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A certain subclass of meromorphically q-starlike functions associated with the Janowski functions

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Abstract

In this paper, the authors introduce a new subclass of meromorphic q-starlike functions which are associated with the Janowski functions. A characterization of meromorphically q-starlike functions associated with the Janowski functions has been obtained when the coefficients in their Laurent series expansion about the origin are all positive. This leads to a study of coefficient estimates, distortion theorems, partial sums, and the radius of starlikeness estimates for this class. It is seen that the class considered demonstrates, in some respects, properties analogous to those possessed by the corresponding class of univalent analytic functions with negative coefficients.

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1 Introduction and definitions

Let the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

be denoted by $\mathcal{H}(\mathbb{U})$, and let $\mathcal{A}$ denote the class of all functions f, which are analytic in the unit disk $\mathbb{U}$ and normalized by

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

Thus, each $f \in \mathcal{A}$ has a Taylor–Maclaurin series representation as follows:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (\forall z \in \mathbb{U}). \tag{1.1}$$

Also, let $\mathcal{S}$ be the subclass of analytic function class $\mathcal{A}$, consisting of all univalent functions in $\mathbb{U}$.

Furthermore, let $\mathcal{P}$ denote the well-known Carathéodory class of functions p, analytic in the open unit disk $\mathbb{U}$, which are normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{1.2}$$

such that

$$\Re\{p(z)\} > 0 \quad (\forall z \in \mathbb{U}).$$

The intrinsic properties of q-analogs, including the applications in the study of quantum groups and q-deformed super-algebras, study of fractals and multi-fractal measures, and in chaotic dynamical systems, are known in the literature. Some integral transforms in the classical analysis have their q-analogues in the theory of q-calculus. This has led various researchers in the field of q-theory to extending all the important results involving the classical analysis to their q-analogs.

For the convenience, we provide some basic definitions and concept details of q-calculus which are used in this paper. Throughout this paper, we will assume that q satisfies the condition $0 < q < 1$. We shall follow the notation and terminology of [7]. We first recall the definitions of fractional q-calculus operators of complex valued function f.

Definition 1 (see [7]) Let $q \in (0,1)$ and define the q-number $[\lambda]_q$ by

$$[\lambda]_q = \begin{cases} \frac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}), \\ \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases}$$

Definition 2 (see [9] and [10]) The q-derivative (or q-difference) D_q of a function f is defined in a given subset of $\mathbb{C}$ by

$$(D_q f)(z) = \begin{cases} \frac{f(qz)-f(z)}{(q-1)z} & (z \neq 0), \\ f'(0) & (z = 0) \end{cases} \tag{1.3}$$

provided that $f'(0)$ exists.

From Definition 2, we can observe that

$$\lim_{q \to 1-} (D_q f)(z) = \lim_{q \to 1-} \frac{f(qz) - f(z)}{(q-1)z} = f'(z)$$

for a differentiable function f in a given subset of $\mathbb{C}$. It is readily known from (1.1) and (1.3) that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \tag{1.4}$$

A number of subclasses of normalized analytic function class $\mathcal{A}$ in geometric function theory have been studied already from different viewpoints. The above defined q-calculus provides an important tool in order to investigate several subclasses of class $\mathcal{A}$. A firm footing usage of the q-calculus in the context of geometric function theory was presented mainly and basic (or q-) hypergeometric functions were first used in geometric function theory in a book chapter by Srivastava (see, for details, [17, pp. 347 *et seq.*]; see also [18]).

Recently Srivastava *et al.* [20] successfully combined the concept of Janowski [11] and the above mentioned q-calculus and defined the following.

