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A CERTAIN EQUIVALENCE ON A SEMIGROUP

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Let S be a periodic semigroup. We shall introduce the equivalence $\overline{\mathbf{K}}$: for $a, b \in S$, $a\overline{\mathbf{K}}b$ if and only if there exists an idempotent e and positive integers m, n such that $a^m = e = b^n$. In [1] J. T. Sedlock studies necessary and sufficient conditions on a periodic semigroup S in order that $\overline{\mathbf{K}}$ coincide with any one of the Green relations [2]. In this paper we consider arbitrary semigroups having similar properties.

I

In this section, S will be a fixed non-empty set. The mapping $\mathbf{U} : \exp S \to \exp S$ is said to be *$\mathscr{C}$-closure operation* if the mapping $\mathbf{U}$ satisfies the following conditions:

(1) $$\mathbf{U}(\emptyset) = \emptyset ;$$

(2) $$A \subset B \subset S \Rightarrow \mathbf{U}(A) \subset \mathbf{U}(B) ;$$

(3) $$A \subset \mathbf{U}(A) \quad \text{for each} \quad A \subset S ;$$

(4) $$\mathbf{U}(\mathbf{U}(A)) = \mathbf{U}(A) \quad \text{for each} \quad A \subset S .$$

For $x \in S$ we write simply $\mathbf{U}(x)$ instead of $\mathbf{U}(\{x\})$. The set of all $\mathscr{C}$-closure operations for a set S will be denoted by $\mathscr{C}(S)$.

A $\mathscr{C}$-closure operation $\mathbf{U}$ is said to be *$\mathscr{Q}$-closure operation* if

(5) $$\mathbf{U}(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} \mathbf{U}(A_i) \quad \text{for} \quad A_i \subset S \ (i \in I \neq \emptyset)$$

holds. Let $\mathscr{Q}(S)$ be the set of all $\mathscr{Q}$-closure operations for a set S. Evidently $\mathscr{Q}(S) \subset \subset \mathscr{C}(S)$.

Let $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$, then we define

(6) $$\mathbf{U} \leqq \mathbf{V} \Leftrightarrow \mathbf{U}(A) \subset \mathbf{V}(A) \quad \text{for each} \quad A \subset S .$$

The ordered set $\mathscr{C}(S)$ is a lattice. If $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$, then

$$(7) \qquad (\mathbf{U} \wedge \mathbf{V})(A) = \mathbf{U}(A) \cap \mathbf{V}(A) \quad \text{for each} \quad A \subset S .$$

If $\mathbf{U}, \mathbf{V} \in \mathscr{Q}(S)$, then

$$(8) \qquad \mathbf{U} \vee \mathbf{V} \in \mathscr{Q}(S) ;$$

$$(9) \qquad \mathbf{U} \leqq \mathbf{V} \Leftrightarrow \mathbf{U}(x) \subset \mathbf{V}(x) \quad \text{for each} \quad x \in S .$$

A subset A of S will be called *$\mathbf{U}$-closed* if $\mathbf{U}(A) = A$. The set of all $\mathbf{U}$-closed subsets of S will be denoted by $\mathscr{F}(\mathbf{U})$. If $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$, then

$$(10) \qquad \mathscr{F}(\mathbf{U} \vee \mathbf{V}) = \mathscr{F}(\mathbf{U}) \cap \mathscr{F}(\mathbf{V}) ;$$

$$(11) \qquad \mathbf{U} \leqq \mathbf{V} \Leftrightarrow \mathscr{F}(\mathbf{V}) \subset \mathscr{F}(\mathbf{U}) .$$

Let $\mathbf{U} \in \mathscr{C}(S)$. We define $\mathbf{U}^* \in \mathscr{Q}(S)$. If $A \subset S$, then $x \in \mathbf{U}^*(A)$ if and only if $\mathbf{U}(x) \cap \cap A \neq \emptyset$. For $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$ we have

$$(12) \qquad \mathbf{U} \leqq \mathbf{V} \Rightarrow \mathbf{U}^* \leqq \mathbf{V}^* ;$$

$$(13) \qquad x \in \mathbf{U}(y) \Leftrightarrow y \in \mathbf{U}^*(x) \quad \text{for each} \quad x, y \in S ;$$

$$(14) \qquad \mathbf{U}(x) = \mathbf{U}^{**}(x) \quad \text{for each} \quad x \in S ;$$

$$(15) \qquad \mathbf{U} = \mathbf{U}^{**} \Leftrightarrow \mathbf{U} \in \mathscr{Q}(S) .$$

(See [3].)

