A CHARACTERISTIC PROPERTY OF THE EUCLIDEAN PLANE

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1. The euclidean plane $\mathbb{R}^2$ has the property that for each family of circles, the length of each circle is a linear function of the radius. It is less obvious that this *linear-growth* condition is satisfied by all curves in the plane, in the following sense: Let γ be a C^∞-curve of finite length and without self-intersection. For each $p \in \gamma$, let σ_p be a line segment lying on a chosen side of γ, and perpendicular to γ. The points on $\sigma(p)$ $(p \in \gamma)$ whose distance from γ is s form a curve γ_s, which is also of class C^∞ for all sufficiently small s. Let $\ell(\gamma_s)$ denote the length of γ_s. Then $\ell(\gamma_s) - \ell(\gamma)$ is a linear function of s (see Section 3).

This suggests a natural problem concerning Riemannian manifolds of arbitrary dimension. Let $\overline{M}$ be a Riemannian manifold, and let M be a compact, orientable submanifold of codimension one (*possibly with boundary*). In short, let M be a compact hypersurface in $\overline{M}$. For sufficiently small s, let M_s be the set of points lying on geodesics normal to M (and on a fixed side of M) at distance s from M. Let $\mathscr{A}$ denote the area (or volume) function, and consider the real-valued function $\phi(s) = \mathscr{A}(M_s) - \mathscr{A}(M)$. How does ϕ grow as a function of s?

In general, ϕ is quite arbitrary. Thus if $\overline{M} = \mathbb{R}^d$ and M is a sphere, then $\phi(s) = cs^{d-1}$, where c is some constant. On the other hand, if $\overline{M} = S^d$ (the d-dimensional sphere) and M is the great sphere in S^d, then the growth of ϕ is dominated by a linear function (see the proposition of Section 2). My problem is to determine all Riemannian manifolds $\overline{M}$ for which ϕ is a *linear* function for every compact (orientable) hypersurface M. A Riemannian manifold $\overline{M}$ with this property is said to have the *linear-growth property*. The solution of the problem is exceedingly simple:

THEOREM. *A Riemannian manifold has the linear-growth property if and only if it is locally the euclidean plane.*

The referee has kindly brought my attention to the fact that what I called M_s in the second paragraph above is usually referred to in the classical literature as a "parallel-body." The behavior of the growth of the volume of M_s when the ambient space $\overline{M}$ is the euclidean space was first considered by J. Steiner in 1840. For further details, the reader is referred to H. Hadwiger [3, p. 213]. The paper [4] by H. Weyl is also relevant here.

2. The proof of our theorem depends on the computation of the second variation of area. Since the situation is different from the standard one in the theory of minimal surfaces, we shall give complete details.

We deal separately with two cases. In the first case, $\dim \overline{M} = 2$. Here M is a finite C^∞-curve, and it is evidently the diffeomorphic image of either $[0, 1]$ or $[0, 1)$. (We usually do not distinguish between the map of a submanifold and its image.) For each $m \in M$, let σ_m denote a geodesic lying on one side of M, emanating at m, and normal to M at m. For convenience, we shall require further that $\sigma_m: [0, 1] \to \overline{M}$ is such that $\sigma_m(0) = m$ and $\|\sigma'_m(0)\| = 1$, where σ'_m denotes

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the tangent vector and $\| \ \|$ the Riemannian norm. Then $M_s = \{\sigma_m(s): m \in M\}$ for sufficiently small s, and each M_s is a C^∞-curve. Using the exponential map of the normal bundle of M in $\overline{M}$, one obtains an injective, nonsingular C^∞-map $f: [0, \varepsilon] \times [0, 1] \to \overline{M}$ from the rectangle in the ST-plane such that for a fixed s, $M_s = \{f(s, t): t \in [0, 1]\}$, and for a fixed t, $\sigma_m(s) = f(s, t)$, where $m = f(0, t)$. We shall use (s, t) as local coordinates. Let $S = \frac{\partial}{\partial s}$, $T = \frac{\partial}{\partial t}$. Evidently, if ℓ denotes the length function and $\ell(s) \equiv \ell(M_s)$, then

$$\ell(s) = \int_0^1 \|T\|(s, t)\,dt \tag{1}$$

Let us summarize our hypotheses and their immediate consequences:

$$\langle T, S \rangle \equiv 0 \qquad (\langle \ , \ \rangle \text{ is the inner product}). \tag{2}$$

$$\langle S, S \rangle \equiv 1. \tag{3}$$

$$D_S S \equiv 0 \qquad (D \text{ is covariant differentiation}). \tag{4}$$

$$[S, T] \equiv 0. \tag{5}$$

The assertion (2) is a consequence of (3), the Gauss lemma, and the fact that $\langle T, S \rangle \mid M = 0$. The statement (4) follows because the s-curves are the geodesics σ_m, and (5) because S and T are coordinate vector fields. Evidently, $\ell'(s) = \int_0^1 (S\|T\|)(s, t)\,dt$. Now, by (5),

$$S\|T\| = \frac{1}{2\|T\|} S\langle T, T\rangle = \frac{1}{\|T\|}\langle D_T S, S\rangle.$$

