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A CHARACTERIZATION OF FINITE PREHOMOGENEOUS VECTOR SPACES OF D_4-TYPE UNDER VARIOUS SCALAR RESTRICTIONS

Dedicated to Professor Tatsuo Kimura on the occasion of his 60th birthday.

By

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Abstract. In the present paper, we give conditions to have only finitely many orbits for prehomogeneous vector spaces of D_4-type. This paper completes the classification of finite prehomogeneous vector spaces of type $(G \times SL_n, \rho \otimes \Lambda_1)$ with $n \geq 2$. We consider everything over the complex number field $\mathbf{C}$.

Introduction

Let $\rho : G \to GL(V)$ be a rational representation of a connected linear algebraic group G on a finite-dimensional vector space V. If V has a Zariski-dense G-orbit, the triplet (G, ρ, V) is called a *prehomogeneous vector space* (abbrev. PV). When V is decomposed into a finite union of G-orbits, it must be a PV. Such a triplet is called a *finite prehomogeneous vector space* (abbrev. FP). When there is no confusion, we sometimes denote it by (G, ρ) instead of (G, ρ, V).

When G is reductive, all FPs have been completely classified under the condition that each irreducible component has an independent scalar multiplication ([KKY]). However if we restrict scalar multiplications, the classification becomes complicated and it has been done only some cases ([NN], [NOT], [KKMOT]).

Let G_i be a general linear algebraic group $GL(m_i)$ or a special linear algebraic group $SL(m_i)$ $(i = 1, \dots, 4)$. Then the group $G = G_1 \times G_2 \times G_3 \times G_4$ acts on $V = M(m_4, m_1) \oplus M(m_4, m_2) \oplus M(m_4, m_3)$ as $\rho(g)v = (g_4 v_1 g_1^{-1}, g_4 v_2 g_2^{-1},$

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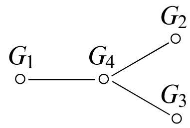
$g_4 v_3 g_3^{-1})$ for $g = (g_1, g_2, g_3, g_4) \in G$ and $v = (v_1, v_2, v_3) \in V$. We call it a triplet of *D_4-type under scalar restriction* and denote it by



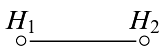
In this paper, we determine the conditions for a triplet of D_4-type under scalar restriction to be an FP by decomposing into the orbits. This method is different from that of [NOT]. This result is useful to study the classification of the FPs of D_r-type $(r \geq 5)$, E_6, E_7 or E_8-type under various scalar restrictions since they contain the diagram of D_4-type as a subdiagram. Together with [KKMOT], this paper completes the classification of FPs of type $(G \times SL(n), \rho \otimes \Lambda_1)$ $(n \geq 2)$ where G is a reductive algebraic group.

1. Preliminaries and Notation

For positive integers m and n, we denote by $M(m,n)$ the totality of $m \times n$ matrices. We also use the notation $M(m,n)' = \{X \in M(m,n) \mid \operatorname{rank} X = \min\{m,n\}\}$ and $M(m,n)'' = \{X \in M(m,n) \mid \operatorname{rank} X < \min\{m,n\}\}$. We denote by I_n the identity matrix of degree n. We write the standard representation of $GL(n)$ on $\mathbf{C}^n$ by Λ_1.

In general, we denote by ρ^* the dual representation of a rational representation ρ. It is known that (H, σ, V) is an FP if and only if (H, σ^*, V^*) is an FP for any algebraic group H, not necessarily reductive (see [P]). Hence $(G, \rho_1^{(*)} \oplus \cdots \oplus \rho_l^{(*)})$ is an FP if and only if $(G, \rho_1 \oplus \cdots \oplus \rho_l)$ is an FP where $\rho^{(*)}$ means ρ or its dual ρ^*. Also if G_1 and G_2 are reductive, then we have $(G_1 \times G_2, \rho_1^{(*)} \otimes \rho_2^{(*)}) \cong (G_1 \times G_2, \rho_1 \otimes \rho_2)$. Using these facts, we do not have to consider the dual representation as far as we deal with D_4-type FPs.

Any subgroup $H_1 \times H_2$ of $GL(m) \times GL(n)$ acts on $M(n,m)$ by $\Lambda_1 \otimes \Lambda_1$. In the following, to simplify the notation, we will express this representation $(H_1 \times H_2, \Lambda_1 \otimes \Lambda_1, M(m,n))$ by the diagram



Since any parabolic subgroup P of $GL(m)$ is conjugate to a standard parabolic subgroup, we may assume that P is a standard parabolic subgroup $P(e_1, \ldots, e_r)$ $(e_1 + \cdots + e_r = m)$ defined as follows:

$$P(e_1,\dots,e_r) = \left\{ \begin{bmatrix} P_{11} & P_{12} & \ddots & P_{1r} \\ 0 & P_{22} & \ddots & \ddots \\ 0 & 0 & \ddots & \ddots \\ 0 & 0 & 0 & P_{rr} \end{bmatrix} \in GL(m) \;\middle|\; \begin{matrix} P_{ij} \in M(e_i, e_j) \\ (1 \le i, j \le m) \end{matrix} \right\}$$

To prove that a triplet is a non FP, the following lemma is fundamental.

LEMMA 1.1 ([K, Proposition 2.4]). *If there exists a non-constant absolute invariant of a triplet (G, ρ, V), then it is a non PV. In particular, it is a non FP.*

EXAMPLE 1.2. Let $F(X) = \det X$ for $X \in M(n,n)$. The diagram $\overset{SL(n)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{SL(n)}{\circ}$ is a non PV since $F(X)$ is a non-constant absolute invariant.

For the A_r-type, the following result is known.

THEOREM 1.3 ([NN, Theorem 4.2]). *Let $d = (d_1, \dots, d_r)$ be an r-tuple of positive integers. Then*

$$\overset{G_1}{\circ}\!-\!\!-\!\!-\!\overset{G_2}{\circ}\!-\!\!-\!\!-\cdots-\!\!-\!\!-\!\overset{G_{r-1}}{\circ}\!-\!\!-\!\!-\!\overset{G_r}{\circ},$$

where $G_k = GL(d_k)$ or $SL(d_k)$, is a non FP if and only if there exist some numbers $u_1, u_2, \dots, u_l$ $(u_1 < \cdots < u_l)$ such that

$$d_{u_1} - d_{u_2} + d_{u_3} - d_{u_4} + d_{u_5} - d_{u_6} + \cdots + (-1)^{l+1} d_{u_l} = 0,$$

$$G_{u_i} = SL(d_{u_i}) \quad \text{for } i = 1, \dots, l,$$

and for $j = 2, \dots, l$,

$$d_{u_{j-1}} - d_{u_{j-2}} + \cdots + (-1)^j d_{u_l} \le \min\{d_{u_{j-1}+1}, d_{u_{j-1}+2}, \dots, d_{u_j}\}.$$

COROLLARY 1.4. *All non FPs of A_3-type under various scalar restrictions are given as follows*:

1. $\overset{SL(m_1)}{\circ}\!-\!\!-\!\!-\!\overset{SL(n)}{\circ}\!-\!\!-\!\!-\!\overset{GL(m_2)}{\circ}$ *with* $n = m_1$,
2. $\overset{SL(m_1)}{\circ}\!-\!\!-\!\!-\!\overset{GL(n)}{\circ}\!-\!\!-\!\!-\!\overset{SL(m_2)}{\circ}$ *with* $n \ge m_1 = m_2$,
3. $\overset{SL(m_1)}{\circ}\!-\!\!-\!\!-\!\overset{SL(n)}{\circ}\!-\!\!-\!\!-\!\overset{SL(m_2)}{\circ}$ *with* $n = m_1$, $n = m_2$, $n = m_1 + m_2$ *or* $n > m_1 = m_2$.

REMARK 1.5. We can also obtain the orbital decomposition of an FP of A_r-type and their isotropy subgroups by [NN]. For our purpose, it is enough to see these results only for A_3-type $\overset{GL(m_1)}{\circ}\!\!-\!\!\!-\!\!\!-\overset{GL(n)}{\circ}\!\!-\!\!\!-\!\!\!-\overset{GL(m_2)}{\circ}$.