Definition 1. Let $\mathbf{U} \in \mathscr{C}(S)$. We shall introduce the equivalence $\overline{\mathbf{U}}$ on S by: for $x, y \in \in S$, $x\overline{\mathbf{U}}y$ if and only if $\mathbf{U}(x) = \mathbf{U}(y)$. For any element x of S, let $\mathbf{U}_x$ denote the $\overline{\mathbf{U}}$-class of S containing x.

Lemma 1. *Let $\mathbf{U} \in \mathscr{C}(S)$. If $x, y \in S$, then $x\overline{\mathbf{U}}y$ if and only if $x \in \mathbf{U}(y)$ and $y \in \mathbf{U}(x)$.*

Proof. If $x\overline{\mathbf{U}}y$, then by (3) $x \in \mathbf{U}(x) = \mathbf{U}(y)$ and $y \in \mathbf{U}(y) = \mathbf{U}(x)$. If $x \in \mathbf{U}(y)$ and $y \in \mathbf{U}(x)$, then by (2), (4) we have $\mathbf{U}(x) \subset \mathbf{U}(\mathbf{U}(y)) = \mathbf{U}(y)$. Similarly we obtain $\mathbf{U}(y) \subset \mathbf{U}(x)$. Thus $\mathbf{U}(x) = \mathbf{U}(y)$ and $x\overline{\mathbf{U}}y$.

Theorem 1. *Let $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$. Then the following conditions are equivalent:*

1. $\overline{\mathbf{U}} \subset \overline{\mathbf{V}}$;
2. *for every $x \in S$, $\mathbf{U}_x \subset \mathbf{V}(x)$;*
3. *for every $A \in \mathscr{F}(\mathbf{V})$, $A = \bigcup_{x \in A} \mathbf{U}_x$.*

Proof. 1 ⇒ 2. Let $x \in S$, then $\mathbf{U}_x \subset \mathbf{V}_x$. If $y \in \mathbf{U}_x$, then $y \in \mathbf{V}_x$. By Definition 1 and (3) we have $y \in \mathbf{V}(y) = \mathbf{V}(x)$. Thus $\mathbf{U}_x \subset \mathbf{V}(x)$.

$2 \Rightarrow 3$. If $x \in A \in \mathscr{F}(\mathbf{V})$, then $\mathbf{V}(x) \subset \mathbf{V}(A) = A$. Hence $\mathbf{U}_x \subset A$. This implies $A = \bigcup_{x \in A} \mathbf{U}_x$.

$3 \Rightarrow 1$. Let $x\overline{\mathbf{U}}y$. Evidently $\mathbf{V}(x) \in \mathscr{F}(\mathbf{V})$ and thus $y \in \mathbf{U}_y = \mathbf{U}_x \subset \mathbf{V}(x)$. Similarly we obtain $x \in \mathbf{U}_y \subset \mathbf{V}(y)$. From Lemma 1 it follows that $x\overline{\mathbf{V}}y$.

Corollary. *If $\mathbf{U} \in \mathscr{C}(S)$, then for every $A \in \mathscr{F}(\mathbf{U})$, $A = \bigcup_{x \in A} \mathbf{U}_x$.*

Theorem 2. *Let $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$. If $\mathbf{U} \leqq \mathbf{V}$, then $\overline{\mathbf{U}} \subset \overline{\mathbf{V}}$.*

Proof. If $x\overline{\mathbf{U}}y$, then by Lemma 1 and (6) we have $x \in \mathbf{U}(y) \subset \mathbf{V}(y)$ and $y \in \mathbf{U}(x) \subset \mathbf{V}(x)$. It follows from Lemma 1 that $x\overline{\mathbf{V}}y$.