Therefore $\ell'(s) = \int_0^1 \left(\frac{1}{\|T\|}\langle D_T S, S\rangle\right)(s, t)\,dt$, and it follows immediately that

$$\ell''(s) = \int_0^1 S\left(\frac{1}{\|T\|}\langle D_T S, S\rangle\right)(s, t)\,dt.$$

Now

$$S\left(\frac{1}{\|T\|}\langle D_T S, S\rangle\right) = -\frac{1}{\|T\|}\langle D_T S, S\rangle^2 + \frac{1}{\|T\|}\{\langle D_S D_T S, T\rangle + \langle D_T S, D_T S\rangle\}.$$

If R denotes the curvature tensor (so that $\langle R_{ST} S, T\rangle$ is a *positive* multiple of the curvature), then

$$R_{ST} S = -D_S D_T S + D_T D_S S + D_{[S,T]} S = -D_S D_T S,$$

because of (4) and (5). Also, observe that $D_S T = D_T S + [S, T] = D_T S$ because of (5). Hence

$$S\left(\frac{1}{\|T\|}\langle D_T S, T\rangle\right)$$

$$= \frac{-1}{\|T\|}\langle R_{ST} S, T\rangle + \left(\frac{1}{\|T\|}\langle D_T S, D_T S\rangle - \frac{1}{\|T\|^3}\langle D_S T, T\rangle^2\right).$$

We claim that the expression in the parentheses on the right vanishes. Let $D_T S = \alpha S + \beta T$; then (2), (3), and (4) imply that $\alpha = \langle D_T S, S\rangle = \frac{1}{2} T\langle S, S\rangle = 0$. Also,

$$\beta = \frac{\langle D_T S, T\rangle}{\langle T, T\rangle} = \frac{1}{\|T\|^2}\langle D_S T, T\rangle .$$

Thus, $\langle D_T S, D_T S\rangle = \beta^2\langle T, T\rangle = \frac{1}{\|T\|^2}\langle D_S T, T\rangle^2$, which proves our claim. Therefore,

$$\ell''(s) = \int_0^1 \frac{-1}{\|T\|}\langle R_{ST} S, T\rangle (s, t)\,dt .$$

Let K denote the curvature function on the surface $\overline{M}$; then

$$\langle R_{ST} S, T\rangle = \langle T, T\rangle K,$$

by virtue of (3). Consequently,

$$\ell''(s) = -\int_0^1 (\|T\| K)(s, t)\,dt. \tag{6}$$

Now we consider the second case, where $\dim \overline{M} \geq 3$. Since $\dim M \geq 2$, it makes sense to speak of the curvature of the induced metric of M. As before, let σ_m denote a geodesic issuing from $m \in M$, lying on one side of M, and such that if σ_m: $[0, 1] \to M$, then

$$\sigma_m(0) = m, \qquad \|\sigma'(0)\| = 1, \qquad \sigma'(0) \perp M_m .$$

If s is sufficiently small, then M_s is the set of all $\sigma_m(s)$ as m varies over M. Using the normal bundle exponential map, we obtain a nonsingular, injective C^∞-map f: $M \times [0, \varepsilon] \to \overline{M}$ (ε small) such that $\sigma_m(s) = f(m, s)$ for all m and for all $s \in [0, \varepsilon]$. We denote by Ω_s the volume form of M_s, and by $\mathscr{A}(s)$ the area (or volume) of M_s. Then

$$\mathscr{A}(s) = \int_{M_s} \Omega_s .$$

To state the formula for $\mathscr{A}''(s)$, we need further notation. Let $\mathscr{R}$ denote the Ricci tensor of $\overline{M}$ ($\mathscr{R}$: $\overline{M}_m \to \overline{M}_m$), and let h: $(M_s)_m \otimes (M_s)_m \to R$ denote the second fundamental form of M_s (there is only one such form, because M_s is a hypersurface). The form h admits an extension to

