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A CHARACTERIZATION OF COLLECTIONS OF TWO-POINT SETS WITH THE UNIQUENESS PROPERTY

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Abstract. We give a sufficient condition for a collection of two-point sets to have the uniqueness property for meromorphic functions.

1. Introduction. For nonconstant meromorphic (or entire) functions f and g on $\boldsymbol{C}$ and a discrete set S in $\hat{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$ (or $\boldsymbol{C}$), we write $f^*(S) = g^*(S)$ if $f^{-1}(S) = g^{-1}(S)$ and if for each $z_0 \in f^{-1}(S)$ two functions $f - f(z_0)$ and $g - g(z_0)$ have a zero point with the same multiplicity at z_0, where a zero point of $f - \infty$ and $g - \infty$ means that of $1/f$ and $1/g$, respectively.

Let $\{S_1, \dots, S_q\}$ be a finite collection of pairwise disjoint discrete sets in $\hat{\boldsymbol{C}}$ (or $\boldsymbol{C}$). If $f^*(S_j) = g^*(S_j)$ $(1 \le j \le q)$ imply $f = g$ for two nonconstant meromorphic functions f and g (or two nonconstant entire functions f and g) on $\boldsymbol{C}$, then the collection is said to have the uniqueness property for meromorphic (or entire) functions (abbreviated to UPM (or UPE)). As examples of such collections, we know Nevanlinna's four values theorem ([N]), uniqueness range sets for meromorphic or entire functions ([LY], [Y2]) and so on ([S], [Y3]). However, there is no result except [N], [T, Theorem 1.4] and [Y1] that characterizes such collections. In this paper we give a characterization for collections of two-point sets with the uniqueness property.

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2. Combinatorial lemmas and Borel's lemma. For a large part of this section we proceed as in [F, §2]. Let G be a torsion-free abelian multiplicative group, and consider a q-tuple $A = (a_1, a_2, \dots, a_q)$ of elements a_i in G. For a subgroup $\tilde{A}$ of G generated by $a_1, a_2, \dots, a_q$, we can take a basis $\{b_1, \dots, b_t\}$ of $\tilde{A}$. Then each a_i can be uniquely represented as

$$a_i = {b_1}^{l_{i1}} {b_2}^{l_{i2}} \cdots {b_t}^{l_{it}}$$

with suitable integers $l_{i\tau}$.

DEFINITION 2.1. We call integers $p_1, p_2, \dots, p_t$ with the following property to be *generic* with respect to $l_{i\tau}$ and call the integers $l_i := \sum_{\tau=1}^{t} l_{i\tau} p_\tau$ *representations* of a_i $(1 \le i \le q)$:

$$\text{if } \ l_i = \pm l_j, \ \text{ then } \ (l_{i1}, l_{i2}, \dots, l_{it}) = \pm(l_{j1}, l_{j2}, \dots, l_{jt}),$$

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where double signs are in the same order.

For example, it is enough to take $p_\tau = p^{\tau-1}$ $(1 \le \tau \le t)$ for an integer $p > 2\max\{|l_{i\tau}|\ ;\ 1 \le i \le q, 1 \le \tau \le t\}$.

DEFINITION 2.2. Let $q \ge r > s \ge 1$ and $A = (a_1, a_2, \dots, a_q)$ a q-tuple of elements a_i in G.

(i) We call A to have the *property* $(P_{r,s})$ if any r elements $a_{\iota(1)}, a_{\iota(2)}, \dots, a_{\iota(r)}$ in A satisfy the condition that, for any given $i_1, i_2, \dots, i_s$ $(1 \le i_1 < \cdots < i_s \le r)$, there exist some other $j_1, j_2, \dots, j_s$ $(1 \le j_1 < \cdots < j_s \le r,\ \{i_1, \dots, i_s\} \ne \{j_1, \dots, j_s\})$ satisfying

$$a_{\iota(i_1)}a_{\iota(i_2)} \cdots a_{\iota(i_s)} = a_{\iota(j_1)}a_{\iota(j_2)} \cdots a_{\iota(j_s)}\,.$$

(ii) We call A to have the *property* (P^*) if each element of $\{\prod_{j=1}^{q} a_j{}^{\varepsilon_j}\ ;\ \varepsilon_j = 0, 1\ (1 \le j \le q)\}$ coinsides with another one of this set, where $a_j{}^0 = 1$ (the unit element of G).