Theorem 3. *Let $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$, then $\overline{\mathbf{U} \wedge \mathbf{V}} = \overline{\mathbf{U}} \cap \overline{\mathbf{V}}$.*

Proof. It follows from Theorem 2 that $\overline{\mathbf{U} \wedge \mathbf{V}} \subset \overline{\mathbf{U}}$, $\overline{\mathbf{U} \wedge \mathbf{V}} \subset \overline{\mathbf{V}}$. This implies $\overline{\mathbf{U} \wedge \mathbf{V}} \subset \overline{\mathbf{U}} \cap \overline{\mathbf{V}}$. If $x(\overline{\mathbf{U}} \cap \overline{\mathbf{V}})\, y$, then $x\overline{\mathbf{U}}y$ and $x\overline{\mathbf{V}}y$. We have thus $\mathbf{U}(x) = \mathbf{U}(y)$ and $\mathbf{V}(x) = \mathbf{V}(y)$ so that $\mathbf{U}(x) \cap \mathbf{V}(x) = \mathbf{U}(y) \cap \mathbf{V}(y)$. By (7) we have $x\overline{\mathbf{U} \wedge \mathbf{V}}y$. Hence $\overline{\mathbf{U}} \cap \overline{\mathbf{V}} \subset \overline{\mathbf{U} \wedge \mathbf{V}}$ which implies $\overline{\mathbf{U} \wedge \mathbf{V}} = \overline{\mathbf{U}} \cap \overline{\mathbf{V}}$.

Theorem 4. *Let $\mathbf{U} \in \mathscr{Q}(S)$. Then the following conditions are equivalent:*

1. $\mathbf{U} = \mathbf{U}^*$;
2. *for every $x \in S$, $\mathbf{U}(x) = \mathbf{U}_x$;*
3. *for every $x \in S$, $\mathbf{U}_x \in \mathscr{F}(\mathbf{U})$.*

Proof. $1 \Rightarrow 2$. It follows from Theorem 1 that $\mathbf{U}_x \subset \mathbf{U}(x)$ for every $x \in S$. Let $y \in \mathbf{U}(x)$. According to (13) we have $x \in \mathbf{U}^*(y) = \mathbf{U}(y)$. Since $\mathbf{U}(x) = \mathbf{U}(y)$, we have $y \in \mathbf{U}_x$, hence $\mathbf{U}(x) \subset \mathbf{U}_x$. This implies $\mathbf{U}(x) = \mathbf{U}_x$.

$2 \Rightarrow 3$. Evident.

$3 \Rightarrow 1$. It follows from (15) that $\mathbf{U} = \mathbf{U}^{**}$. Let $x \in S$. If $y \in \mathbf{U}^*(x)$, then by (13) $x \in \mathbf{U}(y)$. Since $y \in \mathbf{U}_y$, we have $\mathbf{U}(y) \subset \mathbf{U}(\mathbf{U}_y) = \mathbf{U}_y$ so that $x \in \mathbf{U}_y$. This implies $y \in \mathbf{U}(y) = \mathbf{U}(x)$ and $\mathbf{U}^*(x) \subset \mathbf{U}(x)$ for every $x \in S$. It follows from (9) that $\mathbf{U}^* \leqq \mathbf{U}$. By (12) we have $\mathbf{U} = \mathbf{U}^{**} \leqq \mathbf{U}^*$. Hence $\mathbf{U} = \mathbf{U}^*$.

Theorem 5. *Let $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$. If $\mathbf{U} = \mathbf{U}^*$, then $\mathbf{U} \leqq \mathbf{V}$ if and only if $\overline{\mathbf{U}} \subset \overline{\mathbf{V}}$.*

Proof. If $\mathbf{U} \leqq \mathbf{V}$, then by Theorem 2 we have $\overline{\mathbf{U}} \subset \overline{\mathbf{V}}$. Suppose now that $\overline{\mathbf{U}} \subset \overline{\mathbf{V}}$. Evidently $\mathbf{U} = \mathbf{U}^* \in \mathscr{Q}(S)$. Let $A \subset S$. If $y \in \mathbf{U}(A)$, then by (5) we have $y \in \mathbf{U}(x)$ for some $x \in A$. According to Theorem 4, Theorem 1 and (2), we have $y \in \mathbf{U}_x \subset \mathbf{V}(x) \subset \mathbf{V}(A)$. This implies $\mathbf{U}(A) \subset \mathbf{V}(A)$. It follows from (6) that $\mathbf{U} \leqq \mathbf{V}$.

