A certifying algorithm for 3-colorability of P_5-free graphs

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Abstract. We provide a certifying algorithm for the problem of deciding whether a P_5-free graph is 3-colorable by showing there are exactly six finite graphs that are P_5-free and not 3-colorable and minimal with respect to this property.

1 Introduction

An algorithm is *certifying* if it returns with each output a simple and easily verifiable certificate that the particular output is correct. For example, a certifying algorithm for the bipartite graph recognition would return either a 2-coloring of the input graph proving that it is bipartite, or an odd cycle proving it is not bipartite. A certifying algorithm for planarity would return a planar embedding or one of the two Kuratowski subgraphs. The notion of certifying algorithm [9] was developed when researchers noticed that a well known planarity testing program was incorrectly implemented. A certifying algorithm is a desirable tool to guard against incorrect implementation of a particular algorithm. In this paper, we give a certifying algorithm for the problem of deciding whether a P_5-free graph is 3-colorable. We will now discuss the background of this problem.

A class C of graphs is called *hereditary* if for each graph G in C, all induced subgraphs of G are also in $\mathcal{C}$. Every hereditary class of graphs can be described by its *forbidden induced subgraphs*, i.e. the unique set of minimal graphs which do not belong to the class. A comprehensive survey on coloring of graphs in hereditary classes can be found in [12]. An important line of research on colorability of graphs in hereditary classes deals with P_t-free graphs. The induced path on t vertices is called P_t, and a graph is called P_t*-free* if it does not contain P_t as an induced subgraph.

It is known that 4-COLORABILITY is NP-complete for P_9-free graphs [14] and 5-COLORABILITY is NP-complete for P_8-free graphs [10]. And most recently it was proved that 6-COLORABILITY is NP-complete for P_7-free [2]. On the other hand, the k-COLORABILITY problem can be solved in polynomial time for P_4-free graphs (since they are perfect). In [5] and [6], it is shown that k-COLORABILITY can be solved for the class of P_5-free graphs in polynomial time for every particular value of k. For $t = 6, 7$, the complexity of the problem is generally unknown, except for the case

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of 3-COLORABILITY of P_6-free graphs [13]. Known results on the k-COLORABILITY problem in P_t-free graphs are summarized in Table 1 (n is the number of vertices in the input graph, m the number of edges, and α is matrix multiplication exponent known to satisfy $2 \le \alpha < 2.376$ [3]).

$k \backslash t$	3	4	5	6	7	8	9	10	11	12	...
3	$O(m)$	$O(m)$	$O(n^\alpha)$	$O(mn^\alpha)$	?	?	?	?	?	?	...
4	$O(m)$	$O(m)$	P	?	?	?	NP_c	NP_c	NP_c	NP_c	...
5	$O(m)$	$O(m)$	P	?	?	NP_c	NP_c	NP_c	NP_c	NP_c	...
6	$O(m)$	$O(m)$	P	?	NP_c	NP_c	NP_c	NP_c	NP_c	NP_c	...
7	$O(m)$	$O(m)$	P	?	NP_c	NP_c	NP_c	NP_c	NP_c	NP_c	...
...	...	...	...	...	...	...	...	...	...	...	...

Table 1. Known complexities for k-colorability of P_t-free graphs

In this paper, we study the coloring problem for the class of P_5-free graphs. This class has proved resistant with respect to other graph problems. For instance, P_5-free graphs is the unique minimal class defined by a single forbidden induced subgraph with unknown complexity of the MAXIMUM INDEPENDENT SET and MINIMUM INDEPENDENT DOMINATING SET problems. Many algorithmic problems are known to be NP-hard in the class of P_5-free graphs, for example DOMINATING SET [7] and CHROMATIC NUMBER [8]. In contrast to the NP-hardness of finding the chromatic number of a P_5-free graph, it is known [5] that k-COLORABILITY can be solved in this class in polynomial time for every particular value of k. This algorithm produces a k-coloring if one exists, but does not produce an easily verifiable certificate when such coloring does not exist. We are interested in finding a certificate for non-k-colorability of P_5-free graphs. For this purpose, we start with $k = 3$.

Besides [5], there are several polynomial-time algorithms for 3-coloring a P_5-free graph ([6, 11, 14]) but none of them is a certifying algorithm. In this paper, we obtain a certifying algorithm for 3-coloring a P_5-free graphs by proving there are a finite number of minimally non-3-colorable P_5-free graphs and each of these graphs is finite.

Theorem 1.1. *A P_5-free graph is 3-colorable if and only if it does not contain any of the six graphs in Fig. 1 as a subgraph.*

It is an easy matter to verify the graphs in Fig. 1 are not 3-colorable, the rest of the paper involves proving the other direction of the theorem. In the last Section, we will discuss open problems arising from our work.

