w_1))$ and $\nu_{B^t}(h(w_1w_2)) = \nu_{B^t}(h(w_2w_1))$. Implies that $\mu_{h^{-1}(B^t)}(w_1w_2) = \mu_{h^{-1}(B^t)}(w_2w_1)$ and $\nu_{h^{-1}(B^t)}(w_1w_2) = \nu_{h^{-1}(B^t)}(w_2w_1)$. Implies that $w_1w_2 = w_2w_1$. Thus, $C_{\alpha,\beta}(h^{-1}(B^t))$ is an Abelian subgroup of $G_1$, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1$. Hence, $h^{-1}(B^t)$ is an **IFASG** of group $G_1$.

*Example 4.1:* Let $h : Q_8 \to V_4$ be group homomorphism from $Q_8$ to $V_4$, where $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ and $V_4 = \{e, a, b, ab\}$. Let $B^t = (\mu_{B^t}(x), \nu_{B^t}(x))$ be a $t$-IFCSG of $V_4$. Fot $t = 1$ is define as

$$\mu_{B^t}(x) = \begin{cases} 1, & x = e, \\ 0, & x \in \{b, ab\}, \\ 0.3, & x \in \{a\}. \end{cases}$$

and

$$\nu_{B^t}(x) = \begin{cases} 0, & x = e, \\ 0.4, & x \in \{b, ab\}, \\ 0.66, & x \in \{a\}. \end{cases}$$

right.

$$\begin{aligned}
h(1) = h(-1) = e, \quad h(i) = h(-i) = a, \\
h(j) = h(-j) = b, \quad h(k) = h(-k) = ab.
\end{aligned}$$

From Definition 4.1, we have

$$h^{-1}B^t(w) = (\mu_{B^t}(h(w), \nu_{B^t}(h(w).$$

$$\mu_{B^t}(h(w)) = \begin{cases} 1, & w \in \{\pm 1\}, \\ 0, & w \in \{\pm j, \pm k\}, \\ 0.3, & w \in \{\pm i\}. \end{cases}$$

$$\nu_{B^t}(h(w)) = \begin{cases} 0, & w \in \{\pm 1\}, \\ 0.4, & w \in \{\pm j, \pm k\}, \\ 0.66, & w \in \{\pm i\}. \end{cases}$$

$C_{\alpha,\beta}(h^{-1}(B^t)(w)) = \{1, -1\}$ or $\{1, -1, i, -i\}$, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1$. Clearly, all $C_{\alpha,\beta}(h^{-1}(B^t)(w))$ are abelian subgroups of $Q_8$. This implies that $h^{-1}B^t(w)$ is a $t$-IFASG of $Q_8$.

The following result explains that the homomorphic image of $t$-**IFASG** is also $t$-**IFASG**.

*Theorem 4.3:* Let $A^t$ be a $t$-**IFASG** of group $G_1$ and $h$ be a onto homomorphism from $G_1$ into $G_2$. Then $h(A^t)$ is a $t$-**IFASG** of group $G_2$.

*Proof:* Since $A^t$ is a $t$-**IFASG** of group $G_1$. Therefore, $C_{\alpha,\beta}(A^t)$ is an Abelian subgroup of $G_1$, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1$. To prove that $h(A^t)$ is a $t$-**IFASG** of group $G_2$. For this, we show that $C_{\alpha,\beta}(h(A^t))$ is an Abelian subgroup of $G_2$, for all $\alpha, \beta \in (0, 1]$. Let $z_1, z_2 \in C_{\alpha,\beta}(h(A^t))$ be any two elements. Then there exists $w_1, w_2 \in G_1$ such that $h(w_1) = z_1, h(w_2) = z_2$. Then we have

$$\begin{aligned}
&\mu_{h(A^t)}(z_1) \geq \alpha, \quad \nu_{h(A^t)}(z_1) \leq \beta \\
\text{and } &\mu_{h(A^t)}(z_2) \geq \alpha, \quad \nu_{h(A^t)}(z_2) \leq \beta.
\end{aligned}$$