$$((M_s)_m \wedge (M_s)_m) \otimes ((M_s)_m \wedge (M_s)_m) \to \mathbb{R},$$

which we also denote by h. Let dim M = d, and let $\{e_1, \cdots, e_{d-1}\}$ be any orthonormal basis of $(M_s)_m$. Then it is easy to see that $\sum_{i,j} h(e_i \wedge e_j, e_i \wedge e_j)$ is a globally defined function on M_s, independent of the choice of $\{e_1, \cdots, e_{d-1}\}$. Finally, let S denote the unit vector field defined in this neighborhood of M by the tangent vectors $\sigma'_m(s)$ for all $m \in M$, $s \in [0, \varepsilon]$.

PROPOSITION. $\mathscr{A}''(s) = \int_{M_s} \left(\sum_{i,j} h(e_i \wedge e_j, e_i \wedge e_j) - \langle \mathscr{R}(S), S \rangle \right) \Omega_s$.

COROLLARY. *Let* r^s *denote the scalar curvature of* M_s *in the induced metric, and let* $r^{\overline{M}}$ *denote the scalar curvature of* $\overline{M}$. *Then*

$$\mathscr{A}''(s) = \int_{M_s} (r^s - r^{\overline{M}})\Omega_s .$$

We first prove the corollary. The Gauss-Codazzi equation states that

$$h(e_i \wedge e_j, e_i \wedge e_j) = K^s(e_i, e_j) - K^{\overline{M}}(e_i, e_j),$$

where $K^s(e_i, e_j)$ and $K^{\overline{M}}(e_i, e_j)$ denote the sectional curvature of the plane span $\{e_i, e_j\}$ of M_s in the induced metric and in the metric of $\overline{M}$. Then

$$\sum_{i,j} h(e_i \wedge e_j, e_i \wedge e_j) - \langle \mathscr{R}(S), S \rangle$$

$$= \left\{ \sum_{i,j} K^s(e_i, e_j) \right\} - \left\{ \sum_{i,j} K^{\overline{M}}(e_i, e_j) + \sum_i K^{\overline{M}}(S, e_i) \right\}.$$

By definition, these two expressions equal r^s and $r^{\overline{M}}$, respectively.

Now we proceed to the proof of the proposition. In the neighborhood of a point $m \in M_s$, we can use the mapping f above to construct coordinates $\{t_1, \cdots, t_{d-1}, s\}$ so that (1) the s coordinate curves coincide with the geodesics σ_m and (2) for all s_0 close to s, the slice $\{s = s_0\}$ is exactly an open set of M_{s_0} and $\{t_1, \cdots, t_{d-1}, s_0\}$ gives a local coordinate neighborhood of M_{s_0} when s_0 is held fixed. Of course, $S = \frac{\partial}{\partial s}$, and we denote $\frac{\partial}{\partial t_i}$ by T_i. We summarize our assumptions thus far in the four statements

(7) $$\langle T_i, S \rangle \equiv 0 \quad \text{for } i = 1, \cdots, d-1,$$

(8) $$\langle S, S \rangle \equiv 1,$$

(9) $$D_S S \equiv 0,$$

(10) $$[S, T_i] \equiv 0 \quad \text{for } i = 1, \cdots, d-1.$$

Define $g = \langle T_1 \wedge \cdots \wedge T_{d-1}, T_1 \wedge \cdots \wedge T_{d-1} \rangle$; then in this neighborhood

$$\Omega_s = \sqrt{g}\, dt_1 \wedge \cdots \wedge dt_{d-1}. \tag{11}$$

One sees via a partition of unity that $\mathscr{A}''(s) = \int (SS\sqrt{g})\, dt_1 \wedge \cdots \wedge dt_{d-1}$. Therefore we only need to prove that

$$(SS\sqrt{g})\, dt_1 \wedge \cdots \wedge dt_{d-1}$$
$$= \sqrt{g}\left(\sum_{i,j} h(e_i \wedge e_j,\ e_i \wedge e_j) - \left\langle \mathscr{R}(S), S\right\rangle\right) dt_1 \wedge \cdots \wedge dt_{d-1}.$$

To do this, we need only prove the equality at m and for a special choice of $T_1, \cdots, T_{d-1}$ at m. We can certainly arrange things so that

$$\left\langle T_i, T_j\right\rangle(m) = \delta_{ij}. \tag{12}$$

Then $g(m) = 1$, and we are left with the task of proving that

$$(SS\sqrt{g})(m) = \left(\sum_{i,j} h(T_i \wedge T_j,\ T_i \wedge T_j) - \left\langle \mathscr{R}(S), S\right\rangle\right)(m). \tag{13}$$