Definition 3 A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_q^*[A,B]$ if and only if

$$\left|\frac{(B-1)(\frac{z(D_qf)(z)}{f(z)})-(A-1)}{(B+1)(\frac{z(D_qf)(z)}{f(z)})-(A+1)}-\frac{1}{1-q}\right|<\frac{1}{1-q}.$$

From Definition 3, one can easily observe that

1. $\lim_{q\to 1-}\mathcal{S}_q^*[A,B]=\mathcal{S}^*[A,B]$, where $\mathcal{S}^*[A,B]$ is the function class introduced and studied by Janowski [11].
2. $\mathcal{S}_q^*[1,-1]=\mathcal{S}_q^*$, where $\mathcal{S}_q^*$ is the function class introduced and studied by Ismail *et al.* [8].
3. When

$$A=1-2\alpha \quad (0\leq\alpha<1) \quad \text{and} \quad B=-1$$

and let $q\to 1-$, the class $\mathcal{S}_q^*[A,B]$ reduces to the function class $\mathcal{S}^*(\alpha)$, which was introduced and studied by Silverman (see [15]). Moreover, its worthy of note that $\mathcal{S}^*(0)=\mathcal{S}^*$, where $\mathcal{S}^*$ is a well-known function class of starlike functions.

4. When

$$A=1-2\alpha \quad (0\leq\alpha<1) \quad \text{and} \quad B=-1,$$

the class $\mathcal{S}_q^*[A,B]$ reduces to the function class $\mathcal{S}_q^*(\alpha)$, which was introduced and studied recently by Agrawal and Sahoo (see [1]).

Let $\mathcal{M}$ denote the class of functions f of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n, \tag{1.5}$$

that are analytic in the punctured open unit disk

$$\mathbb{U}^*=\{z:z\in\mathbb{C} \text{ and } 0<|z|<1\}=\mathbb{U}\backslash\{0\}$$

with a simple pole at the origin with residue 1 there. Let $\mathcal{MS}^*(\alpha)$ $(0\leq\alpha<1)$ denote the subclasses of $\mathcal{M}$ meromorphically starlike of order α. Analytically f of the form (1.5) is in $\mathcal{MS}^*(\alpha)$ if and only if

$$\Re\left(-\frac{zf'(z)}{f(z)}\right)>\alpha \quad (z\in\mathbb{U}).$$

The sufficient condition for a function f to be in the class $\mathcal{MS}^*(\alpha)$ is given by

$$\sum_{n=1}^{\infty}(n+\alpha)|a_n| \leq 1-\alpha \quad (0 \leq \alpha < 1). \tag{1.6}$$

The class $\mathcal{MS}^*(\alpha)$ and similar other classes have been extensively studied by Pommerenke [13], Clunie [5], Miller [12], Royster [14], and others.

Since to a certain extent the work in the meromorphic univalent case has paralleled that of the analytic univalent case, one is tempted to search results analogous to those of Silverman [16] for meromorphic univalent functions in $\mathbb{U}^*$. Several different subclasses of meromorphic univalent function class $\mathcal{M}$ were introduced and studied analogously by the many authors; see, for example, [3, 4, 6, 19, 21]. However, analogous to Definition 2, we extend the idea of q-difference operator to a function f given by (1.5) from the class $\mathcal{M}$ and also define analogous of meromorphic analogy of the function class $\mathcal{S}_q^*[A,B]$.

Definition 4 For $f \in \mathcal{M}$, let the q-derivative operator (or q-difference operator) be defined by

$$(D_q f)(z) = \frac{f(qz)-f(z)}{(q-1)z} = -\frac{1}{qz^2} + \sum_{n=0}^{\infty}[n]_q a_n z^{n-1} \quad \bigl(\forall z \in \mathbb{U}^*\bigr).$$

Definition 5 A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{MS}_q^*[A,B]$ if and only if

$$\left|\frac{(B-1)(-\frac{z(D_q f)(z)}{f(z)})-(A-1)}{(B+1)(-\frac{z(D_q f)(z)}{f(z)})-(A+1)} - \frac{1}{1-q}\right| < \frac{1}{1-q}.$$

Remark 1 First of all, it is easily seen that

$$\lim_{q\to 1-} \mathcal{MS}_q^*[A,B] = \mathcal{MS}^*[A,B],$$

where $\mathcal{MS}^*[A,B]$ is a function class, introduced and studied by Ali *et al.* [2]. Secondly we have

$$\lim_{q\to 1-} \mathcal{MS}_q^*[1,-1] = \mathcal{MS}^*,$$

where $\mathcal{MS}^*$ is the well-known function class of meromorphic starlike functions. This function class and similar other classes have been extensively studied by Pommerenke [13], Clunie [5], Miller [12], Royster [14], and others.