II

Let now S be an arbitrary semigroup. Let $A \subset S$, $A \neq \emptyset$. Put $\mathbf{L}(A) = S^1A = SA \cup A$ and $\mathbf{R}(A) = AS^1 = AS \cup A$. Finally $\mathbf{L}(\emptyset) = \emptyset = \mathbf{R}(\emptyset)$. It is clear that $\mathbf{L}, \mathbf{R} \in \mathscr{Q}(S)$ and $\mathscr{F}(\mathbf{L})$ is the set of all left ideals of S (including $\emptyset$), $\mathscr{F}(\mathbf{R})$ is the set of all right ideals of S (including $\emptyset$). Put $\mathbf{M} = \mathbf{L} \vee \mathbf{R}$, $\mathbf{H} = \mathbf{L} \wedge \mathbf{R}$. Evidently $\mathbf{M} \in \mathscr{Q}(S)$ and $\mathbf{H} \in \mathscr{C}(S)$. It follows from (10) and (7) that $\mathscr{F}(\mathbf{M})$ is the set of all two-sided ideals of S (including $\emptyset$) and $\mathscr{F}(\mathbf{H})$ is the set of all quasi-ideals of S (including $\emptyset$).

Put $\mathbf{P}(\emptyset) = \emptyset$. If $A \subset S$, $A \neq \emptyset$, then by $\mathbf{P}(A)$ we denote the subsemigroup generated by all elements of A. Evidently $\mathbf{P} \in \mathscr{C}(S)$ and $\mathscr{F}(\mathbf{P})$ is the set of all subsemigroups of S (including $\emptyset$). Clearly $\mathbf{P} \leqq \mathbf{H}$.

Lemma 2. *Let $A \subset S$. Then $A \in \mathscr{F}(P^*)$ if and only if the implication*

$$x^n \in A \Rightarrow x \in A \tag{16}$$

holds for every $x \in S$ and for every positive integer n.

Proof. 1. Let $A \in \mathscr{F}(\mathbf{P}^*)$. If $x^n \in A$ for some $x \in S$ and for some positive integer n, then by (2) and (4) we have $\mathbf{P}^*(x^n) \subset A$. Since $x^n \in \mathbf{P}(x)$, it follows from (13) that $x \in \mathbf{P}^*(x^n) \subset A$.

2. Let (16) hold for every $x \in S$ and for every positive integer n. Evidently $\mathbf{P}^* \in \mathscr{Q}(S)$. If $A \neq \emptyset$, then by (5) we have $\mathbf{P}^*(A) = \bigcup_{x \in A} \mathbf{P}^*(x)$. If $y \in \mathbf{P}^*(A)$, then $y \in \mathbf{P}^*(x)$ for some $x \in A$. According to (13) $x \in \mathbf{P}(y)$ and thus $x = y^n$ for some positive integer n. Since $y^n \in A$, it follows from (16) that $y \in A$. Hence $\mathbf{P}^*(A) \subset A$ so that, by (3), $A = \mathbf{P}^*(A) \in \mathscr{F}(\mathbf{P}^*)$.

Lemma 3. *Let $A \subset S$. Then $A \in \mathscr{F}(\mathbf{P}^{**})$ if and only if the implication*

$$x \in A \Rightarrow x^n \in A$$

holds for every positive integer n.

Proof is analogous to the proof of Lemma 2.

Definition 2. *Put $\mathbf{K} = \mathbf{P}^* \vee \mathbf{P}^{**}$.*

Lemma 4. $\mathbf{K} = \mathbf{K}^*$.

Proof. According to (8) and (15), we have $\mathbf{K} = \mathbf{K}^{**}$. From $\mathbf{P}^* \leqq \mathbf{K}$ and (12) we obtain $\mathbf{P}^{**} \leqq \mathbf{K}^*$. It follows from $\mathbf{P}^{**} \leqq \mathbf{K}$, (12) and (15) that $\mathbf{P}^* = \mathbf{P}^{***} \leqq \mathbf{K}^*$. Thus $\mathbf{K} = \mathbf{P}^* \vee \mathbf{P}^{**} \leqq \mathbf{K}^*$ and by (12) we have $\mathbf{K}^* \leqq \mathbf{K}^{**} = \mathbf{K}$. This implies $\mathbf{K} = \mathbf{K}^*$.