Let us study relations among a_i for a q-tuple $a = (a_1, a_2, \dots, a_q)$ with the property $(P_{r,s})$ or (P^*). To this end, we take representations $l_1, l_2, \dots, l_q$ of $a_1, a_2, \dots, a_q$ for suitably chosen basis and generic integers. We arrange the order of representations to be

$$l_{j_1} \le l_{j_2} \le \cdots \le l_{j_q}\,.$$

LEMMA 2.3 (for the proof, see [F]). *If a q-tuple has the property $(P_{r,s})$, it holds that*

$$l_{j_s} = l_{j_{s+1}} = \cdots = l_{j_{s+u}}\,,$$

and hence

$$a_{j_s} = a_{j_{s+1}} = \cdots = a_{j_{s+u}}$$

for $u := q - r + 1$.

LEMMA 2.4. *If a q-tuple $(a_1, a_2, \dots, a_q)$ has the property (P^*), then at least one a_j is the unit element* 1.

PROOF. Assume that none of $a_1, a_2, \dots, a_q$ is the unit element. Then $l_j \ne 0$, $j = 1, 2, \dots, q$. Let m be the integer such that

$$l_{j_1} \le l_{j_2} \le \cdots \le l_{j_m} < 0 < l_{j_{m+1}} \le \cdots \le l_{j_q}\,.$$

If $m = 0$ or $m = q$, then there is no negative l_j or positive l_j, respectively. Consider the case of $m > 0$. Then $l_{j_1} + l_{j_2} + \cdots + l_{j_m}$ is the minimum partial sum of representations and there is no other such combination with this sum, which contradicts the assumption that $(a_1, a_2, \dots, a_q)$ has the property (P^*). Hence we have $m = 0$. In the case of $m = 0$, we get a contradiction by the same method. In consequence, there is some $l_j = 0$ $(1 \le j \le q)$. □

In the next section we investigate the torsion-free abelian multiplicative group $G = \mathcal{E}/\mathcal{C}$, where $\mathcal{E}$ is the abelian group of entire functions without zeros and $\mathcal{C}$ is the subgroup of non-zero constant functions.

We close this section by the following Borel's Lemma, whose proof can be found, for example, on p. 186 of [L].

LEMMA 2.5. *If entire functions $\alpha_0, \alpha_1, \dots, \alpha_n$ without zeros satisfy*

$$\alpha_0 + \alpha_1 + \cdots + \alpha_n = 0\,,$$

then for each $j = 0, 1, \dots, n$ there exists some $k \neq j$ such that α_j/α_k is constant.

3. Main theorems.

THEOREM 3.1. *Let $\{S_1, \dots, S_q\}$ be a collection of pairwise disjoint two-point sets. Put $S_j = \{\xi_j, \eta_j\} = \{z\,;\ z^2 + a_j z + b_j = 0\}$ $(\xi_j \neq \eta_j)$. Assume that there is no Möbius transformation T such that $T(\xi_j) = \eta_j$ and $T(\eta_j) = \xi_j$ for three distinct j's. If $q \geq 6$, then the collection has UPM.*

LEMMA 3.2. *Let $S_j = \{\xi_j, \eta_j\} = \{z\,;\ z^2 + a_j z + b_j = 0\}$ $(j = 1, 2, 3)$ be pairwise disjoint. Then there exists a Möbius transformation T such that $T(\xi_j) = \eta_j$ and $T(\eta_j) = \xi_j$ $(j = 1, 2, 3)$ if and only if*

$$\begin{vmatrix} 1 & a_1 & b_1 \\ 1 & a_2 & b_2 \\ 1 & a_3 & b_3 \end{vmatrix} = 0\,. \tag{3.3}$$

PROOF. Assume that (3.3) holds. Then there exist constants a, b, c such that $(a, b, c) \neq (0, 0, 0)$ and $a + b(\xi_j + \eta_j) + c\xi_j\eta_j = 0$ for $j = 1, 2, 3$. We can take the Möbius transformation $T(z) = -(bz + a)/(cz + b)$ satisfying $T(\xi_j) = \eta_j,\ T(\eta_j) = \xi_j$ for $j = 1, 2, 3$.

