

A CHARACTERIZATION OF EINSTEIN REAL HYPERSURFACES IN QUATERNIONIC PROJECTIVE SPACE

By

Soo Hyo LEE, Juan de Dios PÉREZ and Young Jin SUH

Abstract. On a real hypersurface of quaternionic projective space QP^m we study the following condition: $\mathfrak{S}(R(X, Y)SZ) = 0$ where \mathfrak{S} denotes the cyclic sum, R , respectively S , the curvature tensor, respectively the Ricci tensor, of the real hypersurface and $X, Y \in \mathcal{D}$, $Z \in \mathcal{D}^\perp$, \mathcal{D} and \mathcal{D}^\perp being certain distributions on the real hypersurface. We prove that such a real hypersurface must be Einstein.

1. Introduction

Let M be a connected real hypersurface of the quaternionic projective space QP^m , $m \geq 3$, endowed with the metric g of constant quaternionic sectional curvature 4. Let N be a unit local normal vector field on M and $U_i = -J_i N$, $i = 1, 2, 3$, where $\{J_i\}_{i=1,2,3}$ is a local basis of the quaternionic structure of QP^m , [2]. Several examples of such real hypersurfaces are well known, see for instance ([1], [3], [4]).

Let S be the Ricci tensor of M . In [4] it is proved that the unique real hypersurfaces of QP^m that are Einstein are geodesic hyperspheres of radius r , $0 < r < (\pi/2)$ and $\cot^2(r) = (1/2m)$.

Recently, in [4] the second author has studied real hypersurfaces of QP^m , $m \geq 2$, such that

$$(1.1) \quad \mathfrak{S}(R(X, Y)SZ) = 0$$

for any X, Y and Z tangent to M , where R denotes the curvature tensor of M and \mathfrak{S} is the cyclic sum on X, Y , and Z , obtaining

THEOREM A. *A real hypersurface M of QP^m , $m \geq 2$ satisfies (1.1) if and only if it is Einstein.*

Now let us define a distribution \mathcal{D} by $\mathcal{D}(x) = \{X \in T_x M : X \perp U_i(x), i = 1, 2, 3\}$, $x \in M$, of a real hypersurface M in QP^m , which is orthogonal to the structure vector fields $\{U_1, U_2, U_3\}$ and invariant with respect to structure tensors $\{\phi_1, \phi_2, \phi_3\}$, and by $\mathcal{D}^\perp = \text{Span}\{U_1, U_2, U_3\}$ its orthogonal complement in TM . In order to obtain a weaker condition than (1.1) it seems natural to propose to study real hypersurfaces of QP^m satisfying

$$(1.2) \quad \mathfrak{S}(R(X, Y)SZ) = 0$$

for any $X, Y \in \mathcal{D}$, and $Z \in \mathcal{D}^\perp$

The purpose of the present paper is to study such a condition. Concretely we shall prove

THEOREM 1. *A real hypersurface M of QP^m , $m \geq 3$, satisfies (1.2) if and only if it is Einstein.*

2. Preliminaries

Let X be a tangent field to M . We write $J_i X = \phi_i X + f_i(X)N$, $i = 1, 2, 3$, where $\phi_i X$ is the tangent component of $J_i X$ and $f_i(X) = g(X, U_i)$, $i = 1, 2, 3$. As $J_i^2 = -id$, $i = 1, 2, 3$, where id denotes the identity endomorphism on TQP^m , we get

$$(2.1) \quad \phi_i^2 X = -X + f_i(X)U_i, \quad f_i(\phi_i X) = 0, \quad \phi_i U_i = 0, \quad i = 1, 2, 3$$

for any X tangent to M . As $J_i J_j = -J_j J_i = J_k$, where (i, j, k) is a cyclic permutation of $(1, 2, 3)$ we obtain

$$(2.2) \quad \phi_i X = \phi_j \phi_k X - f_k(X)U_j = -\phi_k \phi_j X + f_j(X)U_k$$

and

$$(2.3) \quad f_i(X) = f_j(\phi_k X) = -f_k(\phi_j X)$$

for any vector field X tangent to M , where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. It is also easy to see that for any X, Y tangent to M and $i = 1, 2, 3$

$$(2.4) \quad g(\phi_i X, Y) + g(X, \phi_i Y) = 0, \quad g(\phi_i X, \phi_i Y) = g(X, Y) - f_i(X)f_i(Y)$$