Lemma 5. *If $x, y \in S$, then $x\overline{\mathbf{K}}y$ if and only if there exist positive integers n, m such that $x^n = y^m$.*

Proof. 1. Let $x^n = y^m$ for some positive integers n, m. By (14) and (6) we have $y^m \in \mathbf{P}(y) = \mathbf{P}^{**}(y) \subset \mathbf{K}(y)$. This and Lemma 2 implies that $x \in \mathbf{P}^*(x^n) = \mathbf{P}^*(y^m) \subset \subset \mathbf{K}(y^m) \subset \mathbf{K}(y)$. Similarly we obtain $y \in \mathbf{K}(x)$ and thus by Lemma 1 we have $x\overline{\mathbf{K}}y$.

2. If $x\overline{\mathbf{K}}y$, then by Lemma 1 $x \in \mathbf{K}(y)$. Let $A = \{u/u^n = y^m$ for some positive integers $n, m\}$. It follows from Lemma 2, Lemma 3 and (10) that $A \in \mathscr{F}(\mathbf{P}^*) \cap \mathscr{F}(\mathbf{P}^{**}) = = \mathscr{F}(\mathbf{P}^* \vee \mathbf{P}^{**}) = \mathscr{F}(\mathbf{K})$. Since $y \in A$, hence $x \in \mathbf{K}(y) \subset A$. We have thus $x^n = y^m$ for some positive integers n, m.

A semigroup S is called *right regular* (*left regular*) if $x \in x^2S$ ($x \in Sx^2$) for every $x \in S$.

Theorem 6. *The following conditions on a semigroup S are equivalent*:

1. *S is right regular*;
2. $\mathbf{P}^* \leqq \mathbf{R}$;
3. $\mathbf{K} \leqq \mathbf{R}$;
4. $\overline{\mathbf{K}} \subset \overline{\mathbf{R}}$.

Proof. $1 \Rightarrow 2$. Let S be a right regular semigroup. Let A be a right ideal of S. If $x^n \in A$ ($x \in S$, $n \geqq 2$), then there exists $a \in S$ such that $x = x^2a$ and $x^{n-1} = = x^na \in Aa \subset A$. Similarly we obtain $x^{n-i} \in A$ for any positive integer $i < n$. From here it follows that $x \in A$. By Lemma 2 we have $A \in \mathscr{F}(\mathbf{P}^*)$. It follows from (11) that $\mathbf{P}^* \leqq \mathbf{R}$.

$2 \Rightarrow 3$. Suppose $\mathbf{P}^* \leqq \mathbf{R}$. Evidently $\mathbf{P} \leqq \mathbf{R}$. It follows from (12) and (15) that $\mathbf{P}^{**} \leqq \leqq \mathbf{R}^{**} = \mathbf{R}$. Thus $\mathbf{K} = \mathbf{P}^* \vee \mathbf{P}^{**} \leqq \mathbf{R}$.

$3 \Rightarrow 4$. This follows from Theorem 2.

$4 \Rightarrow 1$. If $\overline{\mathbf{K}} \subset \overline{\mathbf{R}}$, then by Lemma 4 and Theorem 5 we have $\mathbf{P}^* \leqq \mathbf{K} \leqq \mathbf{R}$. According to (11) $x^2 \in \mathbf{R}(x^2) \in \mathscr{F}(\mathbf{R}) \subset \mathscr{F}(\mathbf{P}^*)$. It follows from Lemma 2 that $x \in \mathbf{R}(x^2) = = x^2S^1$. We shall show that $x \in x^2S$. Indeed, if $x = x^2$, then $x = x^3 \in x^2S$. Hence, S is right regular.

The following left-right dual of Theorem 6 is also true.

Theorem 7. *The following conditions on a semigroup S are equivalent*:

1. *S is left regular*;
2. $\mathbf{P}^* \leqq \mathbf{L}$;
3. $\mathbf{K} \leqq \mathbf{L}$;
4. $\overline{\mathbf{K}} \subset \overline{\mathbf{L}}$.