Let $D_S T_i = \sum_j \gamma_{ij} T_j + \gamma_i S$. Because of (8) and (10),

$$\gamma_i = \left\langle D_S T_i, S\right\rangle = \left\langle D_{T_i} S, S\right\rangle = \frac{1}{2} T_i \left\langle S, S\right\rangle = 0.$$

Hence

$$D_S T_i = \sum_j \gamma_{ij} T_j. \tag{14}$$

Now

$$S\sqrt{g} = \frac{1}{\sqrt{g}} \sum_i \left\langle T_1 \wedge \cdots \wedge D_S T_i \wedge \cdots \wedge T_{d-1},\ T_1 \wedge \cdots \wedge T_{d-1}\right\rangle$$
$$= \frac{1}{\sqrt{g}}\left(\sum_i \gamma_{ii}\right)\left\langle T_1 \wedge \cdots \wedge T_{d-1},\ T_1 \wedge \cdots \wedge T_{d-1}\right\rangle = \sqrt{g}\left(\sum_i \gamma_{ii}\right)$$

Therefore $SS\sqrt{g} = \sqrt{g}\left(\sum_{i,j} \gamma_{ii}\gamma_{jj} + \sum_i S\gamma_{ii}\right)$. Thus

$$(SS\sqrt{g})(m) = \left(\sum_{i,j} \gamma_{ii}\gamma_{jj} + \sum_i S\gamma_{ii}\right)(m). \tag{15}$$

On the other hand, $h(T_i, T_j) = \left\langle D_{T_i} S, T_j\right\rangle$, in view of (7) and (8). Hence

$$h(T_i \wedge T_j,\ T_i \wedge T_j) = h(T_i, T_i)h(T_j, T_j) - h(T_i, T_j)h(T_j, T_i)$$
$$= \left\langle D_{T_i} S, T_i\right\rangle \left\langle D_{T_j} S, T_j\right\rangle - \left\langle D_{T_i} S, T_j\right\rangle \left\langle D_{T_j} S, T_i\right\rangle.$$

By (12) and (14), this gives the equation

$$\sum_{i,j} h(T_i \wedge T_j, T_i \wedge T_j)(m) = \Big(\sum_{i,j} \gamma_{ii}\gamma_{jj} - \gamma_{ij}\gamma_{ji} \Big)(m).$$

Next, if R denotes the curvature tensor of $\overline{M}$, then by definition and (12),

$$\langle \mathscr{R}(S), S \rangle(m) = \sum_i \langle R_{ST_i} S, T_i \rangle (m) = \sum_i \langle (-D_S D_{T_i} + D_{T_i} D_S + D_{[S,T_i]})S, T_i \rangle (m)$$

$$= - \sum_i \langle D_S D_{T_i} S, T_i \rangle (m),$$

because of (9) and (10). Using (10) and (14), we obtain the equation

$$\langle \mathscr{R}(S), S \rangle (m) = - \sum_i \langle D_S D_S T_i, T_i \rangle (m) = - \sum_i (S\gamma_{ii} + \gamma_{ij}\gamma_{ji})(m).$$

Hence, $\Big(\sum_{i,j} h(T_i \wedge T_j, T_i \wedge T_j) - \langle \mathscr{R}(S), S \rangle \Big)(m) = \Big(\sum_{i,j} \gamma_{ii}\gamma_{jj} + \sum_i S\gamma_{ii} \Big)(m)$, which together with (15) proves (13) and thus establishes the proposition.

3. We can now prove the theorem. First of all, let $\dim \overline{M} = 2$. If $\phi(s) = \ell(s) - \ell(0)$ is linear for every such curve M, then $\ell''(s) = 0$ for *every* admissible curve M (see the Introduction) and for all $s \in [0, \varepsilon]$. By (6), $K \equiv 0$, and therefore M is locally the euclidean plane. Of course, if $\overline{M}$ is locally the euclidean plane, then $K \equiv 0$ and (6) implies $\ell''(s) = 0$. Then $\phi(s) = \ell(s) - \ell(0) = \ell'(0)s$, and ϕ is linear.