2 Definition and Background

Let k and t be positive integers. An MNkPt is a graph G that (i) is not k-colorable and is P_t-free and (ii) every proper subgraph of G is either k-colorable or has a P_t. We will be interested specifically in the case where $k = 3$ and $t = 5$. We will use the following notations. Let G be a simple undirected graph. A set S of vertices of G is *dominating* if

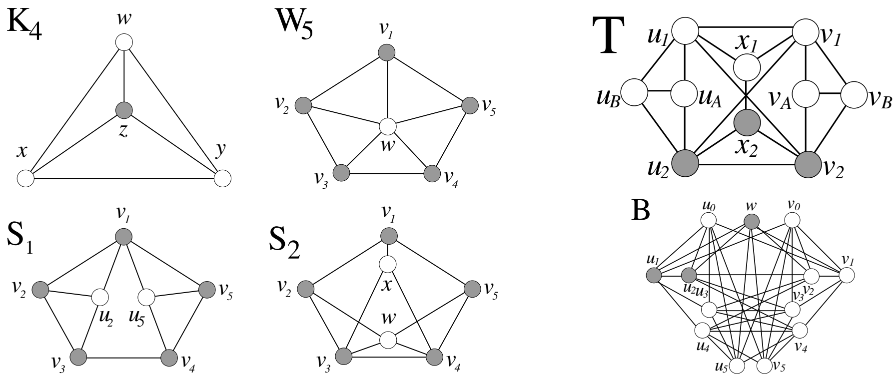

Fig. 1. All 6 MN3P5s

every vertex in $G - S$ has a neighbor in S. A k-clique is a clique on k vertices. $u \sim v$ will mean vertex u is adjacent to vertex v. $u \nsim v$ will mean vertex u is not adjacent to vertex v. For any vertex v, $N(v)$ denotes the set of vertices that are adjacent to v. We write $G \cong H$ to mean G is isomorphic to H. The *clique number* of G, denoted by $\omega(G)$, is the number of vertices in a largest clique of G. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest number of colors needed to color the vertices of G. A *hole* is an induced cycle with at least four vertices, and it is odd (or even) if it has odd (or even) length. An *anti-hole* is the complement of a hole. A k-hole (k-anti-hole) is a hole (anti-hole) on k vertices. A graph G is *perfect* if each induced subgraph H of G has $\chi(G) = \omega(G)$.

Theorem 2.1 (The Strong Perfect Graph Theorem [4]). *A graph is perfect if and only if it does not contain an odd hole or odd anti-hole as an induced subgraph.*

Let $\mathfrak{G} = \{K_4, W_5, S_1, S_2, T, B\}$ be the set of graphs in Fig. 1. We will denote these graphs in the following way.

- $P_5(v_1v_2v_3v_4v_5)$ means there is a P_5 being v_1, v_2, v_3, v_4 and v_5.
- $K_4(wxyz)$ means $\{w, x, y, z\}$ form a K_4.
- $W_5(v_1v_2v_3v_4v_5, w)$ means v_1, v_2, v_3, v_4, v_5 and w form a W_5 where $v_1v_2v_3v_4v_5$ form a 5-cycle and w is adjacent to every other vertex.
- $S_1(v_1v_2v_3v_4v_5, u_2, u_5)$ means $v_1, v_2, v_3, v_4, v_5, u_2, u_5$ form an S_1 where v_1 is the only degree 4 vertex and $N(v_1) = \{u_5, u_2, v_5, v_2\}$. Also $N(v_3) = \{v_4, v_2, u_2\}$ and $N(v_4) = \{v_3, v_5, u_5\}$, and $v_1v_2v_3v_4v_5$ form a 5-cycle.
- $S_2(v_1v_2v_3v_4v_5, w, x)$ means $v_1, v_2, v_3, v_4, v_5, w$ and x form an S_2 where $N(w) = \{v_2, v_3, v_4, v_5\}$, $N(x) = \{v_1, v_3, v_4\}$ and $v_1v_2v_3v_4v_5$ form a 5-cycle.
- $T(u_1u_Au_Bu_2, v_1v_Av_Bv_2, x_1, x_2)$ means a T graph is present as shown previously.
- $B(w, u_0u_1u_2u_3u_4u_5, v_0v_1v_2v_3v_4v_5)$ means a B graph is present as shown previously.

We will rely on the following result.

Theorem 2.2 ([1]). *Every connected P_5-free graph has a dominating clique or a dominating P_3.*

The following lemma is folklore.

Lemma 2.1 (The neighborhood lemma). *Let G be a minimally non k-colorable graph. If u and v are two non-adjacent vertices in G, then $N(u) \nsubseteq N(v)$.*

Proof. Assume $N(u) \subseteq N(v)$. Then the graph $G - v$ admits a k-coloring. By giving u the color of v, we see that G is k-colorable, a contradiction. □

The neighborhood lemma is used predominantly throughout this paper. Writing $\mathbf{N}(\mathbf{v}, \mathbf{w}) \to \mathbf{u}$ will denote the fact that $N(v) \nsubseteq N(w)$ by the neighborhood lemma so there exists a vertex u where $u \sim v$, but $u \nsim w$.