First we consider $\overset{GL(n)}{\circ}\!\!-\!\!\!-\!\!\!-\overset{GL(m_1)}{\circ}$. It is well-known that each orbit is represented by

$$J(r_1) = \begin{bmatrix} I_{r_1} & 0 \\ 0 & 0 \end{bmatrix} \in M(n, m_1) = \mathbf{C}^n \otimes \mathbf{C}^{m_1}$$

with $0 \leq r_1 \leq \min\{n, m_1\}$. Then the $GL(n)$-part of the isotropy subgroup at $J(r_1)$ is given by

$$H_1 = \left\{ \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix} \in GL(n) \,\middle|\, A_1 \in GL(r_1), A_2 \in GL(n - r_1) \right\}.$$

Next we consider $\overset{H_1}{\circ}\!\!-\!\!\!-\!\!\!-\overset{GL(m_2)}{\circ}$. In this case, each orbit is represented by

$$J(r_2, r_3) = \begin{bmatrix} I_{r_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{r_3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in M(n, m_2)$$

which is a block matrix of size $(r_2, r_1 - r_2, r_3, n - r_1 - r_3) \times (r_2, r_1 - r_2, r_3, m_2 - r_1 - r_3)$ with $0 \leq r_2 \leq r_1$ and $0 \leq r_1 + r_3 \leq \min\{n, m_2\}$. For each orbit, the H_1-part of the isotropy subgroup is given as

$$H_2 = \left\{ \begin{bmatrix} B_1 & * & * & * \\ 0 & B_2 & 0 & * \\ 0 & 0 & B_3 & * \\ 0 & 0 & 0 & B_4 \end{bmatrix} \in GL(n) \,\middle|\, \begin{matrix} B_1 \in GL(r_2), \\ B_2 \in GL(r_1 - r_2), \\ B_3 \in GL(r_3), \\ B_4 \in GL(n - r_1 - r_3) \end{matrix} \right\}.$$

The following is a key lemma to classify the FPs under various scalar restrictions.

LEMMA 1.6. *Let $\sigma : H \to GL(m)$ be a representation of an algebraic group H.*

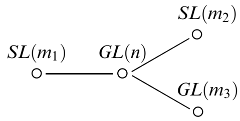
1. *If $m < n$, then $(H \times SL(n), \sigma \otimes \Lambda_1, M(m, n))$ is an FP if and only if $(H \times GL(n), \sigma \otimes \Lambda_1, M(m, n))$ is an FP. In this case they have the same number of orbits.*
2. *If $m \geq n$ and the number of orbits of $H \times SL(n)$ in $M(m, n)'$ is finite, then $(H \times SL(n), \sigma \otimes \Lambda_1, M(m, n))$ is an FP if and only if $(H \times GL(n), \sigma \otimes \Lambda_1, M(m, n))$ is an FP. In this case they have the same number of orbits.*

PROOF. See [KKMOT, Proposition 1.2]. □

2. The FPs of D_4-Type Under Various Scalar Restrictions

In this section, we shall classify FPs of D_4-type under various scalar restrictions. Here we put $m_4 = n$.

Proposition 2.1. *The diagram*



is a non FP if and only if $n \geq m_1 = m_2$.

Proof. If $n \geq m_1 = m_2$, then $\overset{SL(m_1)}{\circ}\text{---}\overset{GL(n)}{\circ}\text{---}\overset{SL(m_2)}{\circ}$ is a non FP by 2 of Corollary 1.4. Therefore our diagram is a non FP.

If $m_1 > n$ or $m_2 > n$, our representation has the same number of orbits as that of D_4-type with full scalar multiplications by 1 of Lemma 1.6.

Suppose that $n \geq m_1, m_2$ and $m_1 \neq m_2$. It follows from 2 of Lemma 1.6 that each orbit contained in $M(n, m_1)''$ of $\overset{GL(n)}{\circ}\text{---}\overset{GL(m_1)}{\circ}$ cannot be decomposed by the scalar-restricted action of $GL(n) \times SL(m_1)$. Therefore our representation has the same number of orbits as that of D_4-type with full scalar multiplications. Hence it is enough to investigate only the orbits of our space related with $M(n, m_1)'$ of $\overset{GL(n)}{\circ}\text{---}\overset{SL(m_1)}{\circ}$.

If $n = m_1 \neq m_2$ (resp. $n = m_2 \neq m_1$), the $GL(n)$-part of the isotropy subgroup of $\overset{GL(n)}{\circ}\text{---}\overset{SL(m_1)}{\circ}$ (resp. $\overset{GL(n)}{\circ}\text{---}\overset{SL(m_2)}{\circ}$) at a generic point is $SL(m_1)$ (resp. $SL(m_2)$). Since $\overset{SL(m_2)}{\circ}\text{---}\overset{SL(m_1)}{\circ}\text{---}\overset{GL(m_3)}{\circ}$ (resp. $\overset{SL(m_1)}{\circ}\text{---}\overset{SL(m_2)}{\circ}\text{---}\overset{GL(m_3)}{\circ}$) is an FP by 1 of Corollary 1.4, our representation is an FP.

If $n > m_1, m_2$ with $m_1 \neq m_2$, the $GL(n)$-part of the isotropy subgroup of $\overset{GL(n)}{\circ}\text{---}\overset{SL(m_1)}{\circ}$ at a generic point is isomorphic to

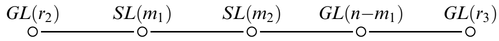
$$H_1 = \left\{ \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix} \in GL(n) \,\middle|\, \begin{array}{l} A_1 \in SL(m_1), \\ A_2 \in GL(n - m_1) \end{array} \right\}.$$

Since $\overset{SL(m_1)}{\circ}\text{---}\overset{GL(n)}{\circ}\text{---}\overset{GL(m_3)}{\circ}$ is an FP, the diagram $\overset{H_1}{\circ}\text{---}\overset{GL(m_3)}{\circ}$ is also an FP. Each orbit of this space is similarly represented by $J(r_2, r_3)$ as in Remark 1.5 ([NN]). The H_1-part of the isotropy subgroup at $J(r_2, r_3)$ contains

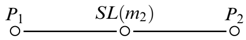
$$H_2 = \left\{ \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \in GL(n) \,\middle|\, B_1 \in P_1, B_2 \in P_2 \right\}.$$

where $P_1 = P(r_2, m_1 - r_2) \cap SL(m_1)$ and $P_2 = P(r_3, n - m_1 - r_3)$.

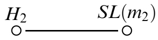
By Theorem 1.3,



is an FP for $m_1 \neq m_2$. In particular,

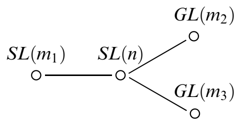


is an FP, and so that



is an FP. Hence our diagram is an FP. □

PROPOSITION 2.2. *The diagram*

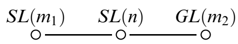


is a non FP if and only if it satisfies at least one of the following conditions:

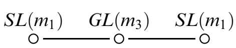
1. $n = m_1$,
2. $n = 2m_1$ *with* $m_1 \leq \min\{m_2, m_3\}$.

PROOF. If $n = m_1$, it is a non FP by Example 1.2. When $n = 2m_1$ with $m_1 \leq \min\{m_2, m_3\}$, take

$$\left(\begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I_{m_1} \end{bmatrix} \right) \in M(n, m_1) \oplus M(n, m_2).$$

The $SL(n)$-part of the isotropy subgroup of  at this point is given by

$$H_1 = \left\{ \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \in SL(n) \middle| A_1, A_2 \in SL(m_1) \right\}.$$

Then $H_1 \times GL(m_3)$ acts on $\begin{bmatrix} X \\ Y \end{bmatrix} \in M(n, m_3)$ with $X, Y \in M(m_1, m_3)$ as , which is a non FP by 2 of Corollary 1.4.

Suppose that the conditions 1 and 2 are not satisfied. If $m_1 > n$, our representation has the same number of orbits as that of D_4-type with full scalar multiplications by 1 of Lemma 1.6. Therefore we may assume, without loss of generality, $n > m_1$ and $m_2 \geq m_3$. It follows from 2 of Lemma 1.6 that each orbit contained in $M(n, m_1)''$ of $\overset{GL(n)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{GL(m_1)}{\circ}$ cannot be decomposed by the scalar-restricted action of $SL(n) \times SL(m_1)$. Therefore our representation has the same number of orbits as that of D_4-type with full scalar multiplications. Hence it is enough to consider only the orbits related with $M(n, m_1)'$ of $\overset{SL(n)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{SL(m_1)}{\circ}$.