and

$$(2.5) \quad \phi_i U_j = -\phi_j U_i = U_k$$

(i, j, k) being a cyclic permutation of $(1, 2, 3)$. From the expression of the curvature tensor of QP^m , $m \geq 2$, we have the equations of Gauss and Codazzi are respectively given by

$$(2.6) \quad R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + \sum_{i=1}^3 \{g(\phi_i Y, Z)\phi_i X - g(\phi_i X, Z)\phi_i Y + 2g(X, \phi_i Y)\phi_i Z\} + g(AY, Z)AX - g(AX, Z)AY,$$

and

$$(2.7) \quad (\nabla_X A)Y - (\nabla_Y A)X = \sum_{i=1}^3 \{f_i(X)\phi_i Y - f_i(Y)\phi_i X + 2g(X, \phi_i Y)U_i\}$$

for any X, Y, Z tangent to M , where R denotes the curvature tensor of M , see [3]. Moreover, the Ricci tensor $S'(Z, Y) = g(SZ, Y) = \text{Trace}\{X \rightarrow R(X, Z)Y\}$ are defined by

$$(2.8) \quad SZ = (4m + 7)Z - 3 \sum_k f_k(Z)U_k + hAZ - A^2Z,$$

respectively.

From the expressions of the covariant derivatives of J_i , $i = 1, 2, 3$, it is easy to see that

$$(2.9) \quad \nabla_X U_i = -p_j(X)U_k + p_k(X)U_j + \phi_i AX$$

and

$$(2.10) \quad (\nabla_X \phi_i)Y = -p_j(X)\phi_k Y + p_k(X)\phi_j Y + f_i(Y)AX - g(AX, Y)U_i$$

for any X, Y tangent to M , (i, j, k) being a cyclic permutation of $(1, 2, 3)$ and p_i , $i = 1, 2, 3$, local 1-forms on QP^m .

3. Key Lemma

Let M be a real hypersurface in a quaternionic projective space QP^m satisfying

$$(3.1) \quad \mathfrak{S}R(X, Y)SZ = 0$$

for any $X, Y \in \mathcal{D}$, and $Z \in \mathcal{D}^\perp$. Now let us take an orthonormal basis

$$\{E_1, \dots, E_{4m-4}, U_1, U_2, U_3\}$$

of the tangent space of $T_x(M)$ at any point $x \in M$. Then for a case where $X = E_i$, $Y = \phi_1 E_i$ and $Z = U_1$ the above formula (3.1) gives

$$(3.2) \quad R(E_i, \phi_1 E_i)SU_1 + R(\phi_1 E_i, U_1)SE_i + R(U_1, E_i)S\phi_1 E_i = 0.$$

Now let us denote by $H = hA - A^2$. Then the first term of the left side of (3.2) becomes

$$\begin{aligned} R(E_i, \phi_1 E_i)SU_1 &= g(\phi_1 E_i, HU_1)E_i - g(E_i, HU_1)\phi_1 E_i \\ &\quad - g(E_i, HU_1)\phi_1 E_i + g(\phi_1 E_i, HU_1)E_i - 2g(\phi_3 E_i, HU_1)\phi_2 E_i \\ &\quad + 2g(\phi_2 E_i, HU_1)\phi_3 E_i - 2\phi_1 HU_1 + g(A\phi_1 E_i, SU_1)AE_i \\ &\quad - g(AE_i, SU_1)A\phi_1 E_i \end{aligned}$$

The second term gives

$$\begin{aligned} R(\phi_1 E_i, U_1)SE_i &= g(HU_1, E_i)\phi_1 E_i - (4m + 7)g(E_i, \phi_1 E_i)U_1 \\ &\quad - g(HE_i, \phi_1 E_i)U_1 + g(U_3, HE_i)\phi_3 E_i \\ &\quad - g(\phi_3 E_i, HE_i)U_3 - g(\phi_2 E_i, HE_i)U_2 \\ &\quad + g(U_2, HE_i)U_2 - g(AU_1, S\phi_1 E_i)AE_i + g(AE_i, S\phi_1 E_i)AU_1. \end{aligned}$$