By Theorem (4.1)(1), we have $h(C_{\alpha,\beta}(A^t)) \subseteq C_{\alpha,\beta}(h(A^t))$. Therefore there exists $w_1, \; w_2 \in G_1$ such that

$$\begin{aligned}
&\mu_{A^t}(w_1) \geq \mu_{h(A^t)}(z_1) \geq \alpha, \; \nu_{A^t}(w_1) \leq \nu_{h(A^t)}(z_1) \leq \beta \\
&\quad \text{and} \\
&\mu_{A^t}(w_2) \geq \mu_{h(A^t)}(z_2) \geq \alpha, \; \nu_{A^t}(w_2) \leq \nu_{h(A^t)}(z_2) \leq \beta. \\
&\Rightarrow \mu_{A^t}(w_1) \geq \alpha, \nu_{A^t}(w_1) \leq \beta \\
&\quad \text{and } \mu_{A^t}(w_2) \geq \alpha, \nu_{A^t}(w_2) \leq \beta, \\
&\Rightarrow \min\{\mu_{A^t}(w_1), \mu_{A^t}(w_2)\} \geq \alpha \\
&\quad \text{and } \max\{\nu_{A^t}(w_1), \nu_{A^t}(w_2)\} \leq \beta.
\end{aligned}$$

As $A^t$ is a $t$-**IFSG** of group $G_1$. Therefore,

$$\begin{aligned}
&\mu_{A^t}(w_1w_2^{-1}) \geq \min\{\mu_{A^t}(w_1), \mu_{A^t}(w_2)\} \geq \alpha, \\
&\text{and } \nu_{A^t}(w_1w_2^{-1}) \leq \max\{\nu_{A^t}(w_1), \nu_{A^t}(w_2)\} \leq \beta. \\
&\Rightarrow \mu_{A^t}(w_1w_2^{-1}) \geq \alpha, \;\; \text{and} \;\; \nu_{A^t}(w_1w_2^{-1}) \leq \beta. \\
&\Rightarrow w_1w_2^{-1} \in C_{\alpha,\beta}(A^t) \\
&\Rightarrow h(w_1w_2^{-1}) \in h(C_{\alpha,\beta}(A^t)) \subseteq C_{\alpha,\beta}(h(A^t)). \\
&\Rightarrow h(w_1)h(w_2^{-1}) \in C_{\alpha,\beta}(h(A^t)) \Rightarrow z_1z_2^{-1} \in C_{\alpha,\beta}(h(A^t)).
\end{aligned}$$

Hence $C_{\alpha,\beta}(h(A^t))$ is subgroup of $G_2$. Further, we prove that commutative property holds in $C_{\alpha,\beta}(h(A^t))$. We have $h(w_1), h(w_2) \in C_{\alpha,\beta}h(A^t)$ Therefore, $\exists \; C_{\delta,\theta}(A^t)$ such that $w_1, w_2 \in C_{\delta,\theta}(A^t)$, where $\delta, \theta \in (0, 1]$ and $0 < \delta + \theta \leq 1$. But $C_{\delta,\theta}(A)$ is an Abelian group. Therefore, $w_1w_2 = w_2w_1$ $\Rightarrow h(w_1w_2) = h(w_2w_1) \Rightarrow h(w_1)h(w_2) = h(w_2)h(w_1)$, $\Rightarrow z_1z_2 = z_2z_1$. Thus $C_{\alpha,\beta}(h(A^t))$ is an Abelian subgroup of $G_2$. Hence $h(A^t)$ is an **IFASG** of group $G_2$.

*Theorem 4.4:* Let $B^t$ be a $t$-**IFCSG** of group $G_2$ and $h$ be a homomorphism from $G_1$ onto $G_2$. Then $h^{-1}(B^t)$ is $t$-**IFCSG** of group $G_1$.

*Proof:* Since $B^t$ is $t$-**IFCSG** of group $G_2$. By definition of $t$-**IFCSG**, we have $C_{\alpha,\beta}(B^t)$ is a cyclic subgroup of $G_2$, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1$. Let $C_{\alpha,\beta}(B^t) = \langle g_2 \rangle$ for some $g_2 \in G_2$. Now for $g_2 \in G_2$, $\exists \; g_1 \in G_1$ such that $h(g_1) = g_2$. Thus $C_{\alpha,\beta}(B^t) = \langle h(g_1) \rangle$. In the view of theorem (4.1), $h^{-1}C_{\alpha,\beta}(B^t) = C_{\alpha,\beta}(h^{-1}(B^t)) = \langle g_1 \rangle$. Hence $h^{-1}(B^t)$ is $t$-**IFCSG** of group $G_1$.