The diagram $\overset{SL(m_1)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{SL(n)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{GL(m_3)}{\circ}$ is an FP by 1 of Corollary 1.4 and each orbit in this case is represented by $J = (J(m_1), J(r_2, r_3)) \in M(n, m_1) \oplus M(n, m_3)$ as in Remark 1.5. The $SL(n)$-part of the isotropy subgroup of $\overset{SL(m_1)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{SL(n)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{GL(m_3)}{\circ}$ at J contains

$$H_2 = \left\{ \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \in SL(n) \,\middle|\, B_1 \in P_1, B_2 \in P_2 \right\}.$$

where $P_1 = P(r_2, m_1 - r_2) \cap SL(m_1)$ and $P_2 = P(r_3, n - m_1 - r_3) \cap SL(n - m_1)$.

By Theorem 1.3,

$$\overset{GL(r_2)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{SL(m_1)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{GL(m_2)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{SL(n-m_1)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{GL(r_3)}{\circ}$$

is an FP for $m_1 \neq n - m_1$ or $m_2 < m_1$, i.e., $2m_1 \neq n$ or $m_2 < m_1$. In particular,

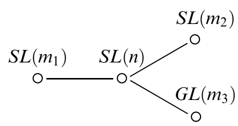
$$\overset{P_1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{GL(m_2)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{P_2}{\circ}$$

is an FP, and so that

$$\overset{H_2}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{GL(m_2)}{\circ}$$

is an FP. Hence we obtain our result. □

PROPOSITION 2.3. *The diagram*



is a non FP if and only if it satisfies at least one of the following conditions:

1. $n = m_1$,
2. $n = m_2$,
3. $n = m_1 + m_2$,
4. $n > m_1 = m_2$,
5. $n = 2m_1$ *with* $m_1 \leq \min\{m_2, m_3\}$,
6. $n = 2m_2$ *with* $m_2 \leq \min\{m_1, m_3\}$.

Proof. If $n = m_1$, $n = m_2$, $n = m_1 + m_2$ or $n > m_1 = m_2$, then $\overset{SL(m_1)}{\circ}\!\!-\!\!-\!\!-\!\!\overset{SL(n)}{\circ}\!\!-\!\!-\!\!-\!\!\overset{SL(m_2)}{\circ}$ is a non FP by 3 of Corollary 1.4. Therefore our diagram is a non FP. If $n = 2m_1$ with $m_1 \leq \min\{m_2, m_3\}$ or $n = 2m_2$ with $m_2 \leq \min\{m_1, m_3\}$, then it is a non FP by Proposition 2.2.

Suppose that the conditions 1 to 6 are not satisfied. If $m_1 > n$, our representation has the same number of orbits as that of D_4-type of Proposition 2.2 by 1 of Lemma 1.6. Hence we assume, without loss of generality, $n > m_1 > m_2$. It follows from 2 of Lemma 1.6 that each orbit contained in $M(n, m_1)''$ of $\overset{GL(n)}{\circ}\!\!-\!\!-\!\!-\!\!\overset{GL(m_1)}{\circ}$ cannot be decomposed by the scalar-restricted action of $SL(n) \times SL(m_1)$. Therefore our representation has the same number of orbits as that of D_4-type of Proposition 2.2. Hence it is enough to investigate the orbits related with $M(n, m_1)'$ of $\overset{SL(n)}{\circ}\!\!-\!\!-\!\!-\!\!\overset{SL(m_1)}{\circ}$. Then each orbit of $\overset{SL(m_1)}{\circ}\!\!-\!\!-\!\!-\!\!\overset{SL(n)}{\circ}\!\!-\!\!-\!\!-\!\!\overset{GL(m_3)}{\circ}$, which is an FP if $n \neq m_1$ by 1 of Corollary 1.4, is represented by $J = (J(m_1), J(r_2, r_3)) \in M(n, m_1) \oplus M(n, m_3)$ as in Remark 1.5. The $SL(n)$-part of the isotropy subgroup at J contains

$$H = \left\{ \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \in SL(n) \,\middle|\, A_1 \in P_1, A_2 \in P_2 \right\}.$$

where $P_1 = P(r_2, m_1 - r_2) \cap SL(m_1)$ and $P_2 = P(r_3, n - m_1 - r_3) \cap SL(n - m_1)$.

By Theorem 1.3,

$$\overset{GL(r_2)}{\circ}\!\!-\!\!-\!\!-\!\!-\!\!\overset{SL(m_1)}{\circ}\!\!-\!\!-\!\!-\!\!-\!\!\overset{SL(m_2)}{\circ}\!\!-\!\!-\!\!-\!\!-\!\!\overset{SL(n-m_1)}{\circ}\!\!-\!\!-\!\!-\!\!-\!\!\overset{GL(r_3)}{\circ}$$

is an FP for $n \neq m_2$, $n \neq m_1 + m_2$, $m_1 \neq m_2$ and ($n \neq 2m_1$ or $m_1 > m_2$ when $n = 2m_1$). In particular,

$$\overset{P_1}{\circ}\!\!-\!\!-\!\!-\!\!-\!\!\overset{SL(m_2)}{\circ}\!\!-\!\!-\!\!-\!\!-\!\!\overset{P_2}{\circ}$$

is an FP. Therefore

$$\overset{H}{\circ}\!\!-\!\!-\!\!-\!\!-\!\!\overset{SL(m_2)}{\circ}$$

is an FP. Hence we obtain our result. □

For Propositions 2.5 and 2.6, we shall prove the next lemma.

Lemma 2.4. *Let G_r be a subgroup of* $((GL(1) \times SL(m_1)) \times (GL(1) \times SL(m_2))) \times SL(n)$ *defined by* $G_r = \{(\alpha, A, \beta, B, C) \mid \alpha, \beta \in GL(1), A \in SL(m_1), B \in SL(m_2), C \in SL(n), \alpha^{m_1} = \beta^{m_2}\}$. *Then* $(G_r, (\Lambda_1 \otimes \Lambda_1 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1)$ *is a non FP if and only if* $n \geq m_1 = m_2$.

Proof. Assume that $n = m_1 = m_2$. The $SL(n)$-part of the isotropy subgroup of $((GL(1) \times SL(m_1)) \times SL(n), (\Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1)$ at I_{m_1} is $SL(m_1)$. Therefore our representation is reduced to $\overset{SL(m_1)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{SL(m_2)}{\circ}$ which is a non FP by Example 1.2.

Assume that $n > m_1 = m_2$. The $SL(n)$-part of the isotropy subgroup of $((GL(1) \times SL(m_1)) \times SL(n), (\Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1)$ at $\begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix} \in M(n, m_1)$ is

$$H_1 = \left\{ \begin{bmatrix} \alpha^{-1}C_1 & * \\ 0 & \gamma C_2 \end{bmatrix} \in SL(n) \middle| \begin{array}{l} C_1 \in SL(m_1), C_2 \in SL(n - m_1), \\ \alpha, \gamma \in GL(1), \alpha^{-m_1} \cdot \gamma^{n-m_1} = 1 \end{array} \right\}.$$

Then $(GL(1) \times SL(m_2)) \times H_1$ acts on $\begin{bmatrix} X \\ 0 \end{bmatrix} \in M(n, m_2)$ with $X \in M(m_1, m_2)$ as $\overset{SL(m_1)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{SL(m_1)}{\circ}$ which is a non FP by Example 1.2.

If $m_1 > n$ or $m_2 > n$, our representation has the same number of orbits as that of an A_3-type with full scalar multiplications by 1 of Lemma 1.6.

Suppose that $n \geq m_1$, $n \geq m_2$ and $m_1 \neq m_2$. It follows from 2 of Lemma 1.6 that each orbit contained in $M(n, m_1)''$ of $\overset{GL(n)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{GL(m_1)}{\circ}$ cannot be decomposed by the scalar-restricted action of $(GL(1) \times SL(m_1)) \times SL(n)$. Therefore our representation has the same number of orbits as that of A_3-type with full scalar multiplications. Hence it is enough to investigate only the orbits related with $M(n, m_1)'$ of $((GL(1) \times SL(m_1)) \times SL(n), (\Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1)$.

If $n = m_1 \neq m_2$, the $SL(n)$-part of the isotropy subgroup of $((GL(1) \times SL(m_1)) \times SL(n), (\Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1)$ at a generic point is $SL(m_1)$. Since $\overset{SL(m_2)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{SL(m_1)}{\circ}$ is an FP, our representation is an FP.