Also the third term of (3.2) gives

$$\begin{aligned} R(U_1, E_i)S\phi_1 E_i &= -g(HU_1, \phi_1 E_i)E_i + (4m + 7)g(E_i, \phi_1 E_i)U_1 \\ &\quad + g(H\phi_1 E_i, E_i)U_1 + g(U_3, H\phi_1 E_i)\phi_2 E_i \\ &\quad - g(U_2, H\phi_1 E_i)\phi_3 E_i - g(\phi_2 E_i, H\phi_1 E_i)U_3 \\ &\quad + g(\phi_3 E_i, H\phi_1 E_i)U_2 - g(AU_1, S\phi_1 E_i)AE_i \\ &\quad + g(AE_i, S\phi_1 E_i)AU_1. \end{aligned}$$

Thus summing up the above formulas, we have

$$\begin{aligned} (3.3) \quad \mathfrak{S}R(E_i, \phi_1 E_i)SU_1 &= \{g(\phi_3 E_i, H\phi_1 E_i) - g(H\phi_2 E_i, E_i)\}U_2 \\ &\quad - \{g(H\phi_3 E_i, E_i) + g(H\phi_2 E_i, \phi_1 E_i)\}U_3 \\ &\quad + g(\phi_1 E_i, HU_1)E_i - g(E_i, HU_1)\phi_1 E_i - 2\phi_1 HU_1 \\ &\quad + \{g(U_2, HE_i) + g(U_3, H\phi_1 E_i) - 2g(\phi_3 E_i, HU_1)\}\phi_2 E_i \\ &\quad + \{g(U_3, HE_i) - g(U_2, H\phi_1 E_i) + 2g(\phi_2 E_i, HU_1)\}\phi_3 E_i \\ &\quad + 3g(AU_1, E_i)A\phi_1 E_i - 3g(AU_1, \phi_1 E_i)AE_i \\ &= 0. \end{aligned}$$

For a case where $j = 2$ we can also calculate the following

$$\begin{aligned}
(3.4) \quad \mathfrak{S}R(E_i, \phi_2 E_i)SU_2 &= \{g(\phi_1 E_i, H\phi_2 E_i) - g(\phi_3 E_i, HE_i)\}U_3 \\
&\quad - \{g(H\phi_1 E_i, E_i) + g(H\phi_3 E_i, \phi_2 E_i)\}U_1 \\
&\quad + g(\phi_2 E_i, HU_2)E_i - g(E_i, HU_2)\phi_2 E_i - 2\phi_2 HU_2 \\
&\quad + \{g(U_3, HE_i) + g(U_1, H\phi_2 E_i) - 2g(\phi_1 E_i, HU_2)\}\phi_3 E_i \\
&\quad + \{g(U_1, HE_i) - g(U_3, H\phi_2 E_i) + 2g(\phi_3 E_i, HU_2)\}\phi_1 E_i \\
&\quad + 3g(AU_2, E_i)A\phi_2 E_i - 3g(AU_2, \phi_2 E_i)AE_i \\
&= 0.
\end{aligned}$$

Similarly, for a case where $j = 3$ we have

$$\begin{aligned}
(3.5) \quad \mathfrak{S}R(E_i, \phi_3 E_i)SU_3 &= \{g(\phi_2 E_i, H\phi_3 E_i) - g(\phi_1 E_i, HE_i)\}U_1 \\
&\quad - \{g(H\phi_2 E_i, E_i) + g(H\phi_1 E_i, \phi_3 E_i)\}U_2 \\
&\quad + g(\phi_3 E_i, HU_3)E_i - g(E_i, HU_3)\phi_3 E_i - 2\phi_3 HU_3 \\
&\quad + \{g(U_1, HE_i) + g(U_2, H\phi_3 E_i) - 2g(\phi_2 E_i, HU_3)\}\phi_1 E_i \\
&\quad + \{g(U_2, HE_i) - g(U_1, H\phi_3 E_i) + 2g(\phi_1 E_i, HU_3)\}\phi_2 E_i \\
&\quad + 3g(AU_3, E_i)A\phi_3 E_i - 3g(AU_3, \phi_3 E_i)AE_i \\
&= 0.
\end{aligned}$$

