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A CHARACTERIZATION OF CERTAIN WEAKLY PSEUDOCONVEX DOMAINS

Dedicated to Professor Fuichi Uchida on his sixtieth birthday

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Abstract. By making use of well-known extension theorems on holomorphic mappings and CR-mappings and applying Webster's CR-invariant metrics, we give a characterization of certain weakly pseudoconvex domains from the viewpoint of biholomorphic automorphism groups.

Introduction. This is a continuation of our previous paper [11], and we retain the terminology and notation there.

Let D be a bounded domain in $\boldsymbol{C}^n$ and let $p \in \partial D$. Then we say that *the condition* $(*)$ *is fulfilled for* (D, p) if

$(*)$ there exists a compact set K in D, a sequence $\{k_\nu\}$ in K and a sequence $\{\varphi_\nu\}$ in $\mathrm{Aut}(D)$ such that $\lim_{\nu\to\infty} \varphi_\nu(k_\nu) = p$.

Now assume that the condition $(*)$ is fulfilled for (D, p). Then *we may ask if it is possible to determine the global structure of D from the local shape of the boundary* ∂D *near p*. Certainly, it is impossible without any further assumption, as one may see in the examples such as the direct product of the open unit disk in $\boldsymbol{C}$ and an arbitrary bounded domain in $\boldsymbol{C}^{n-1}$. As for this problem, it was shown by Wong [25] that if D is a strictly pseudoconvex domain in $\boldsymbol{C}^n$ with smooth boundary and the condition $(*)$ is fulfilled for (D, p) for some $p \in \partial D$, then D is biholomorphically equivalent to the open unit ball B^n in $\boldsymbol{C}^n$. It was later extended by Rosay [20] to the case where ∂D near p is C^2-smooth and strictly pseudoconvex. It is natural to see *what happens when p is a weakly* (*not strictly*) *pseudoconvex boundary point of D*. It was Greene and Krantz [8] who first dealt with this problem in the category of weakly pseudoconvex domains in $\boldsymbol{C}^n$ with globally C^{n+1}-smooth boundaries. As a generalization of their result, we obtained in [11] the following characterization of the weakly pseudoconvex domain

$$E(k, \alpha) = \left\{ z \in \boldsymbol{C}^n \,\middle|\, \sum_{i=1}^{k} |z_i|^2 + \left(\sum_{j=k+1}^{n} |z_j|^2 \right)^{\alpha} < 1 \right\},$$

where $k \in \boldsymbol{Z}$ with $1 \leqq k \leqq n$ and $0 < \alpha \in \boldsymbol{R}$, and it is understood that $E(k, \alpha) = B^n$ if $k = n$:

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THEOREM K (Kodama [11]). *Let D be a bounded domain in $\boldsymbol{C}^n$ satisfying the following conditions:*

(1) $p=(1,0,\dots,0)\in\partial D\cap\partial E(k,\alpha)$;

(2) *there is an open neighborhood U of p in $\boldsymbol{C}^n$ such that $D\cap U=E(k,\alpha)\cap U$;*

(3) *the condition $(*)$ is fulfilled for (D,p).*

Then D is biholomorphically equivalent to the domain $E(k,\alpha)$.

It should be remarked that, in general, $E(k,\alpha)$ is not geometrically convex and, moreover, its boundary is not smooth at every point x of the form $x=(x_1,\dots,x_k,0,\dots,0)$. Also, noting the fact that such a boundary point x is an accumulation point of the $\mathrm{Aut}(E(k,\alpha))$-orbit passing through the origin of $\boldsymbol{C}^n$, one sees that exactly the same conclusion in Theorem K remains valid for an arbitrary point $x=(x_1,\dots,x_k,0,\dots,0)\in\partial D\cap\partial E(k,\alpha)$ as well as $p=(1,0,\dots,0)$. This theorem was later extended by Kodama, Krantz and Ma [15] to a more general domain, called a *generalized complex ellipsoid,*

$$E(n;n_1,\dots,n_s;p_1,\dots,p_s)=\left\{(z_1,\dots,z_s)\in\boldsymbol{C}^{n_1}\times\cdots\times\boldsymbol{C}^{n_s}\;\middle|\;\sum_{i=1}^{s}|z_i|^{2p_i}<1\right\}$$

in $\boldsymbol{C}^n=\boldsymbol{C}^{n_1}\times\cdots\times\boldsymbol{C}^{n_s}$, where $0<p_1,\dots,p_s\in\boldsymbol{R}$ and $0<n_1,\dots,n_s\in\boldsymbol{Z}$ with $n=n_1+\cdots+n_s$, as follows:

THEOREM K-K-M (Kodama, Krantz and Ma [15]). *Let D be a bounded domain in $\boldsymbol{C}^n$ with a point $p\in\partial D$ and E a generalized complex ellipsoid in $\boldsymbol{C}^n$ as above. We assume that*

(1) *$p\in\partial E$ and there is an open neighborhood U of p in $\boldsymbol{C}^n$ such that $D\cap U=E\cap U$;*

(2) *the condition $(*)$ is fulfilled for (D,p) and also for (E,p).*

Then D is biholomorphically equivalent to E. In particular, at least one of the exponents p_i must be equal to 1.

In view of Kodama [12], [13] (in which the structure of generalized complex ellipsoids in $\boldsymbol{C}^n$ with all $n_i=1$ was investigated), it would be natural to ask the following questions: In Theorem K-K-M,

(Q.1) *can we remove the condition $(*)$ for (E,p)?;*

(Q.2) *can we prove that $D=E$ as sets?*

These cannot be answered in full generality at this moment except when all p_i's are positive integers, i.e., the boundary ∂E is real-analytic (cf. [14]). Recall that our proofs there relied heavily upon a result on the localization principle of holomorphic automorphisms of generalized complex ellipsoids E with real analytic boundaries due to Dini and Selvaggi Primicerio [5], [6]. A glance at their proof tells us that the real analyticity of ∂E cannot be avoided with their technique.

The main purpose of this paper is to give partial affirmative answers to the questions (Q.1) and (Q.2) when the boundary ∂E is not necessarily smooth. In fact, we consider here exclusively generalized complex ellipsoids $E(n; k, n-k; 1, \alpha)=E(k, \alpha)$ with arbitrary real numbers $\alpha>0$ and prove the following theorems, which were announced at the POSTECH International Conference on Several Complex Variables in Pohang, South Korea, 1997:

THEOREM 1. *Let $E_1=E(k, \alpha)$, $E_2=E(l, \beta)$ be generalized complex ellipsoids in $\boldsymbol{C}^n$ with arbitrary real numbers α, $\beta>0$ and let $p_1 \in \partial E_1$, $p_2 \in \partial E_2$. We assume that*

(1) *$k \leqq n-2$ and $l \leqq n-2$;*

(2) *there are open neighborhoods U_1 of p_1, U_2 of p_2 in $\boldsymbol{C}^n$ and a biholomorphic mapping $f: U_1 \rightarrow U_2$ such that $f(p_1)=p_2$, $f(U_1 \cap E_1)=U_2 \cap E_2$ and $f(U_1 \cap \partial E_1)=U_2 \cap \partial E_2$.*

Then f extends to a biholomorphic mapping F from E_1 onto E_2. In particular, we have $(k, \alpha)=(l, \beta)$.

Combining this with a result of Bell [2; Theorem 2], we obtain the following:

COROLLARY. *Let $E(k, \alpha)$ and $E(l, \beta)$ be generalized complex ellipsoids in $\boldsymbol{C}^n$ with $k \leqq n-2$, $l \leqq n-2$ and assume that $f: E(k, \alpha) \rightarrow E(l, \beta)$ is a proper holomorphic mapping. Then $(k, \alpha)=(l, \beta)$ and f is a biholomorphic automorphism of $E(k, \alpha)$.*

THEOREM 2. *Let D be a bounded domain in $\boldsymbol{C}^n$ and let $E=E(k, \alpha)$ be a generalized complex ellipsoid in $\boldsymbol{C}^n$ with $0<\alpha \in \boldsymbol{R}$. We assume that*

(1) *there exist a point $p \in \partial D \cap \partial E$ and an open neighborhood U of p in $\boldsymbol{C}^n$ such that $D \cap U=E \cap U$;*

(2) *the condition $(*)$ is fulfilled for (D, p).*

Then we have $D=E$ as sets.