Conversely, assume that a Möbius transformation $T(z) = (az + b)/(cz + d)$ satisfies $T(\xi_j) = \eta_j,\ T(\eta_j) = \xi_j$ for $j = 1, 2, 3$. Then we have $b + a\xi_j - d\eta_j - c\xi_j\eta_j = 0$ and $b + a\eta_j - d\xi_j - c\xi_j\eta_j = 0$ for $j = 1, 2, 3$. By adding these identities, we get $2b + (a - d)(\xi_j + \eta_j) - 2c\xi_j\eta_j = 0$ for $j = 1, 2, 3$. Then (3.3) follows from these, since $b = c = a - d = 0$ is impossible. □

PROOF OF THEOREM 3.1. It suffices to treat the case of $q = 6$. Assume that $f^*(S_j) = g^*(S_j)$ for two nonconstant meromorphic functions f and g. We may write $f = f_1/f_0$ by entire functions f_0, f_1 without common zeors, and $g = g_1/g_0$ in a similar manner. Then there are entire functions α_j without zeros such that

$$f_1{}^2 + a_j f_1 f_0 + b_j f_0{}^2 = \alpha_j (g_1{}^2 + a_j g_1 g_0 + b_j g_0{}^2)\,, \qquad j = 1, \dots, 6$$

by the assumption $f^*(S_j) = g^*(S_j)$. These are expressed as

$$\begin{pmatrix} 1 & a_1 & b_1 & \alpha_1 & a_1\alpha_1 & b_1\alpha_1 \\ 1 & a_2 & b_2 & \alpha_2 & a_2\alpha_2 & b_2\alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_6 & b_6 & \alpha_6 & a_6\alpha_6 & b_6\alpha_6 \end{pmatrix} \begin{pmatrix} f_1{}^2 \\ f_1 f_0 \\ f_0{}^2 \\ -g_1{}^2 \\ -g_1 g_0 \\ -g_0{}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{3.4}$$

where the determinant of the square matrix is identically zero because of $f_1 \not\equiv 0$. In the expansion of the determinant, the coefficient of $\alpha_{j_1}\alpha_{j_2}\alpha_{j_3}$ is given by

$$\pm \begin{vmatrix} 1 & a_{j_1} & b_{j_1} \\ 1 & a_{j_2} & b_{j_2} \\ 1 & a_{j_3} & b_{j_3} \end{vmatrix} \cdot \begin{vmatrix} 1 & a_{k_1} & b_{k_1} \\ 1 & a_{k_2} & b_{k_2} \\ 1 & a_{k_3} & b_{k_3} \end{vmatrix},$$

where $\{j_1, j_2, j_3, k_1, k_2, k_3\} = \{1, 2, \dots, 6\}$. These coefficients are not zero by the assumption together with Lemma 3.2. Also, by using Borel's Lemma for each triple (j_1, j_2, j_3), there exists another triple (k_1, k_2, k_2) such that $(\alpha_{j_1}\alpha_{j_2}\alpha_{j_3})/(\alpha_{k_1}\alpha_{k_2}\alpha_{k_3})$ is constant, where $1 \le j_1 < j_2 < j_3 \le 6,\ 1 \le k_1 < k_2 < k_3 \le 6$ and $\{j_1, j_2, j_3\} \ne \{k_1, k_2, k_3\}$. From Lemma 2.3 we can deduce that there exist $j_1, j_2, 1 \le j_1 < j_2 \le 6$, such that $\alpha_{j_1}/\alpha_{j_2}$ is constant. Without loss of generality, we may assume that $j_1 = 1,\ j_2 = 2$. Put $c = \alpha_1/\alpha_2$ and define a rational function $\varphi(z_0, z_1)$ by

$$\varphi(z_0, z_1) = \frac{{z_1}^2 + a_1 z_1 z_0 + b_1 {z_0}^2}{{z_1}^2 + a_2 z_1 z_0 + b_2 {z_0}^2}.$$

Then we have $\varphi(f_0, f_1) = c\varphi(g_0, g_1)$, i.e.,

$$\frac{f^2 + a_1 f + b_1}{f^2 + a_2 f + b_2} = c\frac{g^2 + a_1 g + b_1}{g^2 + a_2 g + b_2}. \tag{3.5}$$

(i) The case of $c = 1$. In this case, we get from (3.5)

$$(f - g)\{(a_1 - a_2)fg + (b_1 - b_2)(f + g) + (a_2 b_1 - a_1 b_2)\} = 0.$$

Assume that $f \ne g$. Then we have

$$g = -\frac{(b_1 - b_2)f + (a_2 b_1 - a_1 b_2)}{(a_1 - a_2)f + (b_1 - b_2)}.$$

Note that $a_1 = a_2$ and $b_1 = b_2$ imply $S_1 = S_2$, which does not occur.