The following fact is well-known and easy to establish.

Fact 2.1. *In a minimally non k-colorable graph every vertex has degree at least k.* □

3 Intermediate Results

In this section, we establish a number of intermediate results needed for proving the main theorem.

Lemma 3.1. *Let G be an MN3P5 graph with a 5-hole $C = \{v_1, v_2, v_3, v_4, v_5\}$ and a vertex w adjacent to at least 4 vertices of C. Then $G \in \mathfrak{G}$.*

Proof. If w is adjacent to all five vertices of C, then G clearly is isomorphic to W_5. Now, assume $N(w) \cap \{v_1, v_2, v_3, v_4, v_5\} = \{v_2, v_3, v_4, v_5\}$.

We have $\mathbf{N}(\mathbf{v_1}, \mathbf{w}) \to \mathbf{x}$.

Assume for the moment that $x \nsim v_3, v_4$. We have

$x \sim v_5$, otherwise, we have $P_5(xv_1v_5v_4v_3)$.

$x \sim v_2$, otherwise, we have $P_5(xv_1v_2v_3v_4)$.

But then G contains $S_1(v_1v_2v_3v_4v_5, x, w)$. This means $x \sim v_3$ or $x \sim v_4$. By symmetry, we may assume $x \sim v_3$. We have $x \sim v_2$ or $x \sim v_4$, otherwise, G contains $P_5(xv_1v_2wv_4)$. If $x \sim v_2$ then G properly contains $S_1(v_1v_2v_3v_4v_5, x, w)$, a contradiction. This means $x \sim v_4$; so G contains $S_2(v_1v_2v_3v_4v_5, w, x)$ and $G \cong S_2$. □

Theorem 3.1. *Every MN3P5 graph different from K_4 contains a 5-hole.*

Proof. Let G be an MN3P5 graph different from a K_4. We have $\omega(G) \leq 3$ and $\chi(G) \geq 4$. Thus, G is not perfect. By Theorem 2.1, G contains an odd hole or an odd anti-hole H. H cannot be a hole of size 7 or greater because G is P_5-free. We may assume H is an anti-hole of length at least seven, for otherwise we are done (observe that the hole on five vertices is self-complementary). Let v_1, v_2, v_3, v_4, v_5, v_6, v_7 be the cyclic order of the hole in the complement of G. Then G properly contains $S_1(v_4v_6v_3v_5v_2, v_1, v_7)$, a contradiction. □

Lemma 3.2. *Let G be an MN3P5 graph that has a dominating clique $\{a, b, c\}$. Also assume that there is a vertex $v \notin \{a, b, c\}$ adjacent to two vertices from $\{a, b, c\}$. Then $G \in \mathfrak{G}$.*

Proof. The proof is by contradiction. Suppose that $G \notin \mathfrak{G}$. We may assume v is adjacent to b and c. We have $v \nsim a$, otherwise, G contains $K_4(abcv)$. Through repeated applications of the Neighborhood Lemma, we will eventually add nine vertices to G to arrive at a contradiction. In the end, we will obtain the graph B (see Fig. 2 for the order in which vertices are added). Each time we add a vertex we will consider its adjacency to the other vertices of the graph. In every case, the adjacency can be completely determined at each step.

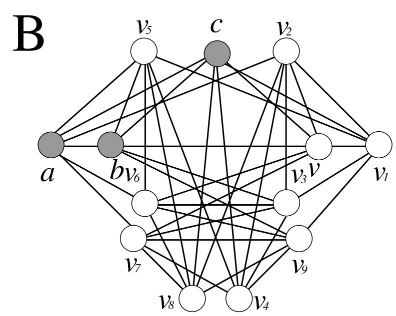

Fig. 2. The graph B obtained in the proof of Lemma 3.2

$\mathbf{N(v, a) \to v_1}$.

- $v_1 \sim c$: since $\{a, b, c\}$ is dominating, v_1 is adjacent to either b or c. Without loss of generality, assume $v_1 \sim c$.
- $v_1 \nsim b$: otherwise, G contains $K_4(bcvv_1)$.

$\mathbf{N(v_1, b) \to v_2}$.

- $v_2 \sim a$: assume $v_2 \nsim a$. We have $v_2 \sim v$, otherwise, G contains $P_5(v_2v_1vba)$. Also, $v_2 \sim c$ since $\{a, b, c\}$ is a dominating set. But then, G contains $K_4(v_1v_2vc)$.
- $v_2 \nsim c$: otherwise, G contains $W_5(abvv_1v_2, c)$.
- $v_2 \sim v$: otherwise, c has four neighbors in the 5-hole v_2abvv_1 contradicting Lemma 3.1.

$\mathbf{N(v_2, c) \to v_3}$.