We may assume that $n > m_1 > m_2$ without loss of generality. The $SL(n)$-part of the isotropy subgroup of $((GL(1) \times SL(m_1)) \times SL(n), (\Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1)$ at a generic point is given by

$$H_2 = \left\{ \begin{bmatrix} \alpha^{-1}C_1 & * \\ 0 & \gamma C_2 \end{bmatrix} \in SL(n) \middle| \begin{array}{l} C_1 \in SL(m_1), C_2 \in SL(n - m_1) \\ \alpha, \gamma \in GL(1), \alpha^{-m_1} \cdot \gamma^{n-m_1} = 1 \end{array} \right\}.$$

By the action of $(GL(1) \times SL(m_2)) \times H_2$, each element $\begin{bmatrix} W \\ Z \end{bmatrix} \in M(n, m_2)$ with $W \in M(m_1, m_2)$, $Z \in M(n - m_1, m_2)$ is transformed to

$$T = \begin{bmatrix} 0 & W' \\ Z' & 0 \end{bmatrix} \in M(n, m_2)$$

with $W' \in M(m_1, m_2 - s)$, $Z' \in M(n - m_1, s)$ and $0 \leq s \leq \min\{n - m_1, m_2\}$.

The isotropy subgroup of $H_2 \times SL(m_2)$ at T contains

$$\left\{ \begin{bmatrix} \alpha^{-1}C_1 & 0 \\ 0 & \gamma C_2 \end{bmatrix} \in SL(n) \,\middle|\, \begin{matrix} C_1 \in SL(m_1), C_2 \in SL(n-m_1) \\ \alpha, \gamma \in GL(1), \alpha^{-m_1} \cdot \gamma^{n-m_1} = 1 \end{matrix} \right\}$$

$$\times \left\{ \begin{bmatrix} \delta_1 B_1 & 0 \\ 0 & \delta_2 B_2 \end{bmatrix} \in SL(m_2) \,\middle|\, \begin{matrix} B_1 \in SL(s), B_2 \in SL(m_2-s), \\ \delta_1, \delta_2 \in GL(1), \delta_1^s \cdot \delta_2^{m_2-s} = 1 \end{matrix} \right\}$$

Hence it is enough to show

$$\begin{bmatrix} 0 & W' \\ Z' & 0 \end{bmatrix} \mapsto \beta \begin{bmatrix} \alpha^{-1}C_1 & 0 \\ 0 & \gamma C_2 \end{bmatrix} \begin{bmatrix} 0 & W' \\ Z' & 0 \end{bmatrix} \begin{bmatrix} \delta_1 B_1 & 0 \\ 0 & \delta_2 B_2 \end{bmatrix}$$

is an FP with $\alpha^{m_1} = \beta^{m_2}$, $\alpha^{-m_1} \cdot \gamma^{n-m_1} = 1$ and $\delta_1^s \cdot \delta_2^{m_2-s} = 1$. If $s = 0$ or m_2, it is clearly an FP. If $0 < s < m_2$,

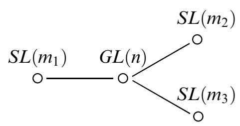
$$M(m_1, m_2 - s) \ni W' \mapsto (\alpha^{-1}C_1)W'(\beta\delta_2 B_2)$$

is an FP since $m_1 > m_2 - s$. Then we can put $\alpha = \beta = \gamma = 1$, and δ_1 runs over $GL(1)$. Therefore

$$M(n - m_1, s) \ni Z' \mapsto (\gamma C_2)Z'(\beta\delta_1 B_1)$$

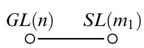
is an FP. Hence we have our results. □

Proposition 2.5. *The diagram*



is a non FP if and only if it satisfies at least one of the following conditions:

1. $n \geq m_1 = m_2$,
2. $n \geq m_2 = m_3$,
3. $n \geq m_3 = m_1$,
4. $n \geq m_1 = m_2 + m_3$,
5. $n \geq m_2 = m_3 + m_1$,
6. $n \geq m_3 = m_1 + m_2$.

Proof. If $n \geq m_1 = m_2$, $n \geq m_2 = m_3$ or $n \geq m_3 = m_1$, then it is a non FP by Proposition 2.1. Assume $n \geq m_1 = m_2 + m_3$. The $GL(n)$-part of the isotropy subgroup of 

at $\begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix} \in M(n, m_1)$ contains

$$H = \begin{bmatrix} SL(m_1) & * \\ 0 & GL(n - m_1) \end{bmatrix} \ (\subset GL(n)).$$

Then $\overset{SL(m_2)}{\circ} \!\!-\!\!\!-\!\!\!-\!\! \overset{H}{\circ} \!\!-\!\!\!-\!\!\!-\!\! \overset{SL(m_3)}{\circ}$ acts on $\left(\begin{bmatrix} X \\ 0 \end{bmatrix}, \begin{bmatrix} Y \\ 0 \end{bmatrix} \right) \in M(n, m_2) \oplus M(n, m_3)$ with $X \in M(m_1, m_2)$, $Y \in M(m_1, m_3)$ as

$$\overset{SL(m_2)}{\circ} \!\!-\!\!\!-\!\!\!-\!\!\!-\!\! \overset{SL(m_1)}{\circ} \!\!-\!\!\!-\!\!\!-\!\!\!-\!\! \overset{SL(m_3)}{\circ}$$

which is a non FP by 3 of Corollary 1.4. When $n \geq m_2 = m_3 + m_1$ or $n \geq m_3 = m_1 + m_2$, we can see similarly that our representation is a non FP.

Assume that the conditions 1 to 6 are not satisfied. If $n < m_1$, $n < m_2$ or $n < m_3$, our representation has the same number of orbits as that of D_4-type of Proposition 2.1 by 1 of Lemma 1.6. Hence we may assume, without loss of generality, $n \geq m_1 > m_2 > m_3$.

It follows from 2 of Lemma 1.6 that each orbit contained in $M(n, m_1)''$ of $\overset{GL(n)}{\circ} \!\!-\!\!\!-\!\!\!-\!\! \overset{GL(m_1)}{\circ}$ cannot be decomposed by the scalar-restricted action of $GL(n) \times SL(m_1)$. Therefore our representation has the same number of orbits as that of D_4-type of Proposition 2.1. Hence it is enough to study only the orbits related with $M(n, m_1)'$ of $\overset{GL(n)}{\circ} \!\!-\!\!\!-\!\!\!-\!\! \overset{SL(m_1)}{\circ}$. Then the orbit $M(n, m_1)'$ is $J(m_1)$ as in Remark 1.5 and we denote by H_1 the $GL(n)$-part of the isotropy subgroup of $\overset{GL(n)}{\circ} \!\!-\!\!\!-\!\!\!-\!\! \overset{SL(m_1)}{\circ}$ at $J(m_1)$.

It follows from 2 of Lemma 1.6 that each orbit contained in $M(n, m_2)''$ of $\overset{H_1}{\circ} \!\!-\!\!\!-\!\!\!-\!\! \overset{GL(m_1)}{\circ}$ cannot be decomposed by the scalar-restricted action of $H_1 \times SL(m_2)$. Therefore our representation has the same number of orbits as that D_4-type of Proposition 2.1. Hence it is enough to see only each orbit related with $M(n, m_2)'$ of $\overset{H_1}{\circ} \!\!-\!\!\!-\!\!\!-\!\! \overset{SL(m_2)}{\circ}$, which are represented by $J(r_2, r_3)$ with $r_2 + r_3 = m_2$ as in Remark 1.5. The H_1-part of the isotropy subgroup of $\overset{H_1}{\circ} \!\!-\!\!\!-\!\!\!-\!\! \overset{SL(m_2)}{\circ}$ at $J(r_2, r_3)$ is isomorphic to

$$H_2 = \begin{bmatrix} H_3 & * \\ 0 & GL(t) \end{bmatrix} \ (\subset H_1)$$

where we put $t = n - m_1 - m_2 + r_2$ and

$$H_3 = \left\{ \begin{bmatrix} \alpha_1 A_1 & 0 & 0 \\ * & \alpha_2 A_2 & * \\ 0 & 0 & \alpha_3 A_3 \end{bmatrix} \in GL(n - t) \,\middle|\, \begin{array}{l} A_1 \in SL(m_1 - r_2), \\ A_2 \in SL(r_2), A_3 \in SL(r_3), \\ \alpha_1, \alpha_2, \alpha_3 \in GL(1), \\ \alpha_1^{m_1 - r_2} \cdot \alpha_2^{r_2} = 1, \\ \alpha_2^{r_2} \cdot \alpha_3^{r_3} = 1 \end{array} \right\}.$$