We would like to remark that the assumption (1) in Theorem 1 is essential. Indeed, consider the generalized complex ellipsoids $E_1=\{(z, w) \in \boldsymbol{C} \times \boldsymbol{C} \mid |z|^2+|w|^{2\alpha}<1\}$, $E_2=B^2$ and a branch f of $(z, w) \mapsto (z, w^\alpha)$ defined in a small neighborhood of a point $p_1=(z_o, w_o) \in \partial E_1$ with $w_o \neq 0$, where $0<\alpha \in \boldsymbol{R}$, $\alpha \neq 1$. Then f gives rise to a biholomorphic equivalence between a neighborhood U_1 of p_1 and a neighborhood U_2 of $p_2:=f(p_1) \in \partial E_2$ satisfying the condition (2) in Theorem 1; however, it is clear that f cannot be continued to a biholomorphic mapping from E_1 onto E_2. Also, considering the special case $\alpha=\beta=1$ in the corollary above, we see that every proper holomorphic self-mapping of the unit ball B^n must be a biholomorphic automorphism of B^n. This is just a well-known theorem of Alexander [1].

In Section 1, by making use of Rudin's extension theorem [21; p. 311] on holomorphic mappings defined near boundary points of B^n, we show some properties of generalized complex ellipsoids $E(k, \alpha)$, which will be a key step to the proofs of our theorems. After this preparation, Theorems 1 and 2 will be proved in Sections 2 and 3, respectively. Our proofs here are based on some extension theorems on proper holomorphic mappings and CR-mappings obtained by Forstnerič and Rosay [7], Pinchuk [18], [19], Bell [3], and also on the existence of Webster's CR-invariant metrics on strictly pseudoconvex real analytic hypersurfaces in C^n without umbilical points [22], [23].

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1. A key lemma. For later purpose, we prove some facts on the structure of the model spaces $E(k, \alpha)$ with arbitrary real numbers $\alpha > 0$.

Throughout the rest of this paper, we use the following notation: For a point $z = (z_1, \dots, z_n) \in C^n$ and for a domain $E(k, \alpha)$, we set $z' = (z_1, \dots, z_k)$, $z'' = (z_{k+1}, \dots, z_n)$, $E = E(k, \alpha)$ and

$$\partial^* E = \{(z', z'') \in C^k \times C^{n-k} \mid z'' \neq 0, |z'|^2 + |z''|^{2\alpha} = 1\}$$

which is an open dense subset of ∂E. Then, by using the facts in the previous paper [11; Section 1], the following assertions are easily proved:

(1.1) $\partial^* E$ is a connected, strictly pseudoconvex, real analytic hypersurface in C^n; moreover, it is simply connected if $k \leqq n-2$ [9; p. 346].

(1.2) $\mathrm{Aut}(E)$ can be regarded as a subgroup of $\mathrm{Aut}(B^k \times C^{n-k})$.

(1.3) $\mathrm{Aut}(E) \cdot \partial^* E = \partial^* E$ and $\mathrm{Aut}(E)$ acts transitively on $\partial^* E$ as a real analytic CR-automorphism group of $\partial^* E$.

The following lemma will play a crucial role in our proofs of Theorems 1 and 2.

LEMMA. *Let $E = E(k, \alpha)$ be a generalized complex ellipsoid in C^n with $k \leqq n-2$ and let $p \in \partial^* E$. Assume that there are an open neighborhood U of p in C^n and a biholomorphic mapping f from U into C^n such that*

$$U \cap \partial E = U \cap \partial^* E, \ f(U \cap \partial^* E) = f(U) \cap \partial B^n \quad \text{and} \quad f(U \cap E) = f(U) \cap B^n .$$

Then f extends to a biholomorphic mapping F: $E \to B^n$. In particular, we have $\alpha = 1$.

PROOF. Since $\partial^* E$ is a connected, strictly pseudoconvex, real analytic hypersurface in C^n by (1.1), it follows from a result of Pinchuk [18], [19; p. 193] that f can be continued along any path lying in $\partial^* E$ as a locally biholomorphic mapping. Since $\partial^* E$ is now simply connected by our assumption $k \leqq n-2$, the monodromy theorem guarantees that f extends to a locally biholomorphic mapping F defined on some connected open neighborhood V of $\partial^* E$ in C^n such that $F(\partial^* E) \subset \partial B^n$ and $F(V \cap E) \subset B^n$. Now we will proceed in several steps.