- $v_3 \sim b$: assume $v_3 \nsim b$. We have $v_3 \sim a$ since $\{a, b, c\}$ is a dominating set. We have $v_3 \nsim v_1$, otherwise, G contains $S_1(vbav_3v_2, c, v_1)$. But then G contains $P_5(v_3v_2v_1cb)$.
- $v_3 \nsim v$; otherwise, G contains $W_5(bcv_1v_2v_3, v)$.
- $v_3 \sim v_1$: otherwise, v has four neighbors in the 5-hole $v_3bcv_1v_2$ contradicting Lemma 3.1.
- $v_3 \nsim a$: otherwise G contains $S_1(v_3acvv_1, b, v_2)$.

$\mathbf{N(v_3, v) \rightarrow v_4}$.

- $v_4 \sim c$: assume $v_4 \nsim c$. Then we have $v_4 \sim v_2$, for otherwise G contains $P_5(v_4v_3v_2vc)$; $v_4 \nsim v_1$, for otherwise G contains $K_4(v_1v_2v_3v_4)$; $v_4 \nsim b$, for otherwise G contains $S_1(v_1v_2v_4bc, v_3, v)$; $v_4 \sim a$ because $\{a, b, c\}$ is dominating. But then G contains $P_5(v_4abvv_1)$.
- $v_4 \nsim v_1$: for otherwise G contains $W_5(v_4v_3v_2vc, v_1)$.
- $v_4 \nsim b$: for otherwise, G contains $S_1(vv_1v_3v_4b, v_2, c)$.
- $v_4 \nsim a$: for otherwise, G contains $P_5(v_4abvv_1)$.
- $v_4 \sim v_2$: for otherwise the vertex v_1 has exactly four neighbors in the 5-hole $v_4v_3v_2vc$ contradicting Lemma 3.1.

$\mathbf{N(a, v) \rightarrow v_5}$.

- $v_5 \nsim v_3$: Assume $v_5 \sim v_3$. Then we have $v_5 \sim v_1$, for otherwise G contains $P_5(av_5v_3v_1v)$; $v_5 \nsim v_2$, for otherwise G contains $K_4(v_1v_2v_3v_5)$; $v_5 \sim c$, for otherwise G contains $P_5(v_5v_3v_2vc)$. But now G contains $W_5(v_5cvv_2v_3, v_1)$.
- $v_5 \sim b$: assume $v_5 \nsim b$. Then we have $v_5 \sim v_1$, for otherwise G contains $P_5(v_5abvv_1)$. But then c has four neighbors in the 5-hole v_5abvv_1 contradicting Lemma 3.1.
- $v_5 \nsim c$: for otherwise G contains $K_4(abcv_5)$.
- $v_5 \sim v_1$: for otherwise G contains $P_5(v_5acv_1v_3)$.
- $v_5 \sim v_4$: for otherwise G contains $P_5(v_3v_4cav_5)$.
- $v_5 \nsim v_2$: for otherwise G contains $S_1(cvv_2v_5a, v_1, b)$.

$\mathbf{N(v_5, c) \rightarrow v_6}$.

- $v_6 \sim v$: assume $v_6 \nsim v$. We have $v_6 \sim a$, for otherwise G contains $P_5(v_6v_5acv)$; $v_6 \nsim b$, for otherwise G contains $K_4(abv_5v_6)$; $v_6 \sim v_1$, for otherwise, G contains $P_5(v_6abvv_1)$. But c has four neighbors in the 5-hole v_6abvv_1 contradicting Lemma 3.1.
- $v_6 \nsim b$: for otherwise G contains $W_5(v_5v_6vca, b)$.
- $v_6 \nsim v_2$: for otherwise G contains $P_5(v_2v_6v_5bc)$.
- $v_6 \sim v_3$: for otherwise G contains $P_5(v_5v_6vv_2v_3)$
- $v_6 \sim a$: for otherwise G contains $P_5(v_3v_6v_5ac)$.
- $v_6 \nsim v_1$: for otherwise G contains $S_1(v_6abcv, v_5, v_1)$.
- $v_6 \nsim v_4$: for otherwise G contains $T(v_6av_5b, v_3v_2v_1v, v_4, c)$.

$\mathbf{N(v_4, v_1) \rightarrow v_7}$.

- $v_7 \sim v$: assume $v_7 \nsim v$. Then we have $v_7 \sim v_3$, for otherwise G contains $P_5(v_7v_4v_3v_1v)$; $v_7 \nsim v_2$, for otherwise G contains $K_4(v_2v_3v_4v_7)$; $v_7 \sim c$, for otherwise G contains $P_5(v_7v_3v_2vc)$. Now, G contains $S_1(v_2vcv_7v_3, v_1, v_4)$.
- $v_7 \nsim v_2$: for otherwise G contains $W_5(vv_1v_3v_4v_7, v_2)$.
- $v_7 \nsim v_6$: for otherwise G contains $P_5(v_6v_7v_4v_2v_1)$.
- $v_7 \sim a$: for otherwise G contains $P_5(v_4v_7vv_6a)$.
- $v_7 \sim v_3$: for otherwise G contains $P_5(av_7vv_1v_3)$.
- $v_7 \nsim c$: for otherwise G contains $S_1(v_3v_4cvv_1, v_7, v_2)$.
- $v_7 \nsim b$: for otherwise G contains $P_5(v_7bcv_1v_2)$.
- $v_7 \nsim v_5$: for otherwise G contains $T(av_7v_5v_4, cvv_1v_2, b, v_3)$.

