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A CHARACTERIZATION IN THE SPACE OF CONVOLUTION OPERATORS

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ABSTRACT. We give a characterization of C^∞ elements in the space of convolution operators θ_c', which belong to the Schwartz space $\mathscr{S}$.

In the space of convolution operators θ_c', also referred to as the space of distributions which are rapidly decreasing at ∞, there are C^∞ elements which are not rapidly decreasing in the sense of the Schwartz space $\mathscr{S}$. For a preliminary discussion of the spaces θ_c' and its image θ_M under the Fourier transform, see Treves [**5**, pp. 314–321], or Schwartz [**4**, Chapter VII, §8].

The purpose of this note is to give a necessary and sufficient condition for C^∞ elements in θ_c' to belong to $\mathscr{S}$. Notation used here but not defined is standard; see Hörmander [**1**, **2**].

Let $\rho(\xi) \in \mathscr{S}^m$ be any "constant coefficient" symbol, that is, $\rho(\xi) \in C^\infty(\mathbf{R}^n)$ and satisfies for each multi-index α the estimate

$$(1) \qquad |D_\xi^\alpha \rho(\xi)| \leqslant c_\alpha (1 + |\xi|)^{m - |\alpha|}, \qquad \xi \in \mathbf{R}^n,$$

where c_α are constants. Let $\rho(D)$ be the pseudo-differential operator corresponding to the symbol $\rho(\xi)$. Then clearly, for $v \in \mathscr{E}'$, one has

$$\widehat{\rho(D) v}(\xi) = \rho(\xi) \hat{v}(\xi) \in \theta_M.$$

Thus $\rho(D)$ may be considered as a map $\rho(D)$: $\mathscr{E}' \to \theta_c'$ and is sometimes referred to as a *Friedrichs operator*. With this notation our main result becomes the following

THEOREM. *Let $f \in \theta_c' \cap C^\infty(\mathbf{R}^n)$. Then $f \in \mathscr{S}$ if and only if there exists a Friedrichs operator $\rho(D)$ such that $f = \rho(D)v$ for some distribution v with compact support.*

REMARK. A weaker form of the Theorem, namely that if $\chi(\xi)$ is a C^∞ function on $\mathbf{R}^n$, positively homogeneous of degree 0 for $|\xi| > 1$ and if $v \in \mathscr{E}'$ is such that $\chi(D)v$ is C^∞, then $|\chi(\xi)\hat{v}(\xi)| = O(|\xi|^{-N})$ for all positive integers N, has been *mentioned* in Nirenberg [**3**, p. 42].

The proof of the Theorem is based on the following

LEMMA. $\{\rho(D)v \colon v \in \mathscr{E}'\} \cap C^\infty(\mathbf{R}^n) \subset \mathscr{S}$.

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PROOF. Suppose $\rho(D)v \in C^\infty(\mathbf{R}^n)$ for some $v \in \mathscr{E}'$, $v \neq 0$. Then we may write (see [**2**, Theorem 2.13, p. 149])

$$\rho(D) = \rho_1 + \rho_2 \tag{2}$$

where ρ_1 is properly supported, that is, maps C_0^∞ into itself and $\mathscr{E}'$ into itself; ρ_2 has a C^∞ kernel and defines a map on $\mathscr{E}' \to C^\infty(\mathbf{R}^n)$. This decomposition is defined by a decomposition of the Schwartz kernel K_ρ associated with $\rho(D)$, in the form

$$K_{\rho_1} = \theta K_\rho \quad \text{and} \quad K_{\rho_2} = (1-\theta)K_\rho \tag{3}$$

where $\theta \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ which takes values in $[0,1]$, equals 1 in a neighbourhood of the diagonal of $\mathbf{R}^n \times \mathbf{R}^n$ and is *properly supported*.