(1) *F extends to a holomorphic mapping $\tilde{F}$ from E into B^n.* To prove this, take an arbitrary r with $0 < r < 1$ and put

$$K_r = \{(z', z'') \in C^k \times C^{n-k} \mid |z'| \leqq r, |z'|^2 + |z''|^{2\alpha} = 1\} .$$

Since $K_r \subset \partial^* E \subset V$ and K_r is compact in V, one can choose a small $\varepsilon = \varepsilon(r) > 0$ in such a way that

$$U_{r,\varepsilon} := \{(z', z'') \in C^k \times C^{n-k} \mid |z'| < r, 1 - \varepsilon < |z'|^2 + |z''|^{2\alpha} < 1 + \varepsilon\} \subset V .$$

Clearly, $U_{r,\varepsilon}$ is a bounded Reinhardt domain in C^n. Moreover, since $k \leqq n-2$, we have $U_{r,\varepsilon} \cap \{z \in C^n \mid z_j = 0\} \neq \varnothing$ for $j = 1, \ldots, n$. Hence, by a well-known fact [16; p. 15] every component function F_j of F has a holomorphic extension F_j^r to the domain

$$\hat{U}_{r,\varepsilon} = \{(z', z'') \in C^k \times C^{n-k} \mid |z'| < r, |z'|^2 + |z''|^{2\alpha} < 1 + \varepsilon\},$$

the smallest complete Reinhardt domain in C^n containing $U_{r,\varepsilon}$. In particular, putting

$$E_r = \{(z', z'') \in C^k \times C^{n-k} \mid |z'| < r, |z'|^2 + |z''|^{2\alpha} < 1\},$$

we see that $F = (F_1, \ldots, F_n)$ has a holomorphic extension $F^r := (F_1^r, \ldots, F_n^r)$ to $E_r \cup V$. Note that $E_r \subset E_s$ for $0 < r < s < 1$, $\bigcup_{0<r<1} E_r = E$ and that the holomorphic extensions F^r are uniquely determined by the values of F on a small neighborhood of the point $(0, \ldots, 0, 1) \in V \cap \partial^* E$. Then, by standard argument, one can define a holomorphic extension $\tilde{F}: E \cup V \to C^n$ of $F: V \to C^n$.

Now we wish to show that $\tilde{F}(E) \subset B^n$. For this let us fix an arbitrary point $z_o = (z_o', z_o'') \in E$ and set

$$E(z_o) = \{(z_o', z'') \in C^k \times C^{n-k} \mid |z_o'|^2 + |z''|^{2\alpha} < 1\},$$

which can be regarded as an open ball in C^{n-k}. Consider the non-constant, continuous plurisubharmonic function $\psi: z'' \mapsto -1 + |\tilde{F}(z_o', z'')|^2$ defined on some open neighborhood of the closure $\overline{E(z_o)}$ of $E(z_o)$ in C^{n-k}. Then $\psi(\partial E(z_o)) = 0$ and $\psi(z'') < 0$ on $E(z_o) \cap V$. This, combined with the maximum principle for plurisubharmonic functions, guarantees that $\psi(z_o'') < 0$, i.e., $\tilde{F}(z_o) \in B^n$ and accordingly $\tilde{F}(E) \subset B^n$.

(2) *There exists a locally injective, real analytic homomorphism* $\Phi: \mathrm{Aut}(E) \to \mathrm{Aut}(B^n)$ *such that* $\Phi(\sigma) \circ \tilde{F} = \tilde{F} \circ \sigma$ *on* E *for all* $\sigma \in \mathrm{Aut}(E)$. Indeed, take an arbitrary $\sigma \in \mathrm{Aut}(E)$. By virtue of (1.2) and (1.3), one can choose an open neighborhood W of the point $p \in \partial^* E$ so small that $W \cup \sigma(W) \subset V$ and $\tilde{F}$ is injective on W and on $\sigma(W)$. Let us consider the biholomorphic mapping $\Psi := \tilde{F} \circ \sigma \circ (\tilde{F} \mid W)^{-1}: \tilde{F}(W) \to \tilde{F}(\sigma(W))$. By an extension theorem due to Rudin [21; p. 311] we obtain an element $\tilde{\Psi} \in \mathrm{Aut}(B^n)$ such that $\tilde{\Psi}(z) = \Psi(z)$ for all $z \in \tilde{F}(W \cap E)$. Note that $W \cap E$ and $\tilde{F}(W \cap E)$ are non-empty open subsets of E and B^n, respectively. Then, by the principle of analytic continuation, we have that $\tilde{\Psi} \circ \tilde{F} = \tilde{F} \circ \sigma$ on E and $\tilde{\Psi}$ is uniquely determined by σ. Accordingly, one can define a mapping

$$\Phi: \mathrm{Aut}(E) \to \mathrm{Aut}(B^n)$$

by setting $\Phi(\sigma) = \tilde{\Psi}$ so that $\Phi(\sigma) \circ \tilde{F} = \tilde{F} \circ \sigma$ on E for all $\sigma \in \mathrm{Aut}(E)$.