$\mathbf{N(v_6, b) \rightarrow v_8}$.

- $v_8 \sim c$: assume $v_8 \nsim c$. Then we have $v_8 \sim a$ because $\{a, b, c\}$ is a dominating set; $v_8 \sim v_5$, for otherwise G contains $P_5(v_8v_6v_5bc)$. But now, G contains $K_4(av_5v_6v_8)$.
- $v_8 \nsim a$: for otherwise G contains $W_5(v_8v_6v_5bc, a)$.
- $v_8 \nsim v_1$: for otherwise G contains $P_5(bav_6v_8v_1)$.
- $v_8 \sim v_2$: for otherwise G contains $P_5(v_2v_1cv_8v_6)$.
- $v_8 \sim v_5$: for otherwise G contains $P_5(v_8v_2v_1v_5b)$.
- $v_8 \nsim v_4$: for otherwise G contains $P_5(v_4v_8v_6ab)$.
- $v_8 \nsim v$: for otherwise G contains $S_1(bcv_8v_6a, v, v_5)$.
- $v_8 \nsim v_3$: for otherwise G contains $T(v_6av_5b, v_3v_2v_1v, v_8, c)$.
- $v_8 \sim v_7$: for otherwise G contains $P_5(v_8v_5bv_3v_7)$.

$\mathbf{N(v_8, a) \rightarrow v_9}$.

- $v_9 \sim b$: assume $v_9 \nsim b$. We have $v_9 \sim v_2$, for otherwise G contains $P_5(v_9v_8v_2ab)$; $v_9 \sim v_6$, for otherwise G contains $P_5(v_9v_8v_6ab)$. This means G contains $T(v_6v_8v_9v_2, abcv, v_5, v_1)$.
- $v_9 \sim v_1$: assume $v_9 \nsim v_1$. We have $v_9 \sim v_2$, for otherwise G contains $P_5(v_9bav_2v_1)$. This means G contains $T(v_2vv_1c, v_8v_6v_5a, v_9, b)$.
- $v_9 \sim v_6$: for otherwise G contains $P_5(v_1v_9bav_6)$.
- $v_9 \sim v_7$: for otherwise G contains $P_5(v_1v_9bav_7)$.
- $v_9 \sim v_4$: assume $v_9 \nsim v_4$. Then we have $v_9 \sim v_2$, for otherwise G contains $P_5(v_9bav_2v_4)$. This means G contains $T(v_6av_5b, v_9v_2v_1v, v_8, c)$.

But this means G contains $B(c, v_5abv_6v_7v_8, v_2v_1vv_3v_9v_4)$, a contradiction. □

Lemma 3.3. *Let G be an MN3P5 with a dominating clique $\{a, b, c\}$. Let $A = N(a) - \{b, c\}$, $B = N(b) - \{a, c\}$ and $C = N(c) - \{a, b\}$. Suppose A, B and C are pairwise disjoint. Then $G \in \mathfrak{G}$.*

Proof. Some observations are necessary for this proof.

Observation 3.1. *Let X and Y be two distinct elements of $\{A, B, C\}$. Let X' be a component in X with at least two vertices, and y be a vertex in Y. Then either y is adjacent to all vertices of X' or to no vertex of X'.*

Proof. Suppose the Observation is false. Then there are adjacent vertices $v_1, v_2 \in X$ such that y is adjacent to exactly one of v_1, v_2. Without loss of generality, we may assume $X = A$ and $Y = B$. Now, $\{c, b, y, v_2, v_1\}$ induces a P_5, a contradiction. □

Observation 3.2. *Every component in A, B or C is a single edge or one vertex.*

Proof. Assume that one of A, B or C contains a vertex of degree 2. Without loss of generality, assume there is such a vertex $a_0 \in A$ that is adjacent to two other distinct vertices a_1 and a_x in A. Now we have $a_1 \nsim a_x$, for otherwise G contains $K_4(a_1a_xa_0a)$. The Neighborhood Lemma implies $\mathbf{N(a_1, a_x) \rightarrow a_2}$ and $\mathbf{N(a_x, a_1) \rightarrow a_y}$. Observation 3.1 implies $a_2, a_y \in A$. We have $a_y \nsim a_0$, for otherwise G contains $K_4(aa_0a_xa_y)$; $a_2 \nsim a_0$, for otherwise G contains $K_4(aa_0a_1a_2)$; $a_y \sim a_2$, for otherwise G contains $P_5(a_ya_xa_0a_1a_2)$. Then G contains $W_5(a_ya_xa_0a_1a_2, a)$, a contradiction. □