By contracting from $i = 1, \dots, 4(m-1)$ the formulas (3.3), (3.4) and (3.5) are reduced by the followings respectively

$$\begin{aligned}
(3.6) \quad &-(4m-5)\phi_1 HU_1 + \phi_2 HU_2 + \phi_3 HU_3 \\
&\quad + 3\{A\phi_1 AU_1 - g(AU_1, U_2)AU_3 + g(AU_1, U_3)AU_2\} = 0,
\end{aligned}$$

$$\begin{aligned}
(3.7) \quad &-(4m-5)\phi_2 HU_2 + \phi_3 HU_3 + \phi_1 HU_1 \\
&\quad + 3\{A\phi_2 AU_2 - g(AU_2, U_3)AU_1 + g(AU_2, U_1)AU_3\} = 0,
\end{aligned}$$

and

$$\begin{aligned}
(3.8) \quad &-(4m-5)\phi_3 HU_3 + \phi_1 HU_1 + \phi_2 HU_2 \\
&\quad + 3\{A\phi_3 AU_3 - g(AU_3, U_1)AU_2 + g(AU_3, U_2)AU_1\} = 0.
\end{aligned}$$

Now let us denote the curvature tensor and the Ricci tensor defined in section 2 respectively by

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + Q(X, Y)Z,$$

and

$$SZ = (4m + 7)Z - 3 \sum_k f_k(Z)U_k + HZ,$$

where

$$\begin{aligned} Q(X, Y)Z = & \sum_{l=1}^3 \{g(\phi_l Y, Z)\phi_l X - g(\phi_l X, Z)\phi_l Y \\ & + 2g(X, \phi_l Y)\phi_l Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

and

$$HZ = hAZ - A^2Z$$

respectively, for any tangent vector fields X , Y and Z of M . In this section we want to prove the following

LEMMA 3.1. *Let M be a real hypersurface in a quaternionic projective space QP^m satisfying (3.1). Then*

$$g(H\mathcal{D}, \mathcal{D}^\perp) = 0.$$

PROOF. The formula (3.1) implies that

$$(3.9) \quad Q(X, Y)SZ + Q(Y, Z)SX + Q(Z, X)SY = 0.$$

for any tangent vector fields $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$. Now let us put $X = E_i$, $Y = \phi_1 E_i$ and $Z = U_2$ in (3.9) and use the basic formulas in section 2, then the first term of the left hand side of (3.9) gives

$$\begin{aligned} (3.10) \quad Q(E_i, \phi_1 E_i)SU_2 = & -g(E_i, HU_2)\phi_1 E_i + g(\phi_1 E_i, HU_2)E_i \\ & - 8(m + 1)U_3 - 2\phi_1 HU_2 - 2g(\phi_3 E_i, HU_2)\phi_2 E_i \\ & + 2g(\phi_2 E_i, HU_2)\phi_3 E_i \\ & + g(A\phi_1 E_i, SU_2)AE_i - g(AE_i, SU_2)A\phi_1 E_i, \end{aligned}$$

where we have used the fact that

$$\phi_1 S U_2 = 4(m+1)U_3 + \phi_1 H U_2.$$

The second term of (3.9) gives

$$(3.11) \quad \begin{aligned} Q(\phi_1 E_i, U_2) S E_i &= -g(U_3, H E_i) E_i + g(E_i, H E_i) U_3 \\ &\quad - g(U_1, H E_i) \phi_2 E_i \\ &\quad + g(\phi_2 E_i, H E_i) U_1 + g(A U_2, S E_i) A \phi_1 E_i \\ &\quad - g(A \phi_1 E_i, S E_i) A U_2 + (4m+7) U_3, \end{aligned}$$

where we have used $S E_i = (4m+7)E_i + H E_i$. Finally the third term of (3.9) is given by the following

$$(3.12) \quad \begin{aligned} Q(U_2, E_i) S \phi_1 E_i &= (4m+7)U_3 + g(\phi_1 E_i, H \phi_1 E_i) U_3 \\ &\quad - g(U_3, H \phi_1 E_i) \phi_1 E_i \\ &\quad - g(\phi_3 E_i, H \phi_1 E_i) U_1 + g(U_1, H \phi_1 E_i) \phi_3 E_i \\ &\quad + g(A E_i, S \phi_1 E_i) A U_2 - g(A U_2, S \phi_1 E_i) A E_i. \end{aligned}$$