It is easy to check that Φ is a group homomorphism. Once it is shown that Φ is continuous at the identity element id_E of $\mathrm{Aut}(E)$, it follows that Φ is real analytic on $\mathrm{Aut}(E)$ (cf. [9; p. 117]). Since the topology of $\mathrm{Aut}(E)$ satisfies the second axiom of countability, we have only to show that Φ is sequentially continuous at id_E. For this let us take an arbitrary sequence $\{\sigma_\nu\}$ in $\mathrm{Aut}(E)$ which converges to id_E and assume that $\{\Phi(\sigma_\nu)\}$ does not converge to the identity element id_{B^n} of $\mathrm{Aut}(B^n)$. Passing to a subsequence, we may assume that there is a neighborhood O of id_{B^n} in $\mathrm{Aut}(B^n)$ such that $\Phi(\sigma_\nu)\notin O$ for all ν. Pick an arbitrary point $x\in E$. Then $\lim_{\nu\to\infty}\Phi(\sigma_\nu)(\tilde{F}(x))=\lim_{\nu\to\infty}\tilde{F}(\sigma_\nu(x))=\tilde{F}(x)\in B^n$, which implies that $\{\Phi(\sigma_\nu)(\tilde{F}(x))\}$ lies in a compact subset of B^n. Hence, after taking a subsequence if necessary, we may assume that $\{\Phi(\sigma_\nu)\}$ converges to some element $g\in\mathrm{Aut}(B^n)$ (cf. [16; p. 82]). Since $g\notin O$, we see that $g\neq\mathrm{id}_{B^n}$. On the other hand, we have $g(\tilde{F}(z))=\lim_{\nu\to\infty}\Phi(\sigma_\nu)(\tilde{F}(z))=\lim_{\nu\to\infty}\tilde{F}(\sigma_\nu(z))=\tilde{F}(z)$ for all $z\in W\cap E$; consequently, $g=\mathrm{id}_{B^n}$ by analytic continuation. This ia a contradiction. Therefore, Φ is continuous at id_E, as desired.

Finally we claim that Φ is locally injective. It suffices to prove that Φ is injective in some neighborhood O of id_E. To this end, let us select a small open neighborhood W of the point $p\in\partial^*E$ in $\boldsymbol{C}^n$ and non-empty open subsets W_1, W_2 of $W\cap E$ with the properties: $\tilde{F}$ is injective on W, and W_1 is a relatively compact subset of W_2. We claim that $O=\{\sigma\in\mathrm{Aut}(E)\,|\,\sigma(\overline{W}_1)\subset W_2\}$ is what is required. Indeed, it is clear that O is an open neighborhood of id_E in $\mathrm{Aut}(E)$. Moreover, assume that $\Phi(\sigma_1)=\Phi(\sigma_2)$ for $\sigma_1,\sigma_2\in O$. It follows that $\tilde{F}(\sigma_1(z))=\Phi(\sigma_1)(\tilde{F}(z))=\Phi(\sigma_2)(\tilde{F}(z))=\tilde{F}(\sigma_2(z))$ for all $z\in E$. Since $\tilde{F}$ is injective on $W_2\subset W$ and since $\sigma_1(z)$, $\sigma_2(z)\in W_2$ for all $z\in W_1$, this says that $\sigma_1=\sigma_2$ on W_1; and hence $\sigma_1=\sigma_2$ on E by analytic continuation. Therefore, we have shown that Φ is locally injective on $\mathrm{Aut}(E)$.