In the present paper, we give a sufficient condition for a function f to be in the class $\mathcal{MS}_q^*[A,B]$, which will be used as a supporting result for further investigation the remainder of this article. Distortion inequalities, as well as results concerning the radius of starlikeness, are obtained. We will investigate the ratio of a function of the form (1.5) to its sequence of partial sums

$$f_k(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n \quad (k \in \mathbb{N}), \tag{1.7}$$

when the coefficients are sufficiently small. Throughout this paper, unless otherwise mentioned, we will assume that

$$-1 \leq B < A \leq 1 \quad \text{and} \quad q \in (0,1).$$

2 Coefficient estimates

In this section, we give a sufficient condition for a function f to be in the class $\mathcal{MS}_q^*[A,B]$, which will work as one of the key results to find other results of this paper.

Theorem 1 *A function $f \in \mathcal{M}$ of the form given by* (1.5) *is in the class $\mathcal{MS}_q^*[A,B]$ if it satisfies the following condition*:

$$\sum_{n=1}^{\infty} \Lambda(n,A,B,q)|a_n| \leq \Upsilon(A,B,q), \tag{2.1}$$

where

$$\Lambda(n,A,B,q) = 2\big([n]_q + 1\big) + \big|(B+1)[n]_q - (A-1)\big|q \tag{2.2}$$

and

$$\Upsilon(A,B,q) = \big|(B+1) - (A+1)q\big| + 2(1-q). \tag{2.3}$$

Proof Assuming that (2.1) holds, it suffices to show that

$$\left|\frac{(B-1)(-\frac{z(D_qf)(z)}{f(z)}) - (A-1)}{(B+1)(-\frac{z(D_qf)(z)}{f(z)}) - (A+1)} - \frac{1}{1-q}\right| < \frac{1}{1-q}.$$

Now we have

$$\begin{aligned}
&\left|\frac{(B-1)(-\frac{z(D_qf)(z)}{f(z)}) - (A-1)}{(B+1)(-\frac{z(D_qf)(z)}{f(z)}) - (A+1)} - \frac{1}{1-q}\right| \\
&\quad \leq \left|\frac{-(B-1)z(D_qf)(z) - (A-1)f(z)}{-(B+1)z(D_qf)(z) - (A+1)f(z)} - 1\right| + \frac{q}{1-q} \\
&\quad = 2\left|\frac{z(D_qf)(z) + f(z)}{-(B+1)z(D_qf)(z) - (A+1)f(z)}\right| + \frac{q}{1-q} \\
&\quad = 2\left|\frac{(q-1) + \sum_{n=1}^{\infty}(1+[n]_q)qa_nz^{n+1}}{(B+1) - (A+1)q - \sum_{n=1}^{\infty}((B+1)[n]_q - (A-1))qa_nz^{n+1}}\right| + \frac{q}{1-q} \\
&\quad \leq 2\frac{(q-1) + \sum_{n=1}^{\infty}(1+[n]_q)q|a_n|}{|(B+1) - (A+1)q| - \sum_{n=1}^{\infty}|(B+1)[n]_q - (A-1)|q|a_n|} + \frac{q}{1-q}.
\end{aligned} \tag{2.4}$$

The last expression in (2.4) is bounded above by $\frac{1}{1-q}$ if

$$\sum_{n=2}^{\infty} \Lambda(n,A,B,q)q|a_n| < \Upsilon(A,B,q),$$

where $\Lambda(n,A,B,q)$ and $\Upsilon(A,B,q)$ are given by (2.2) and (2.3) respectively. The proof of Theorem 1 is thus completed. □

It is easy to deduce the following consequence of Theorem 1.

