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Czechoslovak Mathematical Journal, Vol. 62 (2012), No. 2, 517–525

Persistent URL: http://dml.cz/dmlcz/142843

Czechoslovak Mathematical Journal, 62 (137) (2012), 517–525

A CHARACTERIZATION OF FUCHSIAN GROUPS ACTING ON COMPLEX HYPERBOLIC SPACES

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(Received February 25, 2011)

Abstract. Let $G \subset \mathbf{SU}(2,1)$ be a non-elementary complex hyperbolic Kleinian group. If G preserves a complex line, then G is $\mathbb{C}$-Fuchsian; if G preserves a Lagrangian plane, then G is $\mathbb{R}$-Fuchsian; G is Fuchsian if G is either $\mathbb{C}$-Fuchsian or $\mathbb{R}$-Fuchsian. In this paper, we prove that if the traces of all elements in G are real, then G is Fuchsian. This is an analogous result of Theorem V.G. 18 of B. Maskit, Kleinian Groups, Springer-Verlag, Berlin, 1988, in the setting of complex hyperbolic isometric groups. As an application of our main result, we show that G is conjugate to a subgroup of $\mathbf{S}(U(1) \times U(1,1))$ or $\mathbf{SO}(2,1)$ if each loxodromic element in G is hyperbolic. Moreover, we show that the converse of our main result does not hold by giving a $\mathbb{C}$-Fuchsian group.

Keywords: $\mathbb{R}$-Fuchsian group, $\mathbb{C}$-Fuchsian group, complex line, $\mathbb{R}$-plane, trace

MSC 2010: 30F40, 20H10

1. Introduction

It is known that a Kleinian group G is Fuchsian if there exists a G-invariant disc $\mathbb{D}$ in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \infty$. If we regard $\mathbb{D}$ as $\mathbb{H}^2$, then G is a subgroup of $\mathbf{SL}(2,\mathbb{R})$. The following result due to Maskit is from Theorem V.G. 18 of [5].

Theorem A. *Let* $G \subset \mathbf{SL}(2,\mathbb{C})$ *be a non-elementary Kleinian group in which* $\operatorname{tr}^2(f) \geqslant 0$ *for all* $f \in G$. *Then* G *is Fuchsian.*

The research was partly supported by NSFs of China (No. 11071063) and Shaoxing College of Arts and Sciences (No. 20125009), NSF of Hunan (No. 10JJ4005), Hunan Provincial Education Department (No. 11B019) and the con- struct program of the key discipline in Hunan province.

This result shows that if the traces of all elements in G are real then G preserves a hyperbolic plane which is totally geodesic in $\mathbb{H}^3$. In this note, we will prove a similar result in the setting of complex hyperbolic Kleinian groups of $\mathbf{SU}(2,1)$. Our result is as follows, whose proof will be given in Section 3.

Theorem 1.1. *Let $G \subset \mathbf{SU}(2,1)$ be a non-elementary complex hyperbolic Kleinian group in which $\mathrm{tr}(f) \in \mathbb{R}$ for all $f \in G$. Then G is Fuchsian.*

Note that a loxodromic element in $\mathbf{SU}(2,1)$ is hyperbolic if and only if its trace is real. The proof of Theorem 1.1 easily yields

Corollary 1.2. *Let $G \subset \mathbf{SU}(2,1)$ be a non-elementary group. If each loxodromic element in G is hyperbolic, then G is conjugate to a subgroup of $\mathbf{S}(U(1) \times U(1,1))$ or $\mathbf{SO}(2,1)$.*

As an application of Theorem 1.1, in Section 4, two Fuchsian groups are constructed: one is $\mathbb{C}$-Fuchsian and the other is $\mathbb{R}$-Fuchsian. We also give a $\mathbb{C}$-Fuchsian group which shows that the converse of Theorem 1.1 is not true.