We continue the proof of the Lemma. Assume $G \notin \mathfrak{G}$. Consider the case that two of A, B or C contain an edge. Without loss of generality, assume A contains an edge a_1a_2 and B contains an edge b_1b_2. If a vertex in $\{b_1, b_2\}$ is adjacent to a vertex in $\{a_1, a_2\}$ then by Observation 3.1, G contains $K_4(a_1a_2b_1b_2)$, a contradiction. Suppose some vertex $c_0 \in C$ is adjacent to a vertex in $\{a_1, a_2, b_1, b_2\}$. We may assume $c_0 \sim a_1$. By Observation 3.1, we have $c_0 \sim a_2$. If $c_0 \nsim b_i$ $(i = 1, 2)$ then G contains $P_5(b_ibcc_0a_1)$. So, c_0 is adjacent to all vertices of $\{a_1, a_2, b_1, b_2\}$. But now, G contains $S_1(c_0a_1abb_1, a_2, b_2)$. So, no vertex in C is adjacent to a vertex in $\{a_1, a_2, b_1, b_2\}$. By Fact 2.1 and Observation 3.2, there exists a vertex $a_3 \in A$ with $b_1, b_2 \sim a_3$ and a vertex $b_3 \in B$ with $a_1, a_2 \sim b_3$. Also by Fact 2.1, C contains a vertex c_0. We have $a_3 \sim c_0$, for otherwise G contains $P_5(a_3b_1bcc_0)$; $b_3 \sim c_0$, for otherwise G contains $P_5(b_3a_1acc_0)$; $a_3 \sim b_3$, for otherwise G contains $P_5(b_1a_3c_0b_3a_1)$. But now G contains $T(aa_1a_2b_3, bb_1b_2a_3, c, c_0)$ which is a contradiction. So, at most one of A, B, C contains an edge.

If all of A, B, C is a stable set, then G is obviously 3-colorable. We may assume B, C are stable sets, and A contains an edge. Now there must be one vertex $b_0 \in B$ with $N(b_0)$ contains two adjacent vertices in A. Otherwise, G admits a 3-coloring f as follows. The vertices of C are colored with color 3. Now, for each edge in A, its endpoints are arbitrarily colored with colors 1, 2. The remaining vertices of A are colored with color 1. The vertices of B are colored with color 2 (no vertex of B is adjacent to an endpoint of a edge of A by Observation 3.1), and let $f(a) = 3, f(b) = 1, f(c) = 2$. Thus, f is a 3-coloring which is a contradiction. Therefore, there is a vertex $b_1 \in B$ adjacent to both endpoints in some edge $a_{b1}a_{b2}$ in A. By a similar argument, there is a vertex $c_1 \in C$ adjacent to both endpoints in some edge $a_{c1}a_{c2}$.

Suppose that $a_{b1}a_{b2}$ and $a_{c1}a_{c2}$ are the same edge. For simplicity, write $a_1a_2 = a_{b1}a_{b2} = a_{c1}a_{c2}$. We have $b_1 \nsim c_1$, for otherwise G contains $K_4(a_1a_2b_1c_1)$.

- $\mathbf{N(b_1, a) \to c_2}$. We have $c_2 \in C$ by the fact that B is an independent set.
- $\mathbf{N(c_1, a) \to b_2}$. We have $b_2 \in B$ by the fact that C is an independent set.
- $b_2, c_2 \nsim a_1, a_2$. Otherwise, suppose $b_2 \sim a_1$. Then by Observation 3.1, we have $b_2 \sim a_2$ so G contains $K_4(a_1a_2b_2c_1)$.
- $b_2 \sim c_2$. Otherwise, G contains $P_5(c_1b_2bb_1c_2)$.

Now, G contains $P_5(b_2c_2caa_1)$. Thus, $a_{b1}a_{b2}$ and $a_{c1}a_{c2}$ are distinct edges. We have $b_1 \nsim a_{c1}, a_{c2}$ and $c_1 \nsim a_{b1}, a_{b2}$, for otherwise we are done by the previous case. We have $b_1 \sim c_1$, for otherwise G contains $P_5(b_1a_{b1}aa_{c1}c_1)$. But now G contains $S_1(ab_1a_{b1}b_1c_1a_{c1}, a_{b2}, a_{c2})$, a contradiction. □

Lemma 3.4. *Let G be an MN3P5 with a dominating clique $\{a, b, c\}$. Then $G \in \mathfrak{G}$.*

Proof. If there is a vertex other than a, b and c adjacent to at least two of a, b or c then by Lemma 3.2, $G \in \mathfrak{G}$. Otherwise, the conclusion follows from Lemma 3.3. □

Lemma 3.5. *Let G be an MN3P5 with a dominating clique $\{a, b\}$ of size 2. Then $G \in \mathfrak{G}$.*

Proof. Assume $G \notin \mathfrak{G}$. We may assume G contains no dominating 3-clique, for otherwise we are done by Lemma 3.4. It follows that no vertex v is adjacent to both a, b.

