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A CHARACTERIZATION OF DISCRETE BANACH LATTICES WITH ORDER CONTINUOUS NORMS

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ABSTRACT. We give a characterization of those σ-Dedekind complete Banach lattices for which every continuous linear operator $T: E \to c_0$ is a difference of two positive linear operators from E into c_0.

1. Preliminary remarks. Let E and F be infinite dimensional Banach lattices. In general, the space $L(E,F)$ of all continuous linear operators from E into F is not a Riesz space (= vector lattice) with respect to the natural order, i.e., $T \geq 0$ iff $Tx \geq 0$ for $x \in E_+$, even if F is Dedekind complete. However, the subspace $L^r(E,F)$ of regular operators, i.e., the subspace consisting of operators which are differences of positive linear operators, is a Riesz space under the "pointwise order" provided F is Dedekind complete. Moreover, $L^r(E,F)$ is a Banach lattice for the norm $||T||_r = || \, |T| \, ||$.

A characterization of pairs of Banach lattices E, F for which $L(E,F) = L^r(E,F)$ (or $L(E,F) \equiv L^r(E,F)$, i.e., these spaces are equal and $||T|| = ||T||_r$) is an old problem which, in general, is still not solved. A classical result in this direction says that $L(E,F) \equiv L^r(E,F)$ whenever F is a Dedekind complete AM-space with a strong unit or E is an AL-space and there exists a positive contractive projection $P: F^{**} \to F$. Cartwright and Lotz conjectured in [**4**] that if $L(E,F) = L^r(E,F)$, then E is Riesz isomorphic to an AL-space or F is Riesz isomorphic to an AM-space. They confirmed the conjecture in the case where E^* or F contains a closed sublattice Riesz isomorphic to l^p for some $p \in [1, \infty)$, but Abramovič constructed in [**1**] a pair of Banach lattices E and F with the following properties: E is not Riesz isomorphic to an AL-space, F is not Riesz isomorphic to an AM-space and for any operator $T \in L(E,F)$ the modulus $|T|: E \to F$ exists.

The identity $L(E,F) = L^r(E,F)$ was also considered in [**6**] where the author, among other things, gave a characterization of a compact set X provided $L(C(X), C(Y)) = L^r(C(X), C(Y))$ for every compact set Y.

The space $l^1(A)$ is the unique Banach lattice E (up to a Riesz isomorphism) having the property that $L(E,F) = L^r(E,F)$ for every Banach lattice F. Indeed, it is easy to notice that $L(l^1(A), F) = L^r(l^1(A), F)$ (see for example [**6**, Theorem 2.1]). On the other hand, if $L(E,F) = L^r(E,F)$ for every Banach lattice F then E is an AL-space by the result of Cartwright and Lotz. If E were not discrete then by the famous Carathéodory theorem E would contain a closed Riesz subspace Riesz isomorphic to $L^1(0,1)$. Moreover, there exists a positive projection

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$P\colon E \to L^1(0,1)$, and the operator $T\colon L^1(0,1) \to c_0$ defined by the equality

$$(*) \qquad Tf = \left(\int_0^1 f(t) \sin(nt)\, dt \right)_{n=1}^{\infty}$$

is the well-known example of a continuous and nonregular operator. Therefore $TP \in L(E, c_0) \setminus L^r(E, c_0)$ and we have a contradiction.

For a Banach lattice F with an order continuous norm it is not difficult to check that $L(L^1(0,1), F) = L^r(L^1(0,1), F)$ iff there exists a positive projection $P\colon F^{**} \to F$. Indeed, if there did not exist a projection then F could not be a band in F^{**}. Thus F contains a closed Riesz subspace Riesz isomorphic to c_0. Since the norm on F is order continuous there exists a positive projection $P\colon F \to c_0$ (see [**8**]), and so the operator T defined by $(*)$ is not regular as a map from $L^1(0,1)$ into F.

2. Main result. The purpose of this paper is to characterize those σ-Dedekind complete Banach lattices E for which $L(E, c_0) = L^r(E, c_0)$. The lemma mentioned below is known (see for example [**6**]) but for the sake of completeness and reader's convenience we will present a proof of the lemma using a simpler and different method than in [**6**].

LEMMA. *If K is an infinite compact Hausdorff space then*

$$L(C(K), c_0) \neq L^r(C(K), c_0).$$

PROOF. Choose a weak* null sequence (f_n) of norm-one functionals in the dual space of $C(K)$ (such a sequence exists by the Josefson-Niessenzwieg theorem) and define an operator $T\colon C(K) \to c_0$ by $Tx = (f_n(x))_{n=1}^{\infty}$. Map T is continuous and noncompact. On the other hand, every regular operator from $C(K)$ into c_0 maps the unit ball, which is an order interval, into some order interval in c_0. But order intervals in c_0 are compact, and so every regular operator from $C(K)$ into c_0 must be compact. Therefore $T \in L(C(K), c_0) \setminus L^r(C(K), c_0)$.

COROLLARY. $L(l^\infty, c_0) \neq L^r(l^\infty, c_0)$.

Using the corollary we will prove the following characterization of discrete Banach lattices with order continuous norms.

THEOREM. *Let $(E, ||\cdot||)$ be a σ-Dedekind complete Banach lattice. The following statements are equivalent*:

(a) $L(E, c_0) \equiv L^r(E, c_0)$.
(b) $L(E, c_0) = L^r(E, c_0)$.
(c) *E is discrete and the norm $||\cdot||$ is order continuous.*

PROOF. (a) $\Rightarrow$ (b) obvious; (b) $\Rightarrow$ (c). If the norm $||\cdot||$ is not order continuous then E contains a closed Riesz subspace Riesz isomorphic to l^∞. By injectivity of l^∞ there exists a positive projection $P\colon E \to l^\infty$. Using the corollary choose a nonregular operator $S\colon l^\infty \to c_0$. There are no difficulties in verifying that the operator SP is also nonregular. Therefore $||\cdot||$ is order continuous.