(3) *$\tilde{F}\colon E\to B^n$ is locally injective.* Set $S=\{z\in E\,|\,(J\tilde{F})(z)=0\}$, where $(J\tilde{F})(z)$ denotes the holomorphic Jacobian of $\tilde{F}$ at z. Assume that $S\neq\emptyset$. Then S is a complex analytic subset of E of dimension $n-1$. Once $S\subset\{(z',z'')\in\boldsymbol{C}^k\times\boldsymbol{C}^{n-k}\,|\,z''=0\}\equiv\boldsymbol{C}^k$ is shown, we arrive at a contradiction, since $\dim S=n-1>k=\dim\boldsymbol{C}^k$ by our assumption. Thus we have only to show that $S\subset\boldsymbol{C}^k\times\{0\}$. To this end, take an arbitrary point $x=(x',x'')\in S$ and assume that $x''\neq0$. We may assume that x is a regular point of S. Recall that $\tilde{F}\circ\sigma=\Phi(\sigma)\circ\tilde{F}$ on E for all $\sigma\in\mathrm{Aut}(E)$ by (2). Then

$$(J\tilde{F})(\sigma(x))\cdot(J\sigma)(x)=(J\Phi(\sigma))(\tilde{F}(x))\cdot(J\tilde{F})(x)=0\quad\text{and}\quad(J\sigma)(x)\neq0$$

for all $\sigma\in\mathrm{Aut}(E)$. This means that $\mathrm{Aut}(E)\cdot x$, the $\mathrm{Aut}(E)$-orbit passing through the point x, is contained in S. This is impossible. Indeed, since $x''\neq0$, one can show by using the explicit expression of $\mathrm{Aut}(E(k,\alpha))$ as in [11; Section 1] that the orbit $\mathrm{Aut}(E)\cdot x$ is a real analytic submanifold of E of real dimension $2n-1$; on the other hand, S near x is a real analytic submanifold of E of real dimension $2n-2$. Therefore we conclude that $S\subset\boldsymbol{C}^k\times\{0\}$, completing the proof of (3).

Before proceeding further, we need some preparation. First, notice that B^n is homogeneous and each element $g\in\mathrm{Aut}(B^n)$ extends to a biholomorphic mapping defined in an open neighborhood of $\bar{B}^n$. Thus, shrinking the neighborhood V of ∂^*E and replacing $\tilde{F}$ by a suitable mapping of the form $g\circ\tilde{F}$ with some $g\in\mathrm{Aut}(B^n)$, if necessary, we may assume that the holomorphic mapping $\tilde{F}\colon E\cup V\to\boldsymbol{C}^n$ satisfies an additional condition $\tilde{F}(o)=o$, where o stands for the orgin of $\boldsymbol{C}^n$. Next, let us consider the toral subgroups T_E and T_{B^n} of $\mathrm{Aut}(E)$ and $\mathrm{Aut}(B^n)$, respectively, induced by the rotations on $\boldsymbol{C}^n$ as follows:

$$(z_1, \ldots, z_n) \mapsto ((\exp\sqrt{-1}\,\theta_1)z_1, \ldots, (\exp\sqrt{-1}\,\theta_n)z_n)\,, \qquad (\theta_1, \ldots, \theta_n) \in \boldsymbol{R}^n\,.$$

Then $\Phi(T_E)(o) = \Phi(T_E)(\tilde{F}(o)) = \tilde{F}(T_E(o)) = \tilde{F}(o) = o$, which says that $\Phi(T_E)$ is contained in the unitary group $U(n)$ of degree n (the isotropy subgroup of $\mathrm{Aut}(B^n)$ at the origin o). Since $\Phi(T_E)$ as well as T_{B^n} is now a maximal torus in $U(n)$ by (2), it is well-known that they are conjugate to each other in $U(n)$, that is, there exists an element $\tau \in U(n)$ such that $\tau \cdot \Phi(T_E) \cdot \tau^{-1} = T_{B^n}$. Thus, considering $\tau \circ \tilde{F}$, $\tau \circ \Phi \circ \tau^{-1}$ instead of $\tilde{F}$, Φ if necessary, we may further assume that $\Phi(T_E) = T_{B^n}$. Under these assumptions, we claim the following:

(4) *$\tilde{F}: E \to B^n$ is, in fact, a biholomorphic mapping.* Thanks to the fact (3) one can choose a small open ball $B_\rho = \{z \in \boldsymbol{C}^n \mid |z| < \rho\} \subset E$ on which $\tilde{F}$ is injective. Then, since $\tilde{F}(B_\rho) = \tilde{F}(T_E(B_\rho)) = \Phi(T_E)(\tilde{F}(B_\rho)) = T_{B^n}(\tilde{F}(B_\rho))$, we see that $\tilde{F}(B_\rho)$ is a bounded Reinhardt domain in $\boldsymbol{C}^n$ with center at $\tilde{F}(o) = o$. Therefore, by a well-known theorem of H. Cartan [21; p. 24], the restriction $\tilde{F} | B_\rho: B_\rho \to \tilde{F}(B_\rho)$ is a linear transformation. So we may assume that $\tilde{F} \in \mathrm{Aut}(\boldsymbol{C}^n)$. This, combined with the facts that $\tilde{F}(\partial^* E) \subset \partial B^n$ and $\partial^* E$ is dense in ∂E, guarantees that $\tilde{F}(E) = B^n$; and hence $\tilde{F}: E \to B^n$ is a biholomorphic mapping. Finally, the assertion $\alpha = 1$ follows from a result of Naruki [17]. This completes the proof of the Lemma.

2. Proof of Theorem 1. The proof is divided into three cases as follows:

Case 1. $\alpha = \beta = 1$. We have $E_1 = B^n = E_2$ in this case; hence our theorem follows at once from Rudin's result [21; p. 311].

Case 2. $\alpha \neq 1, \beta = 1$ *or* $\alpha = 1, \beta \neq 1$. We claim that this case does not occur. Indeed, assume the contrary. Since $\partial^* E_1$ and $\partial^* E_2$ are open dense subsets of ∂E_1 and ∂E_2, respectively, and since $f: U_1 \to U_2$ is a biholomorphic mapping, we may assume that

$$p_1 \in \partial^* E_1\,, \quad U_1 \cap \partial E_1 = U_1 \cap \partial^* E_1\,, \quad \alpha \neq 1 \quad \text{and} \quad \beta = 1\,.$$

In particular, we have $E_2 = B^n$. As an immediate consequence of the Lemma in Section 1, we now have $\alpha = 1$, a contradiction.

Case 3. $\alpha \neq 1, \beta \neq 1$. Without loss of generality, we may assume that $p_i \in \partial^* E_i$ and $U_i \cap \partial E_i = U_i \cap \partial^* E_i$ for each $i = 1, 2$. Here, we claim that any strictly pseudoconvex real analytic hypersurface $\partial^* E_i$ has no umbilical points in the sense of CR-geometry; hence, Webster's CR-invariant Riemannian metric g_i can be defined on the whole space $\partial^* E_i$. (For the notion of umbilical points and Webster's CR-invariant metrics in CR-geometry, see [4]; and also, [22], [23], [24].) To prove our claim, assume that there exists an unbilical point on $\partial^* E_i$. Then, all the points of $\partial^* E_i$ are umbilical, since $\mathrm{Aut}(E_i)$ acts transitively on $\partial^* E_i$ by (1.3). Hence, $\partial^* E_i$ must be locally biholomorphically equivalent to the sphere ∂B^n (see, for example, [22; p. 213]). By the Lemma in Section 1 we conclude that $\alpha = 1$ or $\beta = 1$ according as $i = 1$ or $i = 2$. This is a contradiction, as desired. Moreover, we see that $(\partial^* E_i, g_i)$ is complete as a Riemannian manifold, because $\partial^* E_i$ is homogeneous under the CR-automorphism group $\mathrm{Aut}(E_i)$. As a result, each $(\partial^* E_i, g_i)$