Corollary 1 *If a function $f \in \mathcal{M}$ of the form given by (1.5) is in the class $\mathcal{MS}_q^*[A,B]$, then*

$$a_n \leq \frac{\Upsilon(A,B,q)}{\Lambda(n,A,B,q)} \quad (n \in \mathbb{N}) \tag{2.5}$$

with equality for each n, with the function of the form

$$f_n(z) = \frac{1}{z} + \frac{\Upsilon(A,B,q)}{\Lambda(n,A,B,q)} z^n,$$

where $\Upsilon(A,B,q)$ and $\Lambda(n,A,B,q)$ are given by (2.2) and (2.3) respectively.

3 Distortion inequalities

Theorem 2 *If $f \in \mathcal{MS}_q^*[A,B]$, then*

$$\frac{1}{r} - \frac{\Upsilon(A,B,q)}{\Lambda(1,A,B,q)} r \leq \bigl|f(z)\bigr| \leq \frac{1}{r} + \frac{\Upsilon(A,B,q)}{\Lambda(1,A,B,q)} r \quad \bigl(|z| = r\bigr),$$

where equality holds for the function

$$f_1(z) = \frac{1}{z} + \frac{\Upsilon(A,B,q)}{\Lambda(1,A,B,q)} z \quad \text{at } z = ir,$$

with $\Lambda(n,A,B,q)$ and $\Upsilon(A,B,q)$ given by (2.2) and (2.3) respectively.

Proof Let $f \in \mathcal{MS}_q^*[A,B]$. Then, in view of Theorem 1, we have

$$\Lambda(1,A,B,q) \sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} \Lambda(n,A,B,q)|a_n| < \Upsilon(A,B,q),$$

which yields

$$\bigl|f(z)\bigr| \leq \frac{1}{r} + \sum_{n=1}^{\infty} |a_n| r^n \leq \frac{1}{r} + r \sum_{n=1}^{\infty} |a_n| \leq \frac{1}{r} + \frac{\Upsilon(A,B,q)}{\Lambda(1,A,B,q)} r.$$

Similarly, we have

$$\bigl|f(z)\bigr| \geq \frac{1}{r} - \sum_{n=1}^{\infty} |a_n| r^n \geq \frac{1}{r} - r \sum_{n=1}^{\infty} |a_n| \geq \frac{1}{r} - \frac{\Upsilon(A,B,q)}{\Lambda(1,A,B,q)} r.$$

We have thus completed the proof of Theorem 2. □

The following result (Theorem 3) can be proved by using arguments similar to those that have already been presented in the proof of Theorem 2. So we choose to omit the details of our proof of Theorem 3.

Theorem 3 *If $f \in \mathcal{MS}_q^*[A,B]$, then*

$$\frac{1}{r^2} - \frac{2\Upsilon(A,B,q)}{\Lambda(1,A,B,q)} \leq \left|f'(z)\right| \leq \frac{1}{r^2} + \frac{2\Upsilon(A,B,q)}{\Lambda(1,A,B,q)} \quad \left(|z| = r\right)$$

and $\Lambda(k,A,B,q)$ and $\Upsilon(A,B,q)$ are given by (2.2) *and* (2.3) *respectively.*

4 Partial sums for the function class $\mathcal{MS}_q^*[A,B]$

In this section, we examine the ratio of a function of the form (1.5) to its sequence of partial sums

$$f_n(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n$$

when the coefficients of f are sufficiently small to satisfy condition (2.1). We will determine sharp lower bounds for

$$\Re\left(\frac{f(z)}{f_k(z)}\right), \qquad \left(\frac{f_k(z)}{f(z)}\right), \qquad \Re\left(\frac{D_q f(z)}{D_q f_k(z)}\right) \quad \text{and} \quad \Re\left(\frac{(D_q f_k)(z)}{(D_q f)(z)}\right).$$

Unless otherwise stated, we will assume that f is of the form (1.5) and that its sequence of partial sums is denoted by

$$f_k(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n.$$

Theorem 4 *If f of the form* (1.5) *satisfies condition* (2.1), *then*

$$\Re\left(\frac{f(z)}{f_k(z)}\right) \geq 1 - \frac{1}{\xi_{k+1}} \quad (\forall z \in \mathbb{U}) \tag{4.1}$$

and

$$\Re\left(\frac{f_k(z)}{f(z)}\right) \geq \frac{\xi_{k+1}}{1+\xi_{k+1}} \quad (\forall z \in \mathbb{U}), \tag{4.2}$$

where

$$\xi_k = \frac{\Lambda(k,A,B,q)}{\Upsilon(A,B,q)} \tag{4.3}$$

and $\Lambda(k,A,B,q)$ and $\Upsilon(A,B,q)$ are given by (2.2) *and* (2.3) *respectively.*