Suppose E is nondiscrete. Denote by E_a the band generated by discrete elements in E. Let $B(e)$ be the band in E generated by a strictly positive element $e \in E_a^d$ (E_a^d means the orthogonal completion of E_a). The Banach lattice $(B(e), ||\cdot||)$ has an order continuous norm and a weak unit, and so $B(e)$ is Riesz isomorphic to an ideal of some space $L^1(S, \Sigma, \mu) = L^1(\mu)$, where μ is nonatomic and probabilistic (see [**7**, Theorem 1.b.14]). Moreover, we can assume $L^\infty(\mu)$ is contained in the range of $B(e)$ and the characteristic function of the set S is the range of e. We will identify $B(e)$ with its range in further considerations. Choose a separable sub-σ-algebra Σ_0 of Σ such that $\mu_0 = \mu|\Sigma_0$ (the restriction of μ to Σ_0) is nonatomic. The space $L^1(S, \Sigma_0, \mu_0) = L^1(\mu_0)$ is a closed Riesz subspace of $L^1(\mu)$. Therefore, there exists a positive projection $P\colon L^1(\mu) \to L^1(\mu_0)$. Using the famous Carathéodory theorem we can identify $L^1(\mu_0)$ with $L^1(0,1)$.

If P_0 denotes the band projection from E onto $B(e)$ then the composition PP_0 maps continuously E into $L^1(0,1)$ because PP_0 is a positive operator. Define the operator $T\colon E \to c_0$ by the equality $Tx = (\int_0^1 PP_0x(t)\sin(nt)\,dt)_{n=1}^\infty$. If T were regular then the set $T([0,e])$ would be order bounded and therefore conditionally compact. But $T([0,e])$ contains the subset $\{(\int_A \sin(nt)\,dt)_{n=1}^\infty\colon A$ is a Lebesgue measurable subset of $(0,1)\}$ which is not conditionally compact in c_0. Thus T is continuous and nonregular and we have a contradiction. Therefore E is a discrete Riesz space.

(c) $\Rightarrow$ (a). Let $(e_a)_{a\in A}$ be a complete disjoint system in E consisting of discrete elements. For every $x \in E$ there exists the unique net (t_a) of real numbers such that $x = \sum_a t_a e_a$. Let $T\colon E \to c_0$ be a linear continuous operator. Putting $Sx = \sum_a t_a |Te_a|$ we have $S = |T|$ (operator S is well defined because c_0 is an AM-space so the convergence of $\sum_a t_a T(e_a)$ implies the convergence of the series $\sum_a |t_a T(e_a)|$ and this convergence is unconditional). Thus $L(E, c_0) = L^r(E, c_0)$.

On the other hand $|T|x = \sup\{|Ty| : |y| \le x\}$ for $x \in E_+$. Since c_0 is super Dedekind complete there exists a sequence $(y_n) \subset E$ with two properties: $|y_n| \le x$ and $|T|x = \sup_n |Ty_n|$. Using this fact we have for $x \ge 0$,

$$\|\,|T|x\| = \left\|\sup_k(|Ty_1| \vee \cdots \vee |Ty_k|)\right\| = \lim_{k\to\infty} \|\,|Ty_1| \vee \cdots \vee |Ty_k|\,\|$$
$$= \lim_{k\to\infty} \max_{1\le i\le k} \|Ty_i\| \le \|T\|\,\|x\|.$$

Thus $\|T\|_r = \|T\|$ because $\|T\|_r \ge \|T\|$ always holds.

REMARKS. 1. It is clear that every compact operator T from an arbitrary Banach lattice into c_0 is regular (indeed, T maps order intervals into relatively compact sets in c_0 which are order bounded).

2. A similar proof of implication (c) $\Rightarrow$ (a) is presented in [**6**] (see Theorem 2.2).

3. The theorem also gives the following characterization of discrete σ-Dedekind complete Banach lattices with order continuous norms:

(d) The mapping $f \to |f|$ from E^* into E^* is $\sigma(E^*, E)$ sequentially continuous.

Indeed, it is not difficult to notice that statement (b) of the theorem is equivalent to (d) (but the mapping $f \to |f|$ is not $\sigma(E^*, E)$ continuous at zero if $\dim E = \infty$—see [**2**] the proofs of Theorems 6.8 and 6.9).

REFERENCES

1. Ju. A. Abramovič, *The space of operators that act between Banach lattices*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov **73** (1977), 188–192. (Russian)

2. C. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces*, Academic Press, New York and London, 1978.

3. ——, *Positive operators*, Academic Press, New York and London, 1985.
4. D. I. Cartwright and H. P. Lotz, *Some characterizations of AM- and AL-spaces*, Math. Z. **142** (1975), 97–103.
5. D. H. Fremlin, *A characterization of L-spaces*, Indag. Math. **36** (1974), 270–275.
6. Hong-Yun Xiong, *On whether or not* $L(E,F) = L^r(E,F)$ *for some classical Banach lattices* E *and* F, Indag. Math. **46** (1984), 267–282.
7. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces.* II, *Function spaces*, Springer-Verlag, Berlin and New York, 1979.
8. P. Meyer-Nieberg, *Charakterisierung einiger topologischer und ordnung theoretischer Eigenschaften von Banachverbanden mit Hilfe disjunkter Folgen*, Arch. Math. **24** (1973), 640–647.

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