The expression for the kernel K_ρ is given by

$$K_\rho(x, y) = \frac{(2\pi)^{-n}}{(x-y)^\alpha}\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}(-D_\xi)^\alpha \rho(\xi)\,d\xi, \qquad x, y \in \mathbf{R}^n, x \neq y, \tag{4}$$

where the multi-index α is arbitrary but chosen so that the integral converges absolutely. Now by (2), we have

$$\rho(D)v = \rho_1(v) + \rho_2(v) \Rightarrow \rho_1(v) \in C_0^\infty(\mathbf{R}^n). \tag{5}$$

Our aim now is to prove that $\rho_2(v) \in \mathscr{S}$. To this end, we shall first establish the form of the C^∞ function $\rho_2(v)$. If we choose $f \in C_0^\infty(\mathbf{R}^n)$ such that $f = 1$ in a neighbourhood of supp(v), then (see [**2**, Definition 2.11, p. 148])

$$\rho_2(v)(x) = \rho_2(fv)(x) = (2\pi)^{-n}\int_{\mathbf{R}^n} e^{ix\cdot\xi}p_f(x, \xi)\hat{v}(\xi)\,d\xi \tag{6}$$

where p_f stands for a symbol of ρ_2 in $\mathscr{S}^{-\infty}$ and is given by the expression

$$e^{ix\cdot\xi}p_f(x, \xi) = \rho_2(fe^{i\langle\cdot,\xi\rangle})(x) \tag{7}$$

where $e^{i\langle\cdot,\xi\rangle}$ denotes for each ξ the function $e^{ix\cdot\xi}$. Now expressing the action of ρ_2 in terms of its kernel, we have from (3) and (4)

(8)

$$\rho_2(fe^{i\langle\cdot,\xi\rangle})(x) = (2\pi)^{-n}\int_{\mathbf{R}^n}\left[\frac{(1-\theta)}{(x-y)^\alpha}\int_{\mathbf{R}^n} e^{i(x-y)\cdot t}(-D_t)^\alpha\rho(t)\,dt\right]f(y)e^{iy\cdot\xi}\,dy.$$

Noting that $\theta(x, y)f(y)$ has compact support since θ is properly supported and integrating by parts with respect to y, we obtain, for arbitrary multi-indices β_1, β_2

$$\sup_{(x,\xi)\in\mathbf{R}^n\times\mathbf{R}^n}\left|\xi^{\beta_1}D_x^{\beta_2}\rho_2(fe^{i\langle\cdot,\xi\rangle})(x)\right| < \infty \tag{9}$$

when α is chosen so that $|\alpha| > (m + n + |\beta_1| + |\beta_2|)$. Now from (6) and (7) we have

$$\rho_2(v)(x) = (2\pi)^{-n}\int_{\mathbf{R}^n}\hat{v}(\xi)\rho_2(fe^{i\langle\cdot,\xi\rangle})(x)\,d\xi \tag{10}$$

where we note that $\hat{v}$ has polynomial growth, say of order N, and the integral is absolutely convergent because of (9). Differentiating under the integral sign in (10), it is clear that for arbitrary multi-indices μ, ν we have

$$\sup_{x\in\mathbf{R}^n}\left|x^\mu D_x^\nu\rho_2(v)(x)\right| < \infty$$

when the multi-index α appearing in (8) is chosen large enough, that is,

$$|\alpha| > \max\{|\mu|, m + 2n + N + 1 + |\nu|\}.$$

Thus, $\rho_2(v) \in \mathcal{S}$. Hence by (5) $\rho(D)v \in \mathcal{S}$ and the proof of the Lemma is complete.

PROOF OF THE THEOREM. $f \in \mathcal{S} \Leftrightarrow \hat{f}(\xi) \in \mathcal{S} \Leftrightarrow \hat{f}(\xi) \in \mathcal{S}^m\ \forall$ real m. Also we have $f = \hat{f}(D)\delta$, where δ is the Dirac measure on $\mathbf{R}^n$. The rest of the proof follows from the Lemma.

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