*Theorem 4.5:* Let $A^t$ be a $t$-**IFCSG** of group $G_1$ and $h$ be a onto homomorphism from $G_1$ into $G_2$. Then $h(A^t)$ is $t$-**IFCSG** of group $G_2$.

*Proof:* Given that $A^t$ is a $t$-**IFCSG** of group $G_1$. Therefore, $C_{\alpha,\beta}(A^t)$ is a cyclic subgroup of $G_1$, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1$. To prove that $h(A^t)$ is also $t$-**IFCSG** of $G_2$: Let $g \in C_{\alpha,\beta}(h(A^t))$ be any element. As $h$ is surjective, therefore let $g = h(g')$, for some $g' \in G_1$. As $g' \in G_1$, we can find one $C_{\delta,\theta}(A^t)$ which exists, for all $g' \in G_1$ and hence for all $g \in C_{\alpha,\beta}(h(A^t))$ such that $g' \in C_{\delta,\theta}(A^t)$. But $C_{\delta,\theta}(A)$ is a cyclic subgroup of $G_1$. Let $C_{\delta,\theta}(A) = \langle g_1 \rangle$. So $g' = (g_1)^n$.

Thus $g = h(g') = h((g_1)^n) = (h(g_1))^n \Rightarrow C_{\alpha,\beta}(h(A^t))$ is a cyclic subgroup of $G_2$. Hence $h(A^t)$ is $t$-IFCSG of $G_2$.

The following example illustrates that the homomorphic image of $t$-IFCSG of non-cyclic group is $t$- IFCSG.

*Example 4.2:* Let $h : Q_8 \to V_4$ be group homomorphism from $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ to $V_4 = \{e, a, b, ab\}$. Let $A^t = (\mu_{A^t}(w), \nu_{A^t}(w))$ be a $t$-IFCSG of $Q_8$. Fot $t = 1$ is defined as

$$\mu_{A^t}(w) = \begin{cases} 1, & w \in \{\pm 1\}, \\ 0, & w \in \{\pm j, \pm k\}, \\ 0.6, & w \in \{\pm i\}. \end{cases}$$

and

$$\nu_{A^t}(w) = \begin{cases} 0, & w \in \{\pm 1\}, \\ 0.2 & w \in \{\pm j, \pm k\}, \\ 0.7, & w \in \{\pm i\}. \end{cases}$$

$$h(1) = h(-1) = e, \quad h(i) = h(-i) = a,$$
$$h(j) = h(-j) = b, \quad h(k) = h(-k) = ab.$$

In the view of definition 4.1, we get

$$\mu_{h(A^t)}(e) = 1, \ \mu_{h(A^t)}(a) = 0.6,$$
$$\mu_{h(A^t)}(b) = 0 = \mu_{h(A^t)}(ab),$$
$$\nu_{h(A^t)}(e) = 0, \ \nu_{h(A^t)}(a) = 0.5,$$
$$\nu_{h(A^t)}(b) = 0.2 = \nu_{h(A^t)}(ab).$$

Clearly, all $C_{\alpha,\beta}(h(A^t))$ are cyclic subgroups of $V_4$, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1$. Hence, $h(A^t)$ is $t$-IFCSG of $V_4$.

## V. CONCLUSION

Our purpose is to encourage and explore the research of Abelian group by new methodological development on $t$-IFSG, which will be helpful in future. In this manuscript, we have commenced a $t$-intuitionistic fuzzy normalizer and $t$-intuitionistic fuzzy centralizer. Further, we have introduced $t$-IFASG and have developed many significant results about this phenomena. We have proved that homomorphic image and pre-image of $t$-IFASG is $t$-IFASG and some more results about $t$-IFCSG under group homomorphism. In our future work, we will aspire to classify t-IFSG by defining an action of these specific groups on different algebraic structures.

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