By Theorem 3.1, there is 5-hole $C = v_1v_2v_3v_4v_5$ in G because $G \neq K_4$. Clearly C cannot contain both a and b. WLOG, assume that $|N(a) \cap C| \geq |N(b) \cap C|$. If $b \notin C$ then since $\{a, b\}$ is a dominating clique of G we have $|N(a) \cap C| \geq 3$. If $b \in C$, then a must be adjacent to the 2 vertices in C not adjacent to b. Thus, since $a \sim b$ we also have $|N(a) \cap C| \geq 3$. The case when $|N(a) \cap C| \geq 4$ is handled by Lemma 3.1, so WLOG we may assume either $N(a) \cap C = \{v_1, v_2, v_3\}$ or $N(a) \cap C = \{v_1, v_3, v_4\}$.

Suppose $N(a) \cap C = \{v_1, v_2, v_3\}$. Since $\{a, b\}$ is a dominating clique, we have $b \notin C$ and $b \sim v_4, v_5$. Since no vertex is adjacent to both a and b, G contains $P_5(bv_5v_1v_2v_3)$, a contradiction. Now, we may assume $N(a) \cap C = \{v_1, v_3, v_4\}$. There exists a vertex x with $x \nsim a, v_3, v_4$, for otherwise $\{a, v_3, v_4\}$ is dominating 3-clique. If $x \sim v_5$, then $x \sim v_2$, for otherwise G contains $P_5(xv_5v_4v_3v_2)$; but now G contains $P_5(v_2xv_5v_4a)$. Thus, we have $x \nsim v_5$ and by symmetry $x \nsim v_2$. Since $\{a, b\}$ is a dominating clique, we have $x \sim b$, and $b \sim v_2, v_5$. Recall that no vertex is adjacent to both a, b. Now, G contains $P_5(xbv_5v_4v_3)$ which is a contradiction. □

Theorem 3.2. *If G is an MN3P5 with a dominating clique then $G \in \mathfrak{G}$.*

Proof. If G has a dominating clique of size one or two, then it has a dominating clique of size 2 since G contains no isolated vertices. By Lemma 3.5, $G \in \mathfrak{G}$. If G has a dominating clique of size 3, then Lemma 3.4 implies $G \in \mathfrak{G}$. If G has a dominating clique of size 4 or more, then G contains a K_4 so $G = K_4 \in \mathfrak{G}$ by minimality. □

Lemma 3.6. *Let G be an MN3P5 with a dominating 5-hole. Then G has a dominating K_3 or $G \in \mathfrak{G}$.*

Proof. Let $C = v_1v_2v_3v_4v_5$ be an induced 5-hole of G. Assume G does not have a dominating clique. Let X_i be the set of vertices adjacent to v_{i-1} and v_{i+1} and not adjacent to v_{i+2} and v_{i+3} with the subscript taken modulo 5 (i.e., $v_0 = v_5$), for $i = 1, 2, 3, 4, 5$. We now prove every vertex of G belongs to exactly one X_i.

Consider a vertex $w \notin C$. By Lemma 3.1, we have $1 \leq |N(w) \cap C| \leq 3$. If w has one neighbor in C, then G obviously contains a P_5. Suppose w has two neighbors a, b in C. If $a \sim b$, then G obviously contains a P_5. Otherwise, a and b have distance two on C and so w belongs to some X_i. We may now assume w has three neighbors on C. If these three neighbors are consecutive on C, then w belongs to some X_i. Now, we may assume $w \sim v_1, v_3, v_4$. There is a vertex x with $x \nsim w, v_4, v_3$, for otherwise $\{w, v_4, v_3\}$ is a dominating clique. Vertex x must have a neighbor in $\{v_1, v_2, v_5\}$ because C is a dominating set. If $x \sim v_5$, then $x \sim v_2$, for otherwise G contains $P_5(xv_5v_4v_3v_2)$; but now G contains $P_5(v_2xv_5v_4w)$. Thus, we have $x \nsim v_5$ and by symmetry $x \nsim v_2$. Now, we have $x \sim v_1$, and G contains $P_5(xv_1v_5v_4v_3)$. Thus, X_1, X_2, X_3, X_4, X_5 is a partition of $V(G)$.

If there are nonadjacent vertices x_1, x_2 with $x_1 \in X_1, x_2 \in X_2$, then G contains $P_5(x_1v_5v_4v_3x_2)$. Thus, there are all possible edges between X_i and X_{i+1} for all i. If every X_i is a stable set, then G is obviously 3-colorable, a contradiction. So we may assume WLOG X_5 contains an edge ab. Then X_1 is a stable set, for otherwise G contains a K_4 with one edge in X_1 and one edge in X_5. Similarly, X_4 is a stable set. If X_2 contains an edge cd, then G contains $S_1(v_1cv_3v_4a, d, b)$. If X_3 contains an edge fg, then G contains $S_1(v_4fv_2v_1a, g, b)$. Thus, X_i is a stable set for $i = 1, 2, 3, 4$. Consider the subgraph H of G induced by X_5. If H contains an odd cycle D, then $D \cup \{v_1\}$ is a K_4 or W_5, or D contains a P_5. Thus H is bipartite. By coloring X_5 with colors 2,3, $X_1 \cup X_4$ with color 1, X_2 with color 2, X_3 with color 3, we see that G is 3-colorable, a contradiction. □

4 Proof of Theorem 1.1

We can now prove the main theorem.