Now we consider the Möbius transformation

$$T(z) = -\frac{(b_1 - b_2)z + (a_2 b_1 - a_1 b_2)}{(a_1 - a_2)z + (b_1 - b_2)},$$

which exchanges ξ_j with η_j for $j = 1, 2$, and note that $T^{-1}(z) = T(z)$. If $f(z_0) = \xi_j$ for some $j, 3 \le j \le 6$, then by the assumption $g(z_0) = \xi_j$ or $g(z_0) = \eta_j$. However, the latter implies that T exchanges ξ_j with η_j, which contradicts the assumption. The former implies that ξ_j is a fixed point of T. Since T has at most two fixed points, at least six points of $\xi_3, \eta_3, \dots, \xi_6, \eta_6$ are Picard exceptional values of f, which is also a contradiction. In consequence, $f = g$ in this case.

(ii) The case of $c \ne 1$. If there is a point z_0 such that $f(z_0) = g(z_0) \notin S_1 \cup S_2$, then $c = 1$ by (3.5). Therefore $f^{-1}(\xi_j) = g^{-1}(\eta_j)$ and $f^{-1}(\eta_j) = g^{-1}(\xi_j)$ for $j = 3, \dots, 6$. Also, there exist entire functions β_j, γ_j without zeros such that

$$f_1 - \xi_j f_0 = \beta_j(g_1 - \eta_j g_0), \quad g_1 - \xi_j g_0 = \gamma_j(f_1 - \eta_j f_0), \quad j = 3, \dots, 6.$$

By multiplying each side of these equations, we have

$$f_1g_1 - \xi_j(f_1g_0 + f_0g_1) + {\xi_j}^2 f_0g_0 = \delta_j\{f_1g_1 - \eta_j(f_1g_0 + f_0g_1) + {\eta_j}^2 f_0g_0\}$$

where $\delta_j = \beta_j\gamma_j$. Since these identities yields the equation

$$\begin{pmatrix} 1-\delta_3 & \xi_3 - \eta_3\delta_3 & {\xi_3}^2 - {\eta_3}^2\delta_3 \\ 1-\delta_4 & \xi_4 - \eta_4\delta_4 & {\xi_4}^2 - {\eta_4}^2\delta_4 \\ 1-\delta_5 & \xi_5 - \eta_5\delta_5 & {\xi_5}^2 - {\eta_5}^2\delta_5 \end{pmatrix} \begin{pmatrix} fg \\ -(f+g) \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

we have

$$\begin{vmatrix} 1-\delta_3 & \xi_3 - \eta_3\delta_3 & {\xi_3}^2 - {\eta_3}^2\delta_3 \\ 1-\delta_4 & \xi_4 - \eta_4\delta_4 & {\xi_4}^2 - {\eta_4}^2\delta_4 \\ 1-\delta_5 & \xi_5 - \eta_5\delta_5 & {\xi_5}^2 - {\eta_5}^2\delta_5 \end{vmatrix} \equiv 0\,.$$

In the expansion of the determinant, none of the coefficients of each element of $B = \{1, \delta_3, \delta_4, \delta_5, \delta_3\delta_4, \delta_4\delta_5, \delta_3\delta_5, \delta_3\delta_4\delta_5\}$ is zero. Then each of these eight functions has a partner among them such that their ratio is constant by Borel's Lemma. In other words, $C = ([\delta_3], [\delta_4], [\delta_5])$ has the property (P^*), where for $\varphi \in \mathcal{E}$ we express by $[\varphi]$ the class in $\mathcal{E}/\mathcal{C}$ to which φ belongs. Indeed, a product of any three elements in C coincides with some element in B up to multiplying constant, and also it is proportional to another element in B. However the latter coincides with a product of different at most three elements in C or the unit element from the beginning. Thus C has the property (P^*). It follows from Lemma 2.4 that there exists j_3, $3 \le j_3 \le 5$, such that δ_{j_3} is constant, and we may assume that $j_3 = 3$. Put $d = \delta_3$. Then we have

$$f_1g_1 - \xi_3(f_1g_0 + f_0g_1) + {\xi_3}^2 f_0g_0 = d\{f_1g_1 - \eta_3(f_1g_0 + f_0g_1) + {\eta_3}^2 f_0g_0\}$$

and

$$fg - \xi_3(f+g) + {\xi_3}^2 = d\{fg - \eta_3(f+g) + {\eta_3}^2\}\,.$$

Without loss of generality, we may assume that f takes at least three values of $\xi_4, \eta_4, \xi_5, \eta_5$ by assuming that an exceptional value belongs to S_6, if it exists in $S_4 \cup S_5 \cup S_6$. Hence we have

$$\xi_j\eta_j - \xi_3(\xi_j + \eta_j) + {\xi_3}^2 = d\{\xi_j\eta_j - \eta_3(\xi_j + \eta_j) + {\eta_3}^2\}\,, \quad j = 4, 5 \tag{3.6}$$

and

$${\xi_3}^2 + a_j\xi_3 + b_j = d\{{\eta_3}^2 + a_j\eta_3 + b_j\}\,, \quad j = 4, 5\,.$$