Combining (3.10), together with (3.11) and (3.12), we have

$$(3.13) \quad \begin{aligned} 0 &= \mathfrak{S} R(E_i, \phi_1 E_i) S U_2 \\ &= \mathfrak{S} Q(E_i, \phi_1 E_i) S U_2 \\ &= \{g(\phi_2 E_i, H E_i) - g(\phi_3 E_i, H \phi_1 E_i)\} U_1 \\ &\quad + \{g(E_i, H E_i) + 6 + g(\phi_1 E_i, H \phi_1 E_i)\} U_3 \\ &\quad + \{g(\phi_1 E_i, H U_2) - g(U_3, H E_i)\} E_i \\ &\quad - \{g(E_i, H U_2) + g(U_3, H \phi_1 E_i)\} \phi_1 E_i \\ &\quad - \{2g(\phi_3 E_i, H U_2) + g(U_1, H E_i)\} \phi_2 E_i \\ &\quad + \{2g(\phi_2 E_i, H U_2) + g(U_1, H \phi_1 E_i)\} \phi_3 E_i \\ &\quad - 2\phi_1 H U_2 + \{g(A \phi_1 E_i, S U_2) - g(A U_2, S \phi_1 E_i)\} A E_i \\ &\quad + \{g(A U_2, S E_i) - g(A E_i, S U_2)\} A \phi_1 E_i \\ &\quad + \{g(A E_i, S \phi_1 E_i) - g(A \phi_1 E_i, S E_i)\} A U_2. \end{aligned}$$

From this let us take a summation given by $\sum_i = \sum_{i=1}^{4(m-1)}$, then we know the following informations

$$\sum_i g(\phi_2 E_i, H E_i) = 0$$

and

$$\sum_i g(\phi_3 E_i, H \phi_1 E_i) = 0.$$

Moreover, by contracting we also have the followings

$$\sum_i g(E_i, H E_i) = \text{Tr } H - \sum_{i=1}^3 g(U_i, H U_i),$$

$$\sum_i g(\phi_1 E_i, H \phi_1 E_i) = \text{Tr } H - g(\phi_1 U_2, H \phi_1 U_2) - g(\phi_1 U_3, H \phi_1 U_3),$$

$$\sum_i g(\phi_1 E_i, H U_2) E_i = -\phi_1 H U_2 - g(U_3, H U_2) U_2 + g(U_2, H U_2) U_3,$$

$$\sum_i g(U_3, H E_i) E_i = H U_3 - \sum_{i=1}^3 g(U_3, H U_i) U_i,$$

$$\sum_i g(E_i, H U_2) \phi_1 E_i = \phi_1 H U_2 - g(U_2, H U_2) U_3 + g(U_3, H U_2) U_2,$$

$$\sum_i g(U_3, H \phi_1 E_i) = H U_3 - f_1(H U_3) U_1 - g(H U_2, U_3) U_2 - g(H U_3, U_3) U_3,$$

$$2 \sum_i g(\phi_3 E_i, H U_2) \phi_2 E_i = -2\phi_1 H U_2 - 2f_3(H U_2) U_2 + 2g(H U_2, U_2) U_3,$$

and

$$\sum_i g(U_1, H E_i) \phi_2 E_i = \phi_2 H U_1 + g(U_1, H U_1) U_3 - g(U_1, H U_3) U_1.$$

Following with the proof, let us take a contraction to the latter terms of (3.13), then we have

$$\begin{aligned} & \sum_i \{2g(\phi_2 E_i, H U_2) + g(U_1, H \phi_1 E_i)\} \phi_3 E_i \\ &= 2\phi_1 H U_2 - 2f_2(H U_2) U_3 - \phi_2 H U_1 - f_1(H U_1) U_3 \\ & \quad + 2g(H U_2, U_3) U_2 + g(H U_3, U_1) U_1, \end{aligned}$$

$$\begin{aligned} & \sum_i \{g(A\phi_1 E_i, SU_2) - g(AU_2, S\phi_1 E_i)\} A E_i \\ &= 3A\phi_1 AU_2 + 3g(AU_2, U_3)AU_2 - 3g(AU_2, U_2)AU_3, \\ & \sum_i \{g(AU_2, SE_i) - g(AE_i, SU_2)\} A\phi_1 E_i \\ &= -3A\phi_1 AU_2 + 3g(AU_2, U_2)AU_3 + 3g(AU_2, U_3)A\phi_1 U_3, \\ & \sum_i \{g(AE_i, S\phi_1 E_i) - g(A\phi_1 E_i, SE_i)\} AU_2 = 0. \end{aligned}$$

Taking account of these formulas into (3.13), we have

$$\begin{aligned} (3.14) \quad HU_3 &= \phi_1 HU_2 - \phi_2 HU_1 + 2g(HU_2, U_3)U_2 + 2g(HU_1, U_3)U_1 \\ &\quad - \{g(HU_1, U_1) + g(HU_2, U_2) - g(HU_3, U_3)\}U_3 \\ &\quad - 4(m-1)\phi_1 HU_2. \end{aligned}$$