is a connected and simply connected, complete real analytic Riemannian manifold. On the other hand, $f\colon U_1 \cap \partial^* E_1 \to U_2 \cap \partial^* E_2$ is an isometry with respect to the CR-invariant metrics g_1 and g_2. By a well-known fact in Riemannian geometry [10; p. 256], f can now be uniquely extended to a global isometry $F\colon (\partial^* E_1, g_1) \to (\partial^* E_2, g_2)$. It is easily seen that $F\colon \partial^* E_1 \to \partial^* E_2$ is a real analytic CR-diffeomorphism. Accordingly, by a result of Pinchuk [18], [19; p. 186] there are open neighborhoods V_1 of $\partial^* E_1$ and V_2 of $\partial^* E_2$ in $\boldsymbol{C}^n$ such that $F\colon \partial^* E_1 \to \partial^* E_2$ and its inverse $G := F^{-1}\colon \partial^* E_2 \to \partial^* E_1$ extend to locally biholomorphic mappings written in the same notation $F\colon V_1 \to \boldsymbol{C}^n$ and $G\colon V_2 \to \boldsymbol{C}^n$ satisfying $F(V_1 \cap E_1) \subset E_2$ and $G(V_2 \cap E_2) \subset E_1$. Hence, in exactly the same way as in (1) of the proof of the Lemma in Section 1, it can be shown that F and G extend to holomorphic mappings $\tilde{F}\colon E_1 \to \boldsymbol{C}^n$ and $\tilde{G}\colon E_2 \to \boldsymbol{C}^n$. Moreover, replacing $\psi(z'')$ by $\psi_1(z'') = \rho_2(\tilde{F}(z'_o, z''))$ in (1) of the proof of the Lemma in Section 1, we can prove that $\tilde{F}(E_1) \subset E_2$, where ρ_2 is the continuous plurisubharmonic function on $\boldsymbol{C}^n$ defined by $\rho_2(z) = -1 + \sum_{i=1}^{l} |z_i|^2 + (\sum_{j=l+1}^{n} |z_j|^2)^{\beta}$, $z \in \boldsymbol{C}^n$. Analogously, we see that $\tilde{G}(E_2) \subset E_1$. Since $\tilde{G} \circ \tilde{F} = \mathrm{id}_{E_1}$ near $\partial^* E_1$ and $\tilde{F} \circ \tilde{G} = \mathrm{id}_{E_2}$ near $\partial^* E_2$, we conclude by analytic continuation that $\tilde{G} \circ \tilde{F} = \mathrm{id}_{E_1}$ and $\tilde{F} \circ \tilde{G} = \mathrm{id}_{E_2}$; consequently, $\tilde{F}\colon E_1 \to E_2$ is a biholomorphic mapping. Finally the assertion $(k, \alpha) = (l, \beta)$ follows now from Naruki [17], completing the proof of Theorem 1.

3. **Proof of Theorem 2.** The case $k = n-1$ is contained in our previous paper [13]. Thus it suffices to prove Theorem 2 when $k \leqq n-2$. We have two cases to consider:

Case 1. *The point $p \in \partial D$ is a strictly pseudoconvex boundary point.* Hence D is biholomorphically equivalent to B^n by a result of Rosay [20]. Fix a biholomorphic mapping $F\colon D \to B^n$. Using a theorem on the boundary continuity of proper holomorphic mappings due to Forstnerič and Rosay [7], one sees that F extends to a homeomorphism from a connected open neighborhood M of p in $\partial D \cap \partial E$ onto an open subset M' of ∂B^n. Accordingly, by results of Bell [3; Theorem 2], Pinchuk [19; p. 186], the CR-homeomorphism $F\colon M \to M'$ can be extended to a biholomorphism between some open neighborhoods O of M and O' of M' in $\boldsymbol{C}^n$. Hence, $E = B^n$ by the Lemma in Section 1 and F extends to a biholomorphic automorphism Φ of B^n by [21; p. 311]. Set $\Psi = \Phi^{-1} \in \mathrm{Aut}(B^n)$. Then, since $\Psi = F^{-1}$ near M', we have that $\Psi = F^{-1}$ on B^n by analytic continuation. Thus we obtain that $D = F^{-1}(B^n) = \Psi(B^n) = B^n = E$, as desired.

Case 2. *The point $p \in \partial D$ is not a strictly pseudoconvex boundary point.* The point p must be of the form $p = (p_1, \ldots, p_k, 0, \ldots, 0)$ by (1.1). Therefore, it follows at once by Theorem K in the introduction that there exists a biholomorphic mapping $F\colon D \to E$. In exactly the same way as in the proof of [13; Lemma 3], it can be shown that F extends to a homeomorphism from an open subset of $U \cap \partial^* E \cap \partial D$ onto an open subset of $\partial^* E$. By the same reasoning as above, one can now find points $p_1 \in U \cap \partial^* E$, $p_2 \in \partial^* E$, open neighborhoods U_1 of p_1, U_2 of p_2 in $\boldsymbol{C}^n$ and a biholomorphic extension $\tilde{F}\colon U_1 \to U_2$ of F satisfying all the conditions in (2) of Theorem 1. Thus $\tilde{F}$ extends to a biholomorphic automorphism $\tilde{\Phi}$ of E; hence, repeating exactly the same arguments as in Case 1, we can show that $D=E$ as sets. This completes the proof of Theorem 2.

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