It remains to prove that no Riemannian manifold with dimension exceeding 2 can have the linear-growth property. As before, $\phi(s) = \mathscr{A}(s) - \mathscr{A}(0)$ is a linear function of s if and only if $\mathscr{A}''(s) = 0$. By virtue of the proposition, $\mathscr{A}''(s) = 0$ for *every* orientable compact hypersurface if and only if

$$\sum_{i,j} h(e_i \wedge e_j, e_i \wedge e_j) - \langle \mathscr{R}(S), S \rangle \equiv 0$$

on every orientable, compact hypersurface M of $\overline{M}$. We shall show that this is impossible. First pick any point $m \in \overline{M}$ and construct a system of geodesic coordinates around m; call it $\{x_1, \cdots, x_d\}$. We can of course assume that $\Big\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \Big\rangle (m) = \delta_{ij}$. Let M_ℓ be a compact hypersurface which near m has the form $\{(x_1, \cdots, x_d): x_\ell = 0\}$. Since M_ℓ near m consists of geodesics through m, its second fundamental form is zero at m. Hence $h\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big)(m) = 0$ for all $i, j \neq \ell$. Therefore

$$h\Big(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}\Big)(m) = 0 \qquad \text{for all } i, j \neq \ell.$$

Consequently, $\Big\langle \mathscr{R}\Big(\frac{\partial}{\partial x_\ell}\Big), \frac{\partial}{\partial x_\ell} \Big\rangle (m) = 0$. Since this is true for every ℓ and every m, $\overline{M}$ must have identically vanishing Ricci curvature. Consequently, $\sum_{i,j} h(e_i \wedge e_j, e_i \wedge e_j) \equiv 0$ on every compact, orientable hypersurface M of $\overline{M}$. If we can find one such M with positive-definite second fundamental form, then clearly h is also positive-definite on $M_m \wedge M_m$ for every $m \in M$, and we shall arrive at a contradiction. Therefore the following lemma concludes the proof.

LEMMA. *Sufficiently small geodesic spheres all have a positive-definite second fundamental form; that is, if* $x_1, \cdots, x_d$ *are geodesic coordinates around* m *and* $S_\varepsilon = \left\{ \sum_i x_i^2 = \varepsilon^2 \right\}$, *then* S_ε *has a positive-definite second fundamental form if* ε *is small.*

Proof. Let $g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle$ in this geodesic-coordinate neighborhood. It is clear that

$$g_{ij}(0) = \delta_{ij}, \tag{16}$$

$$\frac{\partial g_{ij}}{\partial x_k}(0) = 0 \qquad \text{for all } i, j, k. \tag{17}$$

The *unit* radial vector field is

$$\frac{\partial}{\partial r} \equiv \alpha \sum_k x_k \frac{\partial}{\partial x_k}, \qquad \text{where } \alpha = \left(\sum_{i,j} g_{ij} x_i x_j \right)^{-1/2}$$

Pick a point $p \in S_\varepsilon$ such that $x(p) \neq 0$. The vectors

$$u_i = x_\ell \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_\ell} \qquad \text{for all } i \neq \ell$$

clearly form a basis of the tangent space to S_ε at p. Therefore

$$v_i = \beta \left(x_\ell \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_\ell} \right) \qquad \text{for all } i \neq \ell \text{ with } \beta = (x_\ell^2 g_{ii} + x_i^2 g_{\ell\ell} - 2 x_i x_\ell g_{i\ell})^{-1/2}$$

is a basis consisting of *unit vectors*. If h denotes the second fundamental form of S_ε, then

$$\begin{aligned} h(v_i, v_i)(p) &= \left\langle D_{v_i} \frac{\partial}{\partial r}, v_i \right\rangle(p) \\ &= \beta^2 x_\ell \frac{\partial \alpha}{\partial x_i} \left(x_\ell \sum_k x_k g_{ik} - x_i \sum_k x_k g_{\ell k} \right) + \beta^2 x_i \frac{\partial \alpha}{\partial x_\ell} \left(x_i \sum_k x_k g_{\ell k} - x_\ell \sum_k x_k g_{ik} \right) \\ &\qquad + \alpha\beta^2 (x_\ell^2 g_{ii} + x_i^2 g_{\ell\ell}) + \text{remainder}, \end{aligned}$$

where the remainder approaches zero as $p \to 0$ (we have used the positive definiteness of $\sum_{i,j} g_{ij} x_i x_j$, together with (16) and (17)). It is then clear that

$$\left(\sum_{i,j} g_{ij} x_i x_j \right)^{1/2} h(v_i, v_i)(p) \to 1 \qquad \text{as } p \to 0.$$

Thus $h(v_i, v_i)(p) \to +\infty$ as $p \to 0$. Consequently, since S_ε is compact, there exists an ε_0 such that h is positive-definite on S_ε whenever $\varepsilon < \varepsilon_0$. Q.E.D.

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