It is a routine matter to verify the "only if" part. We only need prove the "if" part. Suppose G does not contain any of the graphs in Fig. 1 but is not 3-colorable. Then G contains an induced subgraph that is minimally not 3-colorable. It follows that we may assume G is a connected MN3P5 graphs. By Theorem 2.2, G contains a dominating clique or P_3. If G contains a dominating clique, then we are done by Theorem 3.2. So, we may assume G contains no dominating clique and thus contains a dominating P_3 with vertices v_1, v_2, v_3 and edges v_1v_2, v_2v_3. There is a vertex v_4 with $v_4 \sim v_3$ and $v_4 \nsim v_1, v_2$ since v_1v_2 is not a dominating edge. Similarly, there is a vertex v_5 with $v_5 \sim v_1$ and $v_5 \nsim v_2, v_3$. We have $v_5 \sim v_4$, for otherwise G contains a P_5. Thus, $v_1v_2v_3v_4v_5$ is a dominating 5-hole of G, and we are done by Lemma 3.6. □

5 Conclusion and Open Problems

In this paper, we provide a certifying algorithm for the problem of 3-coloring a P_5-graph by showing there are exactly six finite minimally non-3-colorable graphs. Previously known algorithms ([6, 11, 14]) provide a yes-certificate by constructing a 3-coloring if one exists. Our algorithm provides a no-certificate by finding one of the six graphs of Fig. 1. Since these graphs are finite, our algorithm runs in polynomial time. We do not know if there is a fast algorithm running in, say, $O(n^4)$ to test if a graph contains one of the six graphs of Fig. 1 as a subgraph. We leave this as an open problem.

In [5, 6], it is shown for every fixed k, determining if a P_5-free graph is k-colorable is polynomial-time solvable. It is tempting to speculate that these two algorithms work because for every fixed k, there is a function $f(k)$ such that every minimally non-k-colorable P_5-free graph has at most $f(k)$ vertices. The result of this paper can be viewed as a first step in this direction.

References

1. G. Bacsó and Z. Tuza, Dominating cliques in P_5-free graphs, Period. Math. Hungar. Vol. 21 No. 4 (1990) 303-308.
2. H.J. Broersma, F.V. Fomin, P.A. Golovach and D. Paulusma, Three complexity results on coloring P_k-free graphs, accepted for IWOCA 2009.
3. D. Coppersmith and S. Winograd, Matrix multiplication via arithmetic progressions. Journal of Symbolic Computation, Vol. 9 No. 3(1990) 251-280.
4. M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas. The strong perfect graph theorem. Annals of Mathematics 164 (1) (2006): 51-229.

5. C. T. Hoàng, M. Kamiński, V. Lozin, J. Sawada, X. Shu, Deciding k-colorability of P5-free graphs in polynomial time, to appear in Algorithmica.
6. C. T. Hoàng, M. Kamiński, V. Lozin, J. Sawada, X. Shu, A Note on k-Colorability of P5-Free Graphs, Lecture Notes In Computer Science; Vol. 5162 (2008) 387 - 394.
7. D.V. Korobitsyn, On the complexity of determining the domination number in monogenic classes of graphs, Diskret. Mat. 2, N 3 (1990), 90-96 in Russian, translation in Discrete Mathematics and Applications, 2 (1992), no. 2, 191-199.
8. D. Kral, J. Kratochvil, Z. Tuza and G. J. Woeginger, Complexity of coloring graphs without forbidden induced subgraphs, in: WG 2001, LNCS 2204, (2001) 254-262.
9. D. Kratsch, R. M. McConnell, K. Mehlhorn, and J. P. Spinrad. Certifying algorithms for recognizing interval graphs and permutation graphs. SIAM J. Comput., 36(2):326-353, 2006.
10. V. Bang Le, B. Randerath, I. Schiermeyer, On the complexity of 4-coloring graphs without long induced paths, Theoretical Computer Science 389 (2007) 330-335.
11. S. Mellin, Polynomielle Färbungsalgorithmen für P_k-freie Graphen, Diplomarbeit am Institut für Informatik, Universität zu Köln, (2002).
12. B. Randerath, I. Schiermeyer, Vertex coloring and forbidden subgraphs – a survey, Graphs and Combinatorics, 20(1) (2004) 1-40.
13. B. Randerath, I. Schiermeyer, 3-colorability $\in \mathcal{P}$ for P_6-free graphs, Discrete Applied Mathematics 136 (2004) 299-313.
14. G. J. Woeginger, J. Sgall, The complexity of coloring graphs without long induced paths, Acta Cybernetica 15(1), (2001) 107-117.