Proof In order to prove inequality (4.1), we set

$$\begin{aligned} \xi_{k+1}\left[\frac{f(z)}{f_j(z)} - \left(1 - \frac{1}{\xi_{k+1}}\right)\right] &= \frac{1 + \sum_{n=1}^{k} a_n z^{n-1} + \xi_{k+1}\sum_{n=k+1}^{\infty} a_n z^{n+1}}{1 + \sum_{n=1}^{k} a_n z^{n+1}} \\ &= \frac{1 + h_1(z)}{1 + h_2(z)}. \end{aligned}$$

If we set

$$\frac{1+h_1(z)}{1+h_2(z)}=\frac{1+w(z)}{1-w(z)},$$

then we find, after some suitable simplification, that

$$w(z)=\frac{h_1(z)-h_2(z)}{2+h_1(z)+h_2(z)}.$$

Thus, clearly, we find that

$$w(z)=\frac{\xi_{k+1}\sum_{n=k+1}^{\infty}a_nz^{n-1}}{2+2\sum_{n=1}^{k}a_nz^{n+1}+\xi_{k+1}\sum_{n=k+1}^{\infty}a_nz^{n+1}}$$

and

$$\left|w(z)\right|\leq\frac{\xi_{k+1}\sum_{n=k+1}^{\infty}|a_n|}{2-2\sum_{n=1}^{k}|a_n|-\xi_{k+1}\sum_{n=k+1}^{\infty}|a_n|}.$$

Now one can see that

$$\left|w(z)\right|\leq 1$$

if and only if

$$2\xi_{k+1}\sum_{n=k+1}^{\infty}|a_n|\leq 2-2\sum_{n=1}^{k}|a_n|,$$

which implies that

$$\sum_{n=1}^{k}|a_n|+\xi_{k+1}\sum_{n=k+1}^{\infty}|a_n|\leq 1. \tag{4.4}$$

Finally, to prove the inequality in (4.1), it suffices to show that the left-hand side of (4.4) is bounded above by $\sum_{n=1}^{\infty}\xi_n|a_n|$, which is equivalent to

$$\sum_{n=1}^{k}(1-\xi_n)|a_n|+\sum_{n=k+1}^{\infty}(\xi_{k+1}-\xi_n)|a_n|\geq 0. \tag{4.5}$$

By virtue of (4.5), the proof of inequality in (4.1) is now completed.

Next, in order to prove inequality (4.2), we set

$$\begin{aligned}(1+\xi_k)\left(\frac{f_k(z)}{f(z)}-\frac{\xi_k}{1+\xi_k}\right)&=\frac{1+\sum_{n=1}^{k}a_nz^{n-1}-\xi_{k+1}\sum_{n=k+1}^{\infty}a_nz^{n-1}}{1+\sum_{n=1}^{\infty}a_nz^{n-1}}\\&=\frac{1+w(z)}{1-w(z)},\end{aligned}$$

where

$$\left|w(z)\right| \leq \frac{(1+\xi_{k+1})\sum_{n=k+1}^{\infty}|a_n|}{2-2\sum_{n=1}^{k}|a_n|-(\xi_{k+1}-1)\sum_{n=k+1}^{\infty}|a_n|} \leq 1. \tag{4.6}$$

This last inequality in (4.6) is equivalent to

$$\sum_{n=1}^{k}|a_n| + \xi_{k+1}\sum_{n=k+1}^{\infty}|a_n| \leq 1. \tag{4.7}$$

Finally, we can see that the left-hand side of the inequality in (4.7) is bounded above by $\sum_{n=1}^{\infty}\xi_n|a_n|$, and so we have completed the proof of (4.2), which completes the proof of Theorem 4. □

We next turn to ratios involving derivatives.