By expressing these as

$$\begin{pmatrix} {\xi_3}^2 + a_4\xi_3 + b_4 & {\eta_3}^2 + a_4\eta_3 + b_4 \\ {\xi_3}^2 + a_5\xi_3 + b_5 & {\eta_3}^2 + a_5\eta_3 + b_5 \end{pmatrix} \begin{pmatrix} 1 \\ -d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

we have

$$\begin{vmatrix} {\xi_3}^2 + a_4\xi_3 + b_4 & {\eta_3}^2 + a_4\eta_3 + b_4 \\ {\xi_3}^2 + a_5\xi_3 + b_5 & {\eta_3}^2 + a_5\eta_3 + b_5 \end{vmatrix} = 0\,.$$

On the other hand, we see that

$$\begin{vmatrix} {\xi_3}^2 + a_4\xi_3 + b_4 & {\eta_3}^2 + a_4\eta_3 + b_4 \\ {\xi_3}^2 + a_5\xi_3 + b_5 & {\eta_3}^2 + a_5\eta_3 + b_5 \end{vmatrix} = \begin{vmatrix} 1 & a_3 & b_3 \\ 1 & a_4 & b_4 \\ 1 & a_5 & b_5 \end{vmatrix} \cdot \begin{vmatrix} {\xi_3}^2 & {\eta_3}^2 & 1 \\ \xi_3 & \eta_3 & 0 \\ 1 & 1 & 0 \end{vmatrix} \neq 0\,,$$

which is a contradiction. This completes the proof.

THEOREM 3.6. *Let $\{S_1, \dots, S_q\}$ be a collection of pairwise disjoint two-point sets. Put $S_j = \{\xi_j, \eta_j\} = \{z\ ;\ z^2 + a_j z + b_j = 0\}$ $(\xi_j \neq \eta_j)$. Assume that there is no Möbius transformation T such that $T(\xi_j) = \eta_j$ and $T(\eta_j) = \xi_j$ for three distinct j's and that $a_j \neq a_k$ $(j \neq k)$. If $q \geq 5$, then the collection has UPE.*

The outline of the proof is the same as that of the proof of Theorem 3.1. There exist two points, which are contained in the sixth set S_6. The first is the point to get (3.4) and the second is the one to get (3.6), where we assumed that one of the exceptional values of f is in S_6 if they exist in $S_4 \cup S_5 \cup S_6$. We can avoid the second, since entire functions have no pole.

Instead of (3.4) we can use the relation

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & a_1 & b_1 & \alpha_1 & a_1\alpha_1 & b_1\alpha_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_5 & b_5 & \alpha_5 & a_5\alpha_5 & b_5\alpha_5 \end{pmatrix} \begin{pmatrix} f^2 \\ f \\ 1 \\ -g^2 \\ -g \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where the determinant of the square matrix is identically zero. Also, we see by the assumptions that in its expansion none of the coefficients of $\alpha_{j_1}\alpha_{j_2}\alpha_{j_3}$, $0 \leq j_1 < j_2 < j_3 \leq 5$, is zero, where $\alpha_0 \equiv 1$. Then there exist j_1, j_2, $0 \leq j_1 < j_2 \leq 5$, such that $\alpha_{j_1}/\alpha_{j_2}$ is constant. If $j_1 \geq 1$, then the rest of the proof proceeds in the same way as that of Theorem 3.1.

Now we consider the case of $j_1 = 0$. We may assume that $j_2 = 1$. Put $c := \alpha_1$(constant). Then we have

$$f^2 + a_1 f + b_1 = c(g^2 + a_1 g + b_1)\,.$$

If $c = 1$ and $f \neq g$, then $f + g + a_1 = 0$. We may assume that neither of ξ_2 and η_2 are exceptional values of f and g. If there is a point z_0 such that $f(z_0) = \xi_2$, $g(z_0) = \eta_2$, then we get $a_1 = a_2$, which is a contradiction. Otherwise, we have $2\xi_2 = -a_1 = 2\eta_2$, which is also a contradiction. Consequently, $f = g$ if $c = 1$.

When $c \neq 1$, we can proceed in the same way as in the proof of the case (ii) of Theorem 3.1, and complete the proof.

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