First we assume that $t\ (= n - m_1 - m_2 + r_2) \neq 0$. By the action of $H_2 \times SL(m_3)$, any element $\begin{bmatrix} W \\ Z \end{bmatrix} \in M(n, m_3)$ with $W \in M(n-t, m_3)$, $Z \in M(t, m_3)$ is transformed to

$$T_1 = \begin{bmatrix} 0 & W' \\ Z' & 0 \end{bmatrix} \in M(n, m_3)$$

with $W' \in M(n-t, m_3 - r_4)$, $Z' \in M(t, r_4)$ and $0 \leq r_4 \leq \min\{t, m_3\}$. Then the isotropy subgroup of $H_2 \times SL(m_3)$ at T_1 contains

$$\begin{bmatrix} H_3 & 0 \\ 0 & GL(t) \end{bmatrix} \times \left\{ \begin{bmatrix} \beta_1 B_1 & 0 \\ 0 & \beta_2 B_2 \end{bmatrix} \middle| \begin{array}{l} B_1 \in SL(r_4), B_2 \in SL(m_3 - r_4), \\ \beta_1, \beta_2 \in GL(1), \beta_1^{r_4} \cdot \beta_2^{m_3 - r_4} = 1 \end{array} \right\}$$

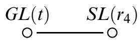
$$(\subset H_2 \times SL(m_3)).$$

Hence it is enough to show

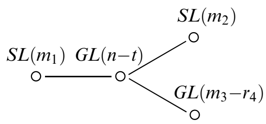
$$\begin{bmatrix} 0 & W' \\ Z' & 0 \end{bmatrix} \mapsto \begin{bmatrix} h & 0 \\ 0 & A_4 \end{bmatrix} \begin{bmatrix} 0 & W' \\ Z' & 0 \end{bmatrix} \begin{bmatrix} \beta_1 B_1 & 0 \\ 0 & \beta_2 B_2 \end{bmatrix}$$

is an FP with $A_4 \in GL(t)$, $h \in H_3$ and $\beta_1^{r_4} \cdot \beta_2^{m_3 - r_4} = 1$, namely

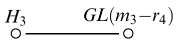
1. $M(t, r_4) \ni Z' \mapsto A_4 Z'(\beta_1 B_1)$ is an FP, and
2. $M(n-t, m_3 - r_4) \ni W' \mapsto hW'(\beta_2 B_2)$ is, at the same time, an FP with $\beta_1^{r_4} \cdot \beta_2^{m_3 - r_4} = 1$.

1 is clearly an FP since 

is an FP. The diagram

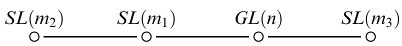


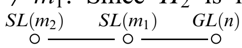
is an FP for $m_1 \neq m_2$ by Proposition 2.1, in particular



is an FP since β_2 runs over $GL(1)$. Hence our representation is an FP.

Next we assume that $t\ (= n - m_1 - m_2 + r_2) = 0$. If $r_2 = m_2$, then $r_3 = 0$, i.e., $n = m_1$. By Theorem 1.3,



is an FP for $m_2 \neq m_1$, $m_2 \neq m_3$, $m_1 \neq m_3$ and $m_2 + m_3 \neq m_1$. Since H_2 is isomorphic to the $GL(n)$-part of the isotropy subgroup of 

at $\left(\begin{bmatrix} I_{m_2} \\ 0 \end{bmatrix}, I_{m_1} \right) \in M(m_1, m_2) \oplus M(n, m_1)$, in particular

$$\overset{H_2}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{SL(m_3)}{\circ}$$

is an FP.

If $r_3 = m_2$, then $r_2 = 0$, i.e., $n = m_1 + m_2$. Then H_2 is

$$\begin{bmatrix} SL(m_1) & 0 \\ 0 & SL(m_2) \end{bmatrix}$$

Since

$$\overset{SL(m_1)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{SL(m_3)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{SL(m_2)}{\circ}$$

is an FP for $m_1 \neq m_3$, $m_1 \neq m_2$, $m_3 \neq m_2$ and $m_1 + m_2 \neq m_3$,

$$\overset{H_2}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{SL(m_3)}{\circ}$$

is an FP.

If $r_2 \neq 0$ and $r_3 \neq 0$, then H_2 is isomorphic to

$$H_4 = \left\{ \begin{bmatrix} \alpha D_1 & * & * \\ 0 & \beta D_2 & 0 \\ 0 & 0 & \gamma D_3 \end{bmatrix} \in GL(n) \,\middle|\, \begin{array}{l} D_1 \in SL(m_1 + m_2 - n), \\ D_2 \in SL(n - m_1), \\ D_3 \in SL(n - m_2), \\ \alpha, \beta, \gamma \in GL(1), \\ \alpha^{m_1+m_2-n} \cdot \beta^{n-m_1} = 1, \\ \alpha^{m_1+m_2-n} \cdot \gamma^{n-m_2} = 1 \end{array} \right\}.$$

We consider the action $H_4 \times SL(m_3)$ on

$$T_2 = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \in M(n, m_3)$$

with $X_1 \in M(m_1 + m_2 - n, m_3)$, $X_2 \in M(n - m_1, m_3)$, $X_3 \in M(n - m_2, m_3)$.

If $X_2 = X_3 = 0$, then $\overset{H_4}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{SL(m_3)}{\circ}$ has the same number of orbits as that of $\overset{GL(m_1+m_2-n)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{SL(m_3)}{\circ}$ which is an FP.

We may suppose that $X_2 \neq 0$ or $X_3 \neq 0$. By the action $H_4 \times SL(m_3)$, an element T_2 is transformed to the

$$T_3 = \begin{bmatrix} 0 & X_1' \\ X_2' & 0 \\ X_3' & 0 \end{bmatrix} \in M(n, m_3)$$

with $X_1' \in M(m_1 + m_2 - n, m_3 - s)$, $X_2' \in M(n - m_1, s)$, $X_3' \in M(n - m_2, s)$ and $s = \max\{\text{rank } X_2, \text{rank } X_3\}$.

Then the isotropy subgroup of $H_2 \times SL(m_3)$ at T_3 contains

$$\left\{ \begin{bmatrix} \alpha D_1 & 0 & 0 \\ 0 & \beta D_2 & 0 \\ 0 & 0 & \gamma D_3 \end{bmatrix} \in H_2 \;\middle|\; \begin{array}{l} D_1 \in SL(m_1 + m_2 - n), \\ D_2 \in SL(n - m_1), D_3 \in SL(n - m_2), \\ \alpha, \beta, \gamma \in GL(1), \\ \alpha^{m_1+m_2-n} \cdot \beta^{n-m_1} = 1, \\ \alpha^{m_1+m_2-n} \cdot \gamma^{n-m_2} = 1 \end{array} \right\}$$

$$\times \left\{ \begin{bmatrix} \delta_1 E_1 & 0 \\ 0 & \delta_2 E_2 \end{bmatrix} \in SL(m_3) \;\middle|\; \begin{array}{l} E_1 \in SL(s), E_2 \in SL(m_3 - s), \\ \delta_1, \delta_2 \in GL(1), \delta_1^s \cdot \delta_2^{m_3-s} = 1 \end{array} \right\}.$$

We put $Y = \begin{bmatrix} X_2' \\ X_3' \end{bmatrix} \in M(2n - m_1 - m_2, s)$, and let $H_5 = \left\{ \begin{bmatrix} \beta D_2 & 0 \\ 0 & \gamma D_3 \end{bmatrix} \right\}$ be the lower reductive part of H_4.

Hence it is enough to show

$$\begin{bmatrix} 0 & X_1' \\ Y & 0 \end{bmatrix} \mapsto \begin{bmatrix} \alpha D_1 & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} 0 & X_1' \\ Y & 0 \end{bmatrix} \begin{bmatrix} \delta_1 E_1 & 0 \\ 0 & \delta_2 E_2 \end{bmatrix}$$

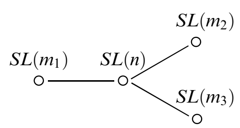
is an FP with $h \in H_5$, $\alpha^{m_1+m_2-n} \cdot \beta^{n-m_1} = 1$, $\alpha^{m_1+m_2-n} \cdot \gamma^{n-m_2} = 1$ and $\delta_1^s \cdot \delta_2^{m_3-s} = 1$, namely

3. $M(m_1 + m_2 - n, m_3 - s) \ni X_1' \mapsto (\alpha D_1) X_1' \delta_2 E_2$ is an FP, and
4. $M(2n - m_1 - m_2, s) \ni Y \mapsto h Y \delta_1 E_1$ is, at the same time, an FP with the conditions $\alpha^{m_1+m_2-n} \cdot \beta^{n-m_1} = 1$, $\alpha^{m_1+m_2-n} \cdot \gamma^{n-m_2} = 1$ and $\delta_1^s \cdot \delta_2^{m_3-s} = 1$.