Theorem 5 *If f of the form* (1.1) *satisfies condition* (2.1), *then*

$$\Re\left(\frac{(D_qf)(z)}{(D_qf_k)(z)}\right) \geq 1-\frac{[k+1]_q}{\xi_{k+1}} \quad (\forall z\in\mathbb{U}) \tag{4.8}$$

and

$$\Re\left(\frac{(D_qf_k)(z)}{(D_qf)(z)}\right) \geq \frac{\xi_{k+1}}{\xi_{k+1}+[k+1]_q} \quad (\forall z\in\mathbb{U}), \tag{4.9}$$

where ξ_k is given by (4.3).

Proof The proof of Theorem 5 is similar to that of Theorem 4, we here choose to omit the analogous details. □

5 Radius of starlikeness

In the following theorem we obtain the radius of q-starlikeness for the class $\mathcal{MS}_q^*[A,B]$, we say that f given by (1.5) is meromorphically starlike of order α $(0\leq\alpha<1)$ in $|z|<r$ when it satisfies condition (1.6) in $|z|<r$.

Theorem 6 *Let the function f given by* (1.5) *be in the class $\mathcal{MS}_q^*[A,B]$. Then, if*

$$\inf_{n\geq 1}\left[\frac{(1-\alpha)\Lambda(n,A,B,q)}{(n+2-\alpha)\Upsilon(A,B,q)}\right]^{\frac{1}{n+1}} = r$$

is positive, then f is meromorphically starlike of order α in $|z|\leq r$, where $\Lambda(n,A,B,q)$ and $\Upsilon(A,B,q)$ are given by (2.2) *and* (2.3) *respectively.*

Proof In order to prove the result, we must show that

$$\left|\frac{zf'(z)}{f(z)}+1\right| \leq 1-\alpha \quad (0\leq\alpha<1) \quad \text{and} \quad |z|\leq r_1.$$

We have

$$\left|\frac{zf'(z)}{f(z)}+1\right| = \left|\frac{\sum_{n=1}^{\infty}(n+\alpha)a_n z^n}{\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n}\right|$$
$$\leq \frac{\sum_{n=1}^{\infty}(n+\alpha)|a_n||z|^{n+1}}{1-\sum_{n=1}^{\infty}|a_n||z|^{n+1}}. \tag{5.1}$$

Hence (5.1) holds true if

$$\sum_{n=1}^{\infty}(n+\alpha)|a_n||z|^{n+1} \leq (1-\alpha)\left(1-\sum_{n=1}^{\infty}|a_n||z|^{n+1}\right). \tag{5.2}$$

The inequality in (5.2) can be written as

$$\sum_{n=1}^{\infty}\left(\frac{n+2-\alpha}{1-\alpha}\right)|a_n||z|^{n+1} \leq 1. \tag{5.3}$$

With the aid of (2.1), inequality (5.3) is true if

$$\left(\frac{n+2-\alpha}{1-\alpha}\right)|z|^{n+1} \leq \frac{\Lambda(n,A,B,q)}{\Upsilon(A,B,q)}. \tag{5.4}$$

Solving (5.4) for $|z|$, we have

$$|z| \leq \left(\frac{(1-\alpha)\Lambda(n,A,B,q)}{(n+2-\alpha)\Upsilon(A,B,q)}\right)^{\frac{1}{n+1}}. \tag{5.5}$$

In view of (5.5), the proof of our theorem is now completed. □

6 Concluding remarks and observations

In our present investigation, we have introduced and studied systematically a new subclass of the class of the meromorphically q-starlike functions, which is associated with the Janowski functions. We have given a characterization of these meromorphically q-starlike functions associated with the Janowski functions when the coefficients in the Laurent series expansion about the origin are all positive. This has led us to a study of coefficient estimates, distortion theorems, partial sums and estimates of the radius of q-starlikeness for this meromorphic function class. We have observed that the class considered in this article demonstrates, in some respects, properties analogous to those possessed by the corresponding class of univalent analytic functions with negative coefficients.

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