Theorem 8. *The following conditions on a semigroup S are equivalent*:

1. *S is a union of groups*;
2. *S is left regular and right regular*;
3. $\mathbf{P}^* \leqq \mathbf{H}$;
4. $\mathbf{K} \leqq \mathbf{H}$;
5. $\overline{\mathbf{K}} \subset \overline{\mathbf{H}}$.

Proof. 1 ⇒ 2. Evident.

2 ⇒ 3 ⇒ 4. This follows from Theorem 6 and Theorem 7.

4 ⇒ 5. This follows from Theorem 2.

5 ⇒ 1. Suppose $\overline{\mathbf{K}} \subset \overline{\mathbf{H}}$. According to Theorem 3, Theorem 6 and Theorem 7, S is right regular and left regular. From here and (7) we obtain $x \in x^2S \cap Sx^2 \subset \subset \mathbf{R}(x^2) \cap \mathbf{L}(x^2) = \mathbf{H}(x^2)$. On the other hand, we have $x^2 \in xS \cap Sx \subset \mathbf{R}(x) \cap \cap \mathbf{L}(x) = \mathbf{H}(x)$. It follows from Lemma 1 and Theorem 3 that $x^2 \in \mathbf{H}_x = \mathbf{R}_x \cap \mathbf{L}_x$. According to [2] S is a union of groups.

A semigroup S is called *intraregular if* $x \in Sx^2S$ for every $x \in S$.

Theorem 9. *The following conditions on a semigroup S are equivalent*:

1. *S is intraregular*;
2. $\mathbf{P}^* \leqq \mathbf{M}$;
3. $\mathbf{K} \leqq \mathbf{M}$;
4. $\overline{\mathbf{K}} \subset \overline{\mathbf{M}}$.

(See [4].)

Proof. 1 ⇒ 2. Let S be an intraregular semigroup. Let A be a two-sided ideal of S. If $x^n \in A$ $(x \in S,\ n \geqq 2)$, then there exist $a, b \in S$ such that $x^{n-1} = ax^{2(n-1)}b \in \in Sx^nS \subset SAS \subset A$. Similarly we obtain $x^{n-i} \in A$ for any positive integer $i < n$. This implies $x \in A$ and it follows from Lemma 2 that $A \in \mathscr{F}(\mathbf{P}^*)$ so that, by (11), $\mathbf{P}^* \leqq \mathbf{M}$.

2 ⇒ 3 ⇒ 4. The proof is analogous to the proof of Theorem 6.

4 ⇒ 1. If $\overline{\mathbf{K}} \subset \overline{\mathbf{M}}$, then by Lemma 4 and Theorem 5 we have $\mathbf{P}^* \leqq \mathbf{K} \leqq \mathbf{M}$. It follows from (11) that $x^2 \in \mathbf{M}(x^2) \in \mathscr{F}(\mathbf{M}) \subset \mathscr{F}(\mathbf{P}^*)$. According to Lemma 2, $x \in \in \mathbf{M}(x^2) = S^1x^2S^1$. We shall prove that $x \in Sx^2S$. If $x \in Sx^2$, then $x = ax^2$ for some $a \in S$, thus $x = a(ax^2)\,x \in Sx^2S$. Similarly, $x \in x^2S$ implies $x \in Sx^2S$. If $x = x^2$, then $x = x^4 \in Sx^2S$. Hence, S is intraregular.

Remark 1. If S is a periodic semigroup, then from Corollary 2.3 [1], Theorem 3.8 [1] we have:

The conditions of Theorems 6, 7, 8 *and* 9 *and the following condition on a periodic semigroup S are equivalent*

$$\overline{\mathbf{K}} = \overline{\mathbf{H}}\,.$$

A semigroup S is called *left* (*right*) *weakly commutative* if for every $a, b \in S$ there exist $x \in S$ and a positive integer k such that $(ab)^k = bx$ $((ab)^k = xa)$.