The space 3 is clearly an FP. Since $n - m_1 \neq n - m_2$, the space 4 is an FP by Lemma 2.4. Hence our representation is an FP. □

Although M. Nagura, S. Otani and D. Takeda independently obtained the same result as the following Proposition 2.6 ([NOT, Theorem 4.1]), we will give our proof here.

Proposition 2.6. *The diagram*



is a non FP if and only if it satisfies at least one of the following conditions:

1. $n = m_1$,
2. $n = m_2$,
3. $n = m_3$,
4. $n > m_1 = m_2$,
5. $n > m_1 = m_3$,
6. $n > m_2 = m_3$,
7. $n = m_1 + m_2$,
8. $n = m_1 + m_3$,
9. $n = m_2 + m_3$,
10. $n > m_1 = m_2 + m_3$,
11. $n > m_2 = m_1 + m_3$,
12. $n > m_3 = m_1 + m_2$,
13. $n = 2m_1$ *with* $m_1 \le \min\{m_2, m_3\}$,
14. $n = 2m_2$ *with* $m_2 \le \min\{m_1, m_3\}$,
15. $n = 2m_3$ *with* $m_3 \le \min\{m_1, m_2\}$,
16. $n + m_1 = m_2 + m_3$ *with* $m_1 < \min\{m_2, m_3\}$,
17. $n + m_2 = m_1 + m_3$ *with* $m_2 < \min\{m_1, m_3\}$,
18. $n + m_3 = m_1 + m_2$ *with* $m_3 < \min\{m_1, m_2\}$,
19. $n = m_1 + m_2 + m_3$,
20. $2n = m_1 + m_2 + m_3$ *with* $n > \max\{m_1, m_2, m_3\}$.

Proof. By Propositions 2.3 and 2.5, the conditions 1 to 15 are sufficient. Assume that $n + m_3 = m_1 + m_2$ with $m_3 < \min\{m_1, m_2\}$. In particular, $n > \max\{m_1, m_2\}$ and $n < m_1 + m_2$. Take $Q_1 = \left(\begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix} \right) \in M(n, m_1) \oplus M(n, m_2)$. The $SL(n)$-part of the isotropy subgroup of $\overset{SL(m_1)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{SL(n)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{SL(m_2)}{\circ}$ at Q_1 is isomorphic to

$$H_1 = \left\{ \begin{bmatrix} A_1 & * & * \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \in SL(n) \;\middle|\; \begin{array}{l} A_1 \in SL(m_1 + m_2 - n), \\ A_2 \in SL(n - m_1), A_3 \in SL(n - m_2) \end{array} \right\}.$$

Then $H_1 \times SL(m_3)$ acts on $Q_2 = \begin{bmatrix} X \\ 0 \end{bmatrix} \in M(n, m_3)$ with $X \in M(m_3, m_3)$ as $\overset{SL(m_3)}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{SL(m_3)}{\circ}$ which is a non FP by Example 1.2.

If $n + m_1 = m_2 + m_3$ with $m_1 < \min\{m_2, m_3\}$ or $n + m_2 = m_3 + m_1$ with $m_2 < \min\{m_3, m_1\}$, we can prove similarly that our representation is a non FP.

If $n = m_1 + m_2 + m_3$, the $SL(n)$-part of the isotropy subgroup of $\overset{SL(m_1)}{\circ}\text{---}\overset{SL(n)}{\circ}\text{---}\overset{SL(m_2)}{\circ}$ at Q_1 is isomorphic to

$$H_2 = \left\{ \begin{bmatrix} B_1 & 0 & 0 \\ * & B_2 & 0 \\ * & 0 & B_3 \end{bmatrix} \in SL(n) \,\middle|\, B_1 \in SL(m_3), B_2 \in SL(m_1), B_3 \in SL(m_2) \right\}.$$

Then $H_2 \times SL(m_3)$ acts on Q_2 as $\overset{SL(m_3)}{\circ}\text{---}\overset{SL(m_3)}{\circ}$ which is a non FP by Example 1.2.

If $2n = m_1 + m_2 + m_3$ with $n > \max\{m_1, m_2, m_3\}$, the $SL(n)$-part of the isotropy subgroup of $\overset{SL(m_1)}{\circ}\text{---}\overset{SL(n)}{\circ}\text{---}\overset{SL(m_2)}{\circ}$ at Q_1 is isomorphic to H_1. Then $H_1 \times SL(m_3)$ acts on

$$\begin{bmatrix} 0 \\ Y_1 \\ Y_2 \end{bmatrix} \in M(n, m_3)$$

with $Y_1 \in M(n - m_1, m_3)$, $Y_2 \in M(n - m_2, m_3)$ as

$$\overset{SL(n-m_1)}{\circ}\text{------}\overset{SL(m_3)}{\circ}\text{------}\overset{SL(n-m_2)}{\circ}$$

which is a non FP for $n - m_1 + n - m_2 = m_3$ by 3 of Corollary 1.4.

Suppose that the conditions 1 to 20 are not satisfied. If $n < m_1$, $n < m_2$ or $n < m_3$, our representation has the same number of orbits as that of D_4-type of Proposition 2.3. Hence we may assume, without loss of generality, $n > m_1 > m_2 > m_3$.

It follows from 2 of Lemma 1.6 that each orbit contained in $M(n, m_1)''$ of $\overset{GL(n)}{\circ}\text{---}\overset{GL(m_1)}{\circ}$ cannot be decomposed by the scalar-restricted action of $SL(n) \times SL(m_1)$. Therefore our representation has the same number of orbits as that of D_4-type of Proposition 2.3. Hence it is enough to study only the orbits related with $M(n, m_1)'$ of $\overset{SL(n)}{\circ}\text{---}\overset{SL(m_1)}{\circ}$. Then the orbit is represented by $J(m_1) \in M(n, m_1)$ as in Remark 1.5 and we denote by H_3 the $SL(n)$-part of the isotropy subgroup of $\overset{SL(n)}{\circ}\text{---}\overset{SL(m_1)}{\circ}$ at $J(m_1)$. It follows from 2 of Lemma 1.6 that each orbit contained in $M(n, m_2)''$ of $\overset{H_3}{\circ}\text{---}\overset{GL(m_2)}{\circ}$ cannot be decomposed by the scalar-restricted action of $H_3 \times SL(m_2)$. Therefore our representation has the same number of orbits as that of D_4-type of Proposition 2.3. Hence it is enough to see only each orbit related with $M(n, m_2)'$ of $\overset{H_3}{\circ}\text{---}\overset{SL(m_2)}{\circ}$, which are represented by $J(r_2, r_3)$ with $r_2 + r_3 = m_2$ as in Remark 1.5.