From this it follows that

$$(3.15) \quad \phi_3 HU_3 = -(4m-5)\phi_2 HU_2 + \phi_1 HU_1.$$

Using the similar method, we have the following

$$(3.16) \quad \phi_2 HU_2 = -(4m-5)\phi_1 HU_1 + \phi_3 HU_3,$$

$$(3.17) \quad \phi_1 HU_1 = -(4m-5)\phi_3 HU_3 + \phi_2 HU_2.$$

Thus summing up (3.15), (3.16) and (3.17), we have

$$\sum_{i=1}^3 \phi_i HU_i = -(4m-5) \sum_{i=1}^3 \phi_i HU_i + \sum_{i=1}^3 \phi_i HU_i,$$

so that

$$\sum_{i=1}^3 \phi_i HU_i = 0.$$

On the other hand, from (3.6) and (3.15) we know that

$$(3.18) \quad 2\phi_3 HU_3 + 3\{A\phi_2 AU_2 - g(AU_2, U_3)AU_1 + g(AU_2, U_1)AU_3\} = 0.$$

Similarly, we can assert that

$$(3.19) \quad 2\phi_1 HU_1 + 3\{A\phi_3 AU_3 - g(AU_3, U_1)AU_2 + g(AU_3, U_2)AU_1\} = 0,$$

and

$$(3.20) \quad 2\phi_2 HU_2 + 3\{A\phi_1 AU_1 - g(AU_1, U_2)AU_3 + g(AU_1, U_3)AU_2\} = 0.$$

On the other hand, putting $\phi_2HU_2 + \phi_3HU_3 = -\phi_1HU_1$ into (3.6) and using (3.20), we have

$$(3.21) \quad -4(m-1)\phi_1HU_1 - 2\phi_2HU_2 = 0.$$

Similarly, we also have

$$(3.22) \quad -4(m-1)\phi_2HU_2 - 2\phi_3HU_3 = 0,$$

and

$$(3.23) \quad -4(m-1)\phi_3HU_3 - 2\phi_1HU_1 = 0.$$

Thus (3.22) and (3.21) imply $\phi_3HU_3 = -2(m-1)\phi_2HU_2 = 4(m-1)^2\phi_1HU_1$. From this, together with (3.23) it follows

$$\{8(m-1)^3 + 1\}\phi_1HU_1 = 0.$$

Similarly, we have $\phi_2HU_2 = 0$, and $\phi_3HU_3 = 0$. From this we complete the proof of our lemma.

4. The Proof of Main Theorem

In section 3 under the condition (3.1) we have proved that $g(H\mathcal{D}, \mathcal{D}^\perp) = 0$ for the distributions \mathcal{D} and $\mathcal{D}^\perp = \text{Span}\{U_1, U_2, U_3\}$ of real hypersurfaces in QP^m , where $H = hA - A^2$. But $HA = AH$. Thus we can find an orthonormal basis of T_xM , for any $x \in M$, such that it diagonalizes simultaneously both H and A . So on this decomposition of T_xM such that $T_xM = \mathcal{D} \oplus \mathcal{D}^\perp$ the fact that $g(H\mathcal{D}, \mathcal{D}^\perp) = 0$ is equivalent to $g(A\mathcal{D}, \mathcal{D}^\perp) = 0$. Then by virtue of a theorem given by J. Berndt [1] we conclude that a real hypersurfaces satisfying (3.1) is locally congruent to one of geodesic hypersphere, a tube over QP^k , $k = 1, \dots, n-1$ with radius $0 < r < (\pi/2)$, or a tube over CP^m with radius $0 < r < (\pi/4)$.