2. Complex hyperbolic geometry

2.1. Complex hyperbolic space. Let $\mathbb{C}^{2,1}$ be the complex vector space of dimension 3 equipped with a non-degenerate, indefinite Hermitian form $\langle .,. \rangle$ of signature $(2,1)$ defined to be

$$\langle z, w \rangle = w^* J z = z_1 \overline{w}_3 + z_2 \overline{w}_2 + z_3 \overline{w}_1$$

with the matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We consider the subspaces

$$V_- = \{\mathbf{z} \in \mathbb{C}^{2,1} \colon \langle \mathbf{z}, \mathbf{z} \rangle < 0\},$$
$$V_0 = \{\mathbf{z} \in \mathbb{C}^{2,1} - \{0\} \colon \langle \mathbf{z}, \mathbf{z} \rangle = 0\}$$

and the canonical projection

$$\mathbb{P} \colon \mathbb{C}^{2,1} - \{0\} \to \mathbb{C}P^2$$

onto the complex projective space. The complex hyperbolic space $\mathbf{H}^{\mathbf{2}}_{\mathbb{C}}$ is defined to be $\mathbb{P}(V_-)$ and its boundary $\partial\mathbf{H}^{\mathbf{2}}_{\mathbb{C}}$ is $\mathbb{P}(V_0)$. That is,

$$\mathbf{H}^{\mathbf{2}}_{\mathbb{C}} = \{(z_1, z_2) \in \mathbb{C}^2 \colon\ 2\Re(z_1) + |z_2|^2 < 0\}$$

and

$$\partial\mathbf{H}^{\mathbf{2}}_{\mathbb{C}} - \{\infty\} = \{(z_1, z_2) \in \mathbb{C}^2 \colon\ 2\Re(z_1) + |z_2|^2 = 0\}.$$

Given a point $z \in \mathbb{C}^2 \subset \mathbb{C}P^2$, we can lift $z = (z_1, z_2)$ to a point $\mathbf{z}$ in $\mathbb{C}^{2,1}$, called the standard lift of z, where

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix}.$$

There are two distinguished points in V_0 which are denoted by $\mathbf{0}$ and ∞, respectively. They are

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } \infty = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

2.2. Isometries. Denote by $\mathbf{U}(2,1)$ the group of unitary matrices for the Hermitian product $\langle .,. \rangle$. Each such matrix A satisfies the relation $A^{-1} = JA^*J$, where A^* is the Hermitian transpose of A. The full group of holomorphic isometries of $\mathbf{H}^{\mathbf{2}}_{\mathbb{C}}$ is the projective unitary group $\mathbf{PU}(2,1) = \mathbf{U}(2,1)/\mathbf{U}(1)$, where $\mathbf{U}(1) = \{e^{i\theta} I \colon\ \theta \in [0, 2\pi)\}$ and I is the 3×3 identity matrix. In this paper, we shall consider the group $\mathbf{SU}(2,1)$ of matrices which are unitary with respect to $\langle .,. \rangle$ and have determinant 1. Following [3], holomorphic isometries of $\mathbf{H}^{\mathbf{2}}_{\mathbb{C}}$ are classified as follows.

(1) An isometry is *elliptic* if it fixes at least one point of $\mathbf{H}^{\mathbf{2}}_{\mathbb{C}}$;
(2) an isometry is *parabolic* if it fixes exactly one point of $\partial\mathbf{H}^{\mathbf{2}}_{\mathbb{C}}$;
(3) an isometry is *loxodromic* if it fixes exactly two points of $\partial\mathbf{H}^{\mathbf{2}}_{\mathbb{C}}$.

See [1], [3], [4], [7] for more details about complex hyperbolic geometry and complex hyperbolic Kleinian groups.

2.3. Totally geodesic manifolds and Fuchsian groups. Unlike the real hyperbolic space, there are two kinds of totally geodesic manifolds with codimension 2 in $\mathbf{H}^{\mathbf{2}}_{\mathbb{C}}$. In the first place there are *complex lines* which have constant curvature -1. Every complex line L is the image of the complex line

$$L_0 = \{(z_1, z_2) \in \mathbf{H}^{\mathbf{2}}_{\mathbb{C}} \colon\ z_2 = 0\}$$

under some element of $\mathbf{SU}(2,1)$. The subgroup of $\mathbf{SU}(2,1)$ stabilizing L is thus conjugate to the subgroup $\mathbf{S}(U(1) \times U(1,1)) \subset \mathbf{SU}(2,1)$. Secondly, we have totally real *Lagrangian planes* which have constant curvature $-\frac{1}{4}$. Every Lagrangian plane is the image of the standard real Lagrangian plane