Lemma 6. *If* $\mathbf{L} \leqq \mathbf{R}$, *then a semigroup S is left weakly commutative.*

Proof. Let $a, b \in S$. By (6) we have $ab \in S^1 b = \mathbf{L}(b) \subset \mathbf{R}(b) = bS^1$. If $ab = bx$ for some $x \in S$, then $(ab)^1 = bx$. If $ab = b$, then $(ab)^2 = b(ab)$. Hence, S is left weakly commutative.

Lemma 7. *If $\mathbf{R} \leqq \mathbf{L}$, then the semigroup S is right weakly commutative.*

Lemma 8. *If S is a right regular and left weakly commutative semigroup, then $\mathbf{L} \leqq \mathbf{R}$.*

Proof. Let $a \in S$. If $x \in Sa$, then $x = ua$ for some $u \in S$. Thus the hypothesis that S is left weakly commutative implies that there exists $v \in S$ and a positive integer k such that $x^k = (ua)^k = av \in aS \in \mathscr{F}(\mathbf{R})$. According to Theorem 6, (11) and Lemma 2, we have $x \in aS$. Hence $Sa \subset aS$. This shows that $\mathbf{L}(a) \subset \mathbf{R}(a)$ for every $a \in S$. Therefore, by (9), we have $\mathbf{L} \leqq \mathbf{R}$.

Lemma 9. *If S is a left regular and right weakly commutative semigroup, then $\mathbf{R} \leqq \mathbf{L}$.*

Theorem 10. *The following conditions on a semigroup S are equivalent:*

1. *S is a semilattice of right groups;*
2. *S is a union of groups and $\mathbf{L} \leqq \mathbf{R}$;*
3. *S is a union of groups and it is left weakly commutative;*
4. *$\mathbf{P}^* \leqq \mathbf{L} \leqq \mathbf{R}$;*
5. *$\mathbf{K} \leqq \mathbf{L} \leqq \mathbf{R}$;*
6. *$\overline{\mathbf{K}} \subset \overline{\mathbf{L}} \subset \overline{\mathbf{R}}$.*

Proof. $1 \Rightarrow 2$. It follows from Theorem 2 [5] that S is a union of groups. Let $a \in S$. If $x \in Sa$, then $x = ua$ for some $u \in S$. Let e and f be an identity for a and for x, respectively. Similarly, let a^{-1} and x^{-1} be an inverse for a and for x, respectively. Since $x = ua = uae = xe$, hence $f = x^{-1}x = x^{-1}xe = fe$. By Theorem 2 [5] we have $f = efe$. Thus $ef = f$. Then $x = fx = efx = ex = (aa^{-1})x = a(a^{-1}x)$. This implies $x \in aS$. Consequently $Sa \subset aS$ and we have thus $\mathbf{L}(a) \subset \mathbf{R}(a)$. By (9) we obtain $\mathbf{L} \leqq \mathbf{R}$.

$2 \Rightarrow 3$. This follows from Lemma 6.

$3 \Rightarrow 4$. This follows from Theorem 8 and Lemma 8.

$4 \Rightarrow 5$. This follows from Theorem 7.

$5 \Rightarrow 6$. This follows from Theorem 2.

$6 \Rightarrow 1$. It follows from Theorem 3 that $\overline{\mathbf{L}} = \overline{\mathbf{H}}$ and $\overline{\mathbf{K}} \subset \overline{\mathbf{H}}$. By Theorem 8, S is a union of groups. Let e and f be idempotents of S. Put $y = fe$. Let g and y^{-1} be an identity and an inverse for y, respectively. Since $y = yg = feg \in Seg$ and $eg = ey^{-1}y \in Sy$, hence $\mathbf{L}(y) = \mathbf{L}(eg)$. Now the hypothesis that $\overline{\mathbf{L}} \subset \overline{\mathbf{R}}$ implies $\mathbf{R}(y) = \mathbf{R}(eg)$. From this it follows that $y = eg$ or $y = egu$ for some $u \in S$. Then $y \in eS$ and therefore $efe = ey = y = fe$. It follows from Theorem 2 [5] that S is a semilattice of right groups.

Remark 2. The following example shows that the implication

$$\overline{\mathbf{L}} \subset \overline{\mathbf{R}} \Rightarrow \mathbf{L} \leqq \mathbf{R}$$

on a semigroup S does not hold in general.

Let $S = \{(i, n - i)/\text{for all positive integers } n \text{ and for } i = 0, 1\}$. Define in S a multiplication by

$$xy = (i, n + m)$$

where $x = (i, n) \in S$ and $y = (j, m) \in S$. Then S is a semigroup (see [6]). It is clear that $\overline{\mathbf{L}} \subset \overline{\mathbf{R}}$. On the other hand, if $a = (1, 0)$, then $\mathbf{R}(a) = aS \subsetneqq S = \mathbf{L}(a)$ and thus $\mathbf{L} \nleqq \mathbf{R}$.

Remark 3. If S a is periodic semigroup, then from Theorem 3 and from Remark 1 we have:

The conditions of Theorem 10 *and the following condition on a periodic semigroup S are equivalent*:

$$\overline{\mathbf{K}} = \overline{\mathbf{L}}.$$

The dual statement reads as follows:

Theorem 11. *The following conditions on a semigroup S are equivalent*:

1. *S is a semilattice of left groups*;
2. *S is a union of groups and* $\mathbf{R} \leqq \mathbf{L}$;
3. *S is a union of groups and it is right weakly commutative*;
4. $\mathbf{P}^* \leqq \mathbf{R} \leqq \mathbf{L}$;
5. $\mathbf{K} \leqq \mathbf{R} \leqq \mathbf{L}$;
6. $\overline{\mathbf{K}} \subset \overline{\mathbf{R}} \subset \overline{\mathbf{L}}$.

Remark 4. *The conditions of Theorem* 11 *and the following condition on a periodic semigroup S are equivalent*:

$$\overline{\mathbf{K}} = \overline{\mathbf{R}}.$$

A semigroup S is called *weakly commutative* if for every $a, b \in S$ there exist $x, y \in S$ and a positive integer k such that

$$(ab)^k = xa = by.$$

Lemma 10. *A semigroup S is weakly commutative if and only if it is left weakly commutative and right weakly commutative.*

Proof. If S is a weakly commutative semigroup, then it is clear that S is left and right weakly commutative.

Suppose that S is left weakly commutative and right weakly commutative. Then there exist $x, y \in S$ and positive integers k, l such that

$$(ab)^k = xa\,, \quad (ab)^l = by\,.$$

This implies that $(ab)^{k+l} = ua = bv$ where $u = (ab)^l x$, $v = y(ab)^k$.

Theorem 12. *The following conditions on a semigroup S are equivalent:*

1. *S is a semilattice of groups;*
2. *S is a union of groups and $\mathbf{L} = \mathbf{R}$;*
3. *S is a union of groups and it is weakly commutative;*
4. $\mathbf{P}^* \leqq \mathbf{L} = \mathbf{R}$;
5. $\mathbf{K} \leqq \mathbf{L} = \mathbf{R}$;
6. $\overline{\mathbf{K}} \subset \overline{\mathbf{L}} = \overline{\mathbf{R}}$.

Proof follows from Theorem 2 [5], Corollary 2 [5], Theorem 10, Theorem 11 and Lemma 10.

Remark 5. *The conditions of Theorem* 12 *and the following conditions on a periodic semigroup S are equivalent:*

1. $\overline{\mathbf{K}} = \overline{\mathbf{L}} = \overline{\mathbf{R}}$;
2. $\overline{\mathbf{K}} = \overline{\mathbf{M}}$.

Proof. Conditions of Theorem 12 $\Leftrightarrow$ 1. This follows from Theorem 12, Remark 3 and Remark 4.

1 $\Rightarrow$ 2. It follows from Theorem 12 that $\mathbf{L} = \mathbf{R}$. Then $\mathbf{L} = \mathbf{M}$ and thus, by Remark 3, $\overline{\mathbf{K}} = \overline{\mathbf{L}} = \overline{\mathbf{M}}$.

2 $\Rightarrow$ 1. According to Remark 1, we have $\overline{\mathbf{M}} = \overline{\mathbf{K}} = \overline{\mathbf{H}}$. It follows from Theorem 2 that $\overline{\mathbf{K}} = \overline{\mathbf{L}} = \overline{\mathbf{R}}$.

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