We put $t = n - m_1 - m_2 + r_2$. The H_3-part of the isotropy subgroup of $\overset{H_3}{\circ}\!\!\frac{\quad SL(m_2)\quad}{}\!\!\circ$ at $J(r_2, r_3)$ is isomorphic to

$$H_4 = \left\{ \begin{bmatrix} \alpha_1 C_1 & * & * & * \\ 0 & \alpha_2 C_2 & 0 & * \\ 0 & 0 & \alpha_3 C_3 & * \\ 0 & 0 & 0 & \alpha_4 C_4 \end{bmatrix} \in SL(n) \,\middle|\, \begin{array}{l} C_1 \in SL(r_2), \\ C_2 \in SL(m_1 - r_2), \\ C_3 \in SL(m_2 - r_2), \\ C_4 \in SL(t), \\ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in GL(1), \\ \alpha_1^{r_2} \cdot \alpha_2^{m_1 - r_2} = 1, \\ \alpha_1^{r_2} \cdot \alpha_3^{m_2 - r_2} = 1, \\ \alpha_2^{m_1 - r_2} \cdot \alpha_4^{t} = 1, \\ \alpha_3^{m_2 - r_2} \cdot \alpha_4^{t} = 1 \end{array} \right\}.$$

First we assume that $t\ (= n - m_1 - m_2 + r_2) \neq 0$. By the action of $H_4 \times SL(m_3)$, an element $\begin{bmatrix} W \\ Z \end{bmatrix} \in M(n, m_3)$ with $W \in M(n - t, m_3)$, $Z \in M(t, m_3)$ is transformed to

$$T_1 = \begin{bmatrix} 0 & W' \\ Z' & 0 \end{bmatrix} \in M(n, m_3)$$

with $W' \in M(n - t, m_3 - r_4)$, $Z' \in M(t, r_4)$ and $0 \le r_4 \le \min\{t, m_3\}$. Let

$$K_1 = \left\{ \begin{bmatrix} \alpha_1 C_1 & * & * \\ 0 & \alpha_2 C_2 & 0 \\ 0 & 0 & \alpha_3 C_3 \end{bmatrix} \in GL(n - t) \,\middle|\, \begin{array}{l} C_1 \in SL(r_2), \\ C_2 \in SL(m_1 - r_2), \\ C_3 \in SL(m_2 - r_2), \\ \alpha_1, \alpha_2, \alpha_3 \in GL(1), \\ \alpha_1^{r_2} \cdot \alpha_2^{m_1 - r_2} = 1, \\ \alpha_1^{r_2} \cdot \alpha_3^{m_2 - r_2} = 1 \end{array} \right\}$$

be the upper $(n - t) \times (n - t)$-part of H_4. Then the isotropy subgroup of H_4 at T_1 contains

$$\left\{ \begin{bmatrix} h & 0 \\ 0 & \alpha_4 C_4 \end{bmatrix} \in H_4 \,\middle|\, \begin{array}{l} h \in K_1, \\ \alpha_1^{r_2} \cdot \alpha_2^{m_1 - r_2} = 1, \\ \alpha_1^{r_2} \cdot \alpha_3^{m_2 - r_2} = 1, \\ \alpha_2^{m_1 - r_2} \cdot \alpha_4^{t} = 1, \\ \alpha_3^{m_2 - r_2} \cdot \alpha_4^{t} = 1 \end{array} \right\}$$

and the isotropy subgroup of $SL(m_3)$ at T_1 contains

$$L = \left\{ \begin{bmatrix} \beta_1 D_1 & 0 \\ 0 & \beta_2 D_2 \end{bmatrix} \in SL(m_3) \,\middle|\, \begin{array}{l} D_1 \in SL(r_4), D_2 \in SL(m_3 - r_4), \\ \beta_1, \beta_2 \in GL(1), \beta_1^{r_4} \cdot \beta_2^{m_3 - r_4} = 1 \end{array} \right\}.$$

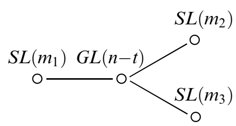
Hence it is enough to show

$$\begin{bmatrix} 0 & W' \\ Z' & 0 \end{bmatrix} \mapsto \begin{bmatrix} h & 0 \\ 0 & \alpha_4 C_4 \end{bmatrix} \begin{bmatrix} 0 & W' \\ Z' & 0 \end{bmatrix} \begin{bmatrix} \beta_1 D_1 & 0 \\ 0 & \beta_2 D_2 \end{bmatrix}$$

is an FP with $\alpha_1^{r_2} \cdot \alpha_2^{m_1 - r_2} = 1$, $\alpha_1^{r_2} \cdot \alpha_3^{m_2 - r_2} = 1$, $\alpha_2^{m_1 - r_2} \cdot \alpha_4^t = 1$, $\alpha_3^{m_2 - r_2} \cdot \alpha_4^t = 1$ and $\beta_1^{r_4} \cdot \beta_2^{m_3 - r_4} = 1$, namely

1. $M(t, r_4) \ni Z' \mapsto (\alpha_4 C_4) Z' (\beta_1 D_1)$ is an FP, and
2. $M(n - t, m_3 - r_4) \ni W' \mapsto h W' (\beta_2 D_2)$ is, at the same time, an FP with the conditions of $\alpha_1^{r_2} \cdot \alpha_2^{m_1 - r_2} = 1$, $\alpha_1^{r_2} \cdot \alpha_3^{m_2 - r_2} = 1$, $\alpha_2^{m_1 - r_2} \cdot \alpha_4^t = 1$, $\alpha_3^{m_2 - r_2} \cdot \alpha_4^t = 1$ and $\beta_1^{r_4} \cdot \beta_2^{m_3 - r_4} = 1$.

If $r_4 = 0$, the space 2 has the same number of orbits as that of

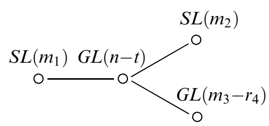


which is an FP by Proposition 2.5.

If $0 < r_4 < \min\{t, m_3\}$, the space Z' is transformed to the form

$$\begin{bmatrix} I_{r_4} \\ 0 \end{bmatrix} \in M(t, r_4).$$

Then α_4 and β_1 independently run over $GL(1)$, and 2 has the same number of orbits as that of



which is an FP by Proposition 2.1.

If $r_4 = m_3 \leq t$, its orbit is represented by

$$\begin{bmatrix} 0 \\ I_{m_3} \end{bmatrix} \in M(n, m_3).$$

Suppose that $r_4 = t < m_3$. The space 1 is clearly an FP. Then $\beta_1 = \alpha_4^{-1}$. By the conditions of L we have $\beta_2^{m_3 - t} = \alpha_4^t$. Therefore $\alpha_2^{m_1 - r_2} \cdot \beta_2^{m_3 - t} = 1$, $\alpha_3^{m_2 - r_2} \cdot \beta_2^{m_3 - t} = 1$ and $\alpha_1^{r_2} = \beta_2^{m_3 - t}$. Let

$$L_1 = \{[\beta_2 D_2] \in GL(1) \times SL(m_3 - t)\}$$

be the lower reductive part of L. We consider the action $K_1 \times L_1$ on

$$W' = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \in M(n-t, m_3-t)$$

with $X_1 \in M(r_2, m_3-t)$, $X_2 \in M(m_1-r_2, m_3-t)$, $X_3 \in M(m_2-r_2, m_3-t)$.

If $X_2 = X_3 = 0$, then 2 is an FP. We may suppose that $X_2 \neq 0$ or $X_3 \neq 0$. The action $K_1 \times L_1$ transforms W' to the form

$$W'' = \begin{bmatrix} 0 & X_1' \\ X_2' & 0 \\ X_3' & 0 \end{bmatrix} \in M(n-t, m_3-t)$$

with $X_1' \in M(r_2, m_3-t-s)$, $X_2' \in M(m_1-r_2, s)$, $X_3' \in M(m_2-r_2, s)$ and $s = \max\{\operatorname{rank} X_2, \operatorname{rank} X_3\}$. The isotropy subgroup of $K_1 \times L_1$ at W'' contains

$$K_2 = \left\{ \begin{bmatrix} \alpha_1 C_1 & 0 & 0 \\ 0 & \alpha_2 C_2 & 0 \\ 0 & 0 & \alpha_3 C_3 \end{bmatrix} \in K_1 \;\middle|\; \begin{array}{l} C_1 \in SL(r_2), C_2 \in SL(m_1-r_2), \\ C_3 \in SL(m_2-r_2), \alpha_1, \alpha_2, \alpha_3 \in GL(1), \\ \alpha_1^{r_2} = \beta_2^{m_3-t}, \\ \alpha_2^{m_1-r_2} \cdot \beta_2^{m_3-t} = 1, \\ \alpha_3^{m_2-r_2} \cdot \beta_2^{m_3-t} = 1, \end{array} \right\}$$

and

$$L_2 = \left\{ \begin{bmatrix} \beta_2 \cdot \gamma_1 E_1 & 0 \\ 0 & \beta_2 \cdot \gamma_2 E_2 \end{bmatrix} \in L_1 \;\middle|\; \begin{array}{l} E_1 \in SL(s), E_2 \in SL(m_3-t-s) \\ \beta_2, \gamma_1, \gamma_2 \in GL(1), \\ \gamma_1^s \cdot \gamma_2^{m_3-t-s} = 1 \end{array} \right\}.$$