$$R_{\mathbb{R}} = \{(z_1, z_2) \in \mathbf{H}^{\mathbf{2}}_{\mathbb{C}} \colon \ z_i = x_i \in \mathbb{R}, 2x_1 + x_2^2 < 0\}$$

under some element of $\mathbf{SU}(2,1)$. The group stabilizing $R_{\mathbb{R}}$ is denoted by $\mathbf{SO}(2,1)$, which is the subgroup of $\mathbf{SU}(2,1)$ comprising elements with real entries. We say a group G is *non-elementary* if there are two loxodromic elements in G with distinct fixed points. Following [2], for any non-elementary complex hyperbolic Kleinian group $G \subset \mathbf{SU}(2,1)$,

(1) G is called $\mathbb{C}$-*Fuchsian* if it preserves a complex line;
(2) G is called $\mathbb{R}$-*Fuchsian* if it preserves a Lagrangian plane;
(3) otherwise, G is called *non-Fuchsian.*

We call a non-elementary Kleinian group G *Fuchsian* if G is either $\mathbb{C}$-Fuchsian or $\mathbb{R}$-Fuchsian.

2.4. Cartan's angular invariant and the cross-ratio variety. Let z_1, z_2, z_3 be three distinct points in $\partial\mathbf{H}^{\mathbf{2}}_{\mathbb{C}}$ with lifts $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{z}_3$, respectively. Cartan's angular invariant $\mathbb{A}$ is defined to be

$$\mathbb{A}(z_1, z_2, z_3) = \arg(-\langle \mathbf{z}_1, \mathbf{z}_2\rangle \langle \mathbf{z}_2, \mathbf{z}_3\rangle \langle \mathbf{z}_3, \mathbf{z}_1\rangle).$$

It is known that $\mathbb{A}$ is invariant under the elements of $\mathbf{SU}(2,1)$. The following is a useful property of $\mathbb{A}$ which was proved by Goldman, see Section 7.1 of [3].

Theorem B. *Let z_1, z_2, z_3 be three distinct points of $\partial\mathbf{H}^{\mathbf{2}}_{\mathbb{C}}$ and let $\mathbb{A} = \mathbb{A}(z_1, z_2, z_3)$ denote their angular invariant. Then*

(1) $\mathbb{A} \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$;
(2) $\mathbb{A} = \pm\frac{1}{2}\pi$ *if and only if* z_1, z_2, z_3 *all lie on a chain;*
(3) $\mathbb{A} = 0$ *if and only if* z_1, z_2, z_3 *all lie on an* $\mathbb{R}$*-circle.*

Here we call the boundary of a complex line a *chain* and the boundary of a Lagrangian plane an $\mathbb{R}$-*circle.*

Proposition 2.1. *Let $G \subset \mathbf{SU}(2,1)$ be a non-elementary complex hyperbolic Kleinian group. Then G is $\mathbb{C}$-Fuchsian ($\mathbb{R}$-Fuchsian) if and only if the fixed points of all loxodromic elements in G are contained in a chain (an $\mathbb{R}$-circle).*

P r o o f. First, it is obvious that if G is $\mathbb{C}$-Fuchsian ($\mathbb{R}$-Fuchsian) then any loxodromic element U in G must preserve the invariant complex line (the Lagrangian plane) and so its fixed points must be on the boundary chain (the $\mathbb{R}$-circle). Conversely, suppose G is non-elementary and contains loxodromic elements U and V with distinct fixed points. Suppose the fixed points of all loxodromic elements of G lie on a chain (an $\mathbb{R}$-circle). In particular, there is a unique complex line L (a unique Lagrangian plane R) such that the fixed points of U and V lie in ∂L (∂R). Let A be any element of G. Then the fixed points of AUA^{-1} and AVA^{-1} lie on the boundary of the complex line $A(L)$ (the Lagrangian plane $A(R)$). By hypothesis, they also lie on the boundary of L (R). Since four distinct points lie on at most one chain ($\mathbb{R}$-circle), we see that A sends L (R) to itself (as a set). This is true for all elements of G, and so G is $\mathbb{C}$-Fuchsian ($\mathbb{R}$-Fuchsian). □

Let z_1, z_2, z_3, z_4 be four distinct points of $\partial\mathbf{H}^2_{\mathbb{C}}$ and $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4$ their corresponding lifts in $V_0 \subset \mathbb{C}^{2,1}$, respectively. Then their *complex cross ratio* is defined to be

$$\mathbb{X} = [z_1, z_2, z_3, z_4] = \frac{\langle \mathbf{z}_3, \mathbf{z}_1\rangle\langle \mathbf{z}_4, \mathbf{z}_2\rangle}{\langle \mathbf{z}_4, \mathbf{z}_1\rangle\langle \mathbf{z}_3, \mathbf{z}_2\rangle}.$$

It is easy for us to know that $\mathbb{X}$ is neither 0 nor ∞. By changing the order of the four points we can define the following three different cross-ratios:

$$\mathbb{X}_1 = [z_1, z_2, z_3, z_4],\ \mathbb{X}_2 = [z_1, z_3, z_2, z_4] \text{ and } \mathbb{X}_3 = [z_2, z_3, z_1, z_4].$$

The following lemma which is crucial for us follows from Propositions 5.12, 5.13 and 5.14 of [6].

Lemma 2.2. *Let z_1, z_2, z_3, z_4 be four distinct points of $\partial\mathbf{H}^2_{\mathbb{C}}$. Then all z_i ($i = 1, 2, 3, 4$) lie on a chain or an $\mathbb{R}$-circle if and only if all $\mathbb{X}_j$ ($j = 1, 2, 3$) are real.*

P r o o f. It follows from

$$\mathbb{X}_1 = [z_1, z_2, z_3, z_4] = \frac{\langle \mathbf{z}_3, \mathbf{z}_1\rangle\langle \mathbf{z}_4, \mathbf{z}_2\rangle}{\langle \mathbf{z}_4, \mathbf{z}_1\rangle\langle \mathbf{z}_3, \mathbf{z}_2\rangle} = \frac{\langle \mathbf{z}_1, \mathbf{z}_2\rangle\langle \mathbf{z}_2, \mathbf{z}_3\rangle\langle \mathbf{z}_3, \mathbf{z}_1\rangle|\langle \mathbf{z}_2, \mathbf{z}_4\rangle|^2}{\langle \mathbf{z}_1, \mathbf{z}_2\rangle\langle \mathbf{z}_2, \mathbf{z}_4\rangle\langle \mathbf{z}_4, \mathbf{z}_1\rangle|\langle \mathbf{z}_2, \mathbf{z}_3\rangle|^2}$$

that

$$\begin{aligned}\arg(\mathbb{X}_1) &= \arg(-\langle \mathbf{z}_1, \mathbf{z}_2\rangle\langle \mathbf{z}_2, \mathbf{z}_3\rangle\langle \mathbf{z}_3, \mathbf{z}_1\rangle) - \arg(-\langle \mathbf{z}_1, \mathbf{z}_2\rangle\langle \mathbf{z}_2, \mathbf{z}_4\rangle\langle \mathbf{z}_4, \mathbf{z}_1\rangle)\\ &= \mathbb{A}(z_1, z_2, z_3) - \mathbb{A}(z_1, z_2, z_4).\end{aligned}$$

Since all z_i ($i = 1, 2, 3, 4$) lie on a chain or an $\mathbb{R}$-circle, by Theorem B we know that $\mathbb{X}_1$ is real. Similar discussions yield that $\mathbb{X}_2$ and $\mathbb{X}_3$ are real.

Now we prove the sufficiency. It suffices to consider the case that all $\mathbb{X}_j$ ($j = 1, 2, 3$) are positive since if one of $\mathbb{X}_j$ is negative, then by [6, Proposition 5.1] we know that all z_i lie on a chain. It follows that

$$\mathbb{A}(z_1, z_2, z_4) = \mathbb{A}(z_1, z_2, z_3),\ \mathbb{A}(z_1, z_3, z_2) = \mathbb{A}(z_1, z_