Firstly, let us consider the case where M is a geodesic hypersphere. Then its principal curvatures are given by $\alpha = 2 \cot 2r$, $\cot r$ with multiplicities 3 and $4(m-1)$ respectively. That is, $AU_i = \alpha U_i$, $i = 1, 2, 3$ and $AX = \cot r X$ for any $X \in \mathcal{D}$. From this the Ricci tensor S , for any X in \mathcal{D} , is given by

$$\begin{aligned} SX &= [(4m+7) + \{(4m-1)\cot r - 3\tan r\}\cot r - \cot^2 r]X \\ &= [4m+7 + (4m-1)\cot^2 r - 3 - \cot^2 r]X \\ &= [4m+4 + (4m-2)\cot^2 r]X. \end{aligned}$$

On $\mathcal{D}^\perp = \text{Span}\{U_1, U_2, U_3\}$ we have

$$\begin{aligned} SU_i &= [4m + 4 + \{(4m - 1) \cot r - 3 \tan r\}(\cot r - \tan r) - (\cot r - \tan r)^2] U_i \\ &= \{4 + (4m - 2) \cot^2 r + 2 \tan^2 r\} U_i. \end{aligned}$$

On the other hand, the condition (3.1) implies that

$$(4.1) \quad \mathfrak{S}R(X, Y)SU_i = R(X, Y)SU_i + R(Y, U_i)SX + R(U_i, X)SY = 0,$$

for $i = 1, 2, 3$. Thus on this geodesic hypersphere we can put $SU_i = \gamma U_i$ and $SX = \delta X$ for any X in \mathcal{D} . Then it can be easily verify that γ and δ could be equal to each other. This means that the geodesic hypersphere M in QP^m is Einstein. From this and the above expression of the Ricci tensor we have

$$4 + (4m - 2) \cot^2 r + 2 \tan^2 r - (4m + 4) - (4m - 2) \cot^2 r = 0.$$

That is, M is a Einstein real hypersurface in QP^m , which is congruent to a tube of radius r such that $\cot^2 r = (1/2m)$.

For the case where M is congruent to a tube over QP^k , $k = 1, 2, \dots, m - 1$. Its principal curvatures are also given by $\cot r$, $-\tan r$ and $2 \cot 2r$ with their multiplicities $4m - 4k - 4$, $4k$ and 3 , respectively. Thus h is given by

$$\begin{aligned} h = \text{Tr } A &= (4m - 4k - 4) \cot r - 4k \tan r + 3(\cot r - \tan r) \\ &= (4m - 4k - 1) \cot r - (4k + 3) \tan r. \end{aligned}$$

Now let us take principal vectors such that $X \in V_{\cot r}$, $Y \in V_{-\tan r}$ and $U_i \in \mathcal{D}^\perp$, where the distribution \mathcal{D} is given by $\mathcal{D} = V_{\cot r} \oplus V_{-\tan r}$. Then we have the following

$$(4.2) \quad \begin{aligned} SX &= (4m + 7)X + \{(4m - 4k - 1) \cot r - (4k + 3) \tan r\} \cot r X - \cot^2 r X \\ &= \{(4m - 4k + 4) + (4m - 4k - 2) \cot^2 r\} X, \end{aligned}$$

$$(4.3) \quad \begin{aligned} SY &= (4m + 7)Y - \{(4m - 4k - 1) \cot r - (4k + 3) \tan r\} \tan r Y - \tan^2 r Y \\ &= \{4k + 8 + (4k + 2) \tan^2 r\} Y, \end{aligned}$$

and

$$\begin{aligned}
(4.4) \quad SU_i &= (4m + 4)U_i + (\cot r - \tan r)\{(4m - 4k - 1)\cot r - (4k + 3)\tan r \\
&\quad - (\cot r - \tan r)\}U_i \\
&= \{4 + (4m - 4k - 2)\cot^2 r + (4k + 2)\tan^2 r\}U_i.
\end{aligned}$$

Thus if we put $SX = \gamma X$, $SY = \delta Y$ for any $X \in V_{\cot r}$ and $Y \in V_{-\tan r}$, and $SU_i = \beta U_i$, then the condition (4.1) implies that $\gamma = \beta = \delta$. Thus subtracting (4.2) and (4.3) from (4.4) respectively, then it follows respectively that

$$(4k + 2)\tan^2 r = 4m - 4k$$

and

$$(4m - 4k - 2)\cot^2 r = 4k + 4.$$

These imply $(4m - 4k)(4k + 4) = (4m - 4k - 2)(4k + 2)$. Thus $8m = -4$. This makes also a contradiction. Thus this case does not appear.

Finally let us consider for the case where M is congruent to a tube over CP^n . Then its principal curvatures are given by $\cot r$, $-\tan r$, $2\cot 2r$ and $-2\tan 2r$ with multiplicities $2(m - 1)$, $2(m - 1)$, 1 and 2 respectively. Then the trace of the second fundamental form A is given by

$$\begin{aligned}
h &= 2(m - 1)(\cot r - \tan r) + 2\cot 2r - 4\tan 2r \\
&= (2m - 1)(\cot r - \tan r) - 4\tan 2r.
\end{aligned}$$

Now let us denote by its corresponding principal curvature vectors $X \in V_{\cot r}$, $Y \in V_{-\tan r}$, $U_1 \in V_{2\cot 2r}$, and $U_2, U_3 \in V_{-2\tan 2r}$. Then we have the following

$$\begin{aligned}
SX &= (4m + 7)X + \{(2m - 1)(\cot r - \tan r) - 4\tan 2r\}\cot r X - \cot^2 r X \\
&= \{2m + 8 + 2(m - 1)\cot^2 r - 4\tan 2r \cot r\}X,
\end{aligned}$$

$$\begin{aligned}
SY &= (4m + 7)Y - \{(2m - 1)(\cot r - \tan r) - 4\tan 2r\}\tan r Y - \tan^2 r Y \\
&= \{2m + 8 + 2(m - 1)\tan^2 r + 4\tan 2r \tan r\}Y,
\end{aligned}$$

$$\begin{aligned}
SU_1 &= (4m + 4)U_1 + (\cot r - \tan r)\{(2m - 1)(\cot r - \tan r) - 4\tan 2r\}U_1 \\
&\quad - 4\cot^2 2r U_1,
\end{aligned}$$

$$SU_k = (-4m + 8 + 4\tan^2 2r)U_k, \quad k = 2, 3.$$

On the other hand, let us put $X \in V_{\cot r}$, $\phi_2 X \in V_{-\tan r}$ in (3.1). Then we have

$$\begin{aligned} &R(X, \phi_2 X)SU_1 + R(\phi_2 X, U_1)SX + R(U_1, X)S\phi_2 X \\ &= \{(4m - 4) + (2m - 2)(\cot r - \tan r)^2\}R(X, \phi_2 X)U_1 \\ &\quad + \{(2m + 8) + (2m - 2)\cot^2 r - 4 \tan 2r \cot r\}R(\phi_2 X, U_1)X \\ &\quad + \{(2m + 8) + (2m - 2)\tan^2 r + 4 \tan 2r \tan r\}R(U_1, X)\phi_2 X \\ &= 2\{(4m - 4) + 2(m - 1)(\cot r - \tan r)^2\}U_3 \\ &\quad - 2\{(m + 4) + (m - 1)\cot^2 r - 2 \tan 2r \cot r\}U_3 \\ &\quad - 2\{(m + 4) + (m - 1)\tan^2 r + 2 \tan 2r \tan r\}U_3 \\ &= 0. \end{aligned}$$

So it follows

$$\begin{aligned} &2\{4(m - 1) + 2(m - 1)(\cot r - \tan r)^2\} - 2(2m + 8) - 4(m - 1)(\cot^2 r + \tan^2 r) \\ &\quad + 4 \tan 2r(\cot r - \tan r) = 0. \end{aligned}$$

Thus $-4m - 8 = 0$. This is impossible. Thus this case also can not occur.

Summing up this result, we conclude that a real hypersurface in QP^m satisfying (3.1) is Einstein and it is congruent to a geodesic hypersphere, that is a tube over one point with radius r such that $\cot^2 r = (1/2m)$. This completes the proof of our assertion.

REMARK. But if we consider the above situation for the shape operator A of M in a quaternionic projective space QP^m , we can verify that QP^m do not admit any real hypersurfaces satisfying the corresponding condition. Using the same method as in the proof of Theorem 1, we can assert this as follows:

THEOREM 2. *There do not exist any real hypersurfaces M in a quaternionic projective space QP^m , $m \geq 2$, satisfying $\mathfrak{S}R(X, Y)AZ = 0$ for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$, where \mathfrak{S} denotes the cyclic sum of X, Y and Z and R is the curvature tensor of M .*

COROLLARY 3. *There do not exist any real hypersurfaces M in QP^m , $m \geq 2$, satisfying $\mathfrak{S}R(X, Y)AZ = 0$ for any X, Y and Z tangent to M , where \mathfrak{S} denotes the cyclic sum of X, Y and Z .*

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Department of Mathematics
Kyungpook National University
Taegu, 702-701, KOREA

Departamento de Geometría y Topología
Facultad de Ciencias
Universidad de Granada
18071-Granada, SPAIN

Department of Mathematics
Kyungpook National University
Taegu, 702-701, KOREA