We put $Y = \begin{bmatrix} X_2' \\ X_3' \end{bmatrix} \in M(m_1+m_2-2r_2, s)$, and let

$$K_3 = \left\{ \begin{bmatrix} \alpha_2 C_2 & 0 \\ 0 & \alpha_3 C_3 \end{bmatrix} \in GL(m_1+m_2-2r_2) \;\middle|\; \begin{array}{l} C_2 \in SL(m_1-r_2), \\ C_3 \in SL(m_2-r_2), \\ \alpha_2, \alpha_3 \in GL(1) \\ \alpha_2^{m_1-r_2} \cdot \beta_2^{m_3-t} = 1, \\ \alpha_3^{m_2-r_2} \cdot \beta_2^{m_3-t} = 1 \end{array} \right\}$$

be the middle reductive part of K_2. Hence it is enough to show

$$\begin{bmatrix} 0 & X_1' \\ Y & 0 \end{bmatrix} \mapsto \begin{bmatrix} \alpha_1 C_1 & 0 \\ 0 & h' \end{bmatrix} \begin{bmatrix} 0 & X_1' \\ Y & 0 \end{bmatrix} \begin{bmatrix} \beta_2 \cdot \gamma_1 E_1 & 0 \\ 0 & \beta_2 \cdot \gamma_2 E_2 \end{bmatrix}$$

is an FP with $h' \in K_3$, $\alpha_1^{r_2} = \beta_2^{m_3-t}$, $\alpha_2^{m_1-r_2} \cdot \beta_2^{m_3-t} = 1$, $\alpha_3^{m_2-r_2} \cdot \beta_2^{m_3-t} = 1$ and $\gamma_1^s \cdot \gamma_2^{m_3-t-s} = 1$, namely

3. $M(r_2, m_3 - t - s) \ni X_1' \mapsto (\alpha_1 C_1) X_1' (\beta_2 \cdot \gamma_2 E_2)$ is an FP, and
4. $M(m_1 + m_2 - 2r_2, s) \ni Y \mapsto h' Y(\beta_2 \cdot \gamma_1 E_1)$ is, at the same time, an FP with the conditions $\alpha_1^{r_2} = \beta_2^{m_3 - t}$, $\alpha_2^{m_1 - r_2} \cdot \beta_2^{m_3 - t} = 1$, $\alpha_3^{m_2 - r_2} \cdot \beta_2^{m_3 - t} = 1$ and $\gamma_1^s \cdot \gamma_2^{m_3 - t - s} = 1$.

The space 3 is clearly an FP. Then the space 4 is an FP by Lemma 2.4 since $m_1 - r_2 \neq m_2 - r_2$. Hence our representation is an FP.

Next we assume that $t \ (= n - m_1 - m_2 + r_2) = 0$. The isotropy subgroup H_4 is isomorphic to

$$H_4' = \left\{ \begin{bmatrix} C_1' & 0 & 0 \\ * & C_2' & * \\ 0 & 0 & C_3' \end{bmatrix} \in SL(n) \;\middle|\; \begin{array}{l} C_1' \in SL(n - m_2), \\ C_2' \in SL(m_1 + m_2 - n), \\ C_3' \in SL(n - m_1), \end{array} \right\}.$$

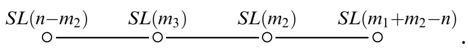
Then H_4' contains

$$H_5 = \begin{bmatrix} SL(n - m_2) & 0 \\ 0 & K_4 \end{bmatrix} \ (\subset H_4')$$

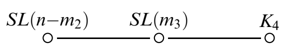
where

$$K_4 = \left\{ \begin{bmatrix} C_2' & * \\ 0 & C_3' \end{bmatrix} \in SL(m_2) \;\middle|\; \begin{array}{l} C_2' \in SL(m_1 + m_2 - n), \\ C_3' \in SL(n - m_1) \end{array} \right\}.$$

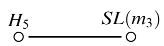
By Theorem 1.3, we can see the conditions to be an FP of



In particular



is an FP. Therefore



is an FP, except $n - m_2 = m_1 + m_2 - n$, i.e., $2n = m_1 + 2m_2$.

On the other hand, the isotropy subgroup H_4' contains

$$H_6 = \begin{bmatrix} K_5 & 0 \\ 0 & SL(n - m_1) \end{bmatrix} \ (\subset H_4')$$

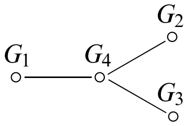
where

$$K_5 = \left\{ \begin{bmatrix} C_1' & 0 \\ * & C_2' \end{bmatrix} \in SL(m_1) \;\middle|\; \begin{array}{l} C_1' \in SL(n - m_2), \\ C_2' \in SL(m_1 + m_2 - n) \end{array} \right\}.$$

We can show similarly this case to be an FP, except the case of $2n = 2m_1 + m_2$. Therefore it remains the case of $2n = m_1 + 2m_2 = 2m_1 + m_2$, i.e., $m_1 = m_2$. However $m_1 = m_2$ contradicts the assumption of $m_1 \neq m_2$. Hence we obtain our results. □

By Propositions 2.1 to 2.3, 2.5 and 2.6, we have the following theorem.

THEOREM 2.7. *The diagram*



where $G_i = GL(m_i)$ *or* $SL(m_i)$ *for* $i = 1, 2, 3, 4$, *is a non FP if and only if it satisfies at least one of the following conditions*:

1. $m_4 = m_1$ *with* $G_1 = SL(m_1)$ *and* $G_4 = SL(m_4)$,
2. $m_4 = m_2$ *with* $G_2 = SL(m_2)$ *and* $G_4 = SL(m_4)$,
3. $m_4 = m_3$ *with* $G_3 = SL(m_3)$ *and* $G_4 = SL(m_4)$,
4. $m_4 > m_1 = m_2$ *with* $G_i = SL(m_i)$ *for* $i = 1, 2$,
5. $m_4 > m_1 = m_3$ *with* $G_i = SL(m_i)$ *for* $i = 1, 3$,
6. $m_4 > m_2 = m_3$ *with* $G_i = SL(m_i)$ *for* $i = 2, 3$,
7. $m_4 = m_1 + m_2$ *with* $G_i = SL(m_i)$ *for* $i = 1, 2$ *and* $G_4 = SL(m_4)$,
8. $m_4 = m_1 + m_3$ *with* $G_i = SL(m_i)$ *for* $i = 1, 3$ *and* $G_4 = SL(m_4)$,
9. $m_4 = m_2 + m_3$ *with* $G_i = SL(m_i)$ *for* $i = 2, 3$ *and* $G_4 = SL(m_4)$,
10. $m_4 \geq m_1 = m_2 + m_3$ *with* $G_i = SL(m_i)$ *for* $i = 1, 2, 3$,
11. $m_4 \geq m_2 = m_1 + m_3$ *with* $G_i = SL(m_i)$ *for* $i = 1, 2, 3$,
12. $m_4 \geq m_3 = m_1 + m_2$ *with* $G_i = SL(m_i)$ *for* $i = 1, 2, 3$,
13. $m_4 = 2m_1$ *with* $m_1 \leq \min\{m_2, m_3\}$, $G_1 = SL(m_1)$ *and* $G_4 = SL(m_4)$,
14. $m_4 = 2m_2$ *with* $m_2 \leq \min\{m_1, m_3\}$, $G_2 = SL(m_2)$ *and* $G_4 = SL(m_4)$,
15. $m_4 = 2m_3$ *with* $m_3 \leq \min\{m_1, m_2\}$, $G_3 = SL(m_3)$ *and* $G_4 = SL(m_4)$,
16. $m_4 + m_1 = m_2 + m_3$ *with* $m_1 < \min\{m_2, m_3\}$ *and* $G_i = SL(m_i)$ *for* $i = 1, 2, 3, 4$,
17. $m_4 + m_2 = m_1 + m_3$ *with* $m_2 < \min\{m_1, m_3\}$ *and* $G_i = SL(m_i)$ *for* $i = 1, 2, 3, 4$,
18. $m_4 + m_3 = m_1 + m_2$ *with* $m_3 < \min\{m_1, m_2\}$ *and* $G_i = SL(m_i)$ *for* $i = 1, 2, 3, 4$,
19. $m_4 = m_1 + m_2 + m_3$ *with* $G_i = SL(m_i)$ *for* $i = 1, 2, 3, 4$,
20. $2m_4 = m_1 + m_2 + m_3$ *with* $m_4 > \max\{m_1, m_2, m_3\}$ *and* $G_i = SL(m_i)$ *for* $i = 1, 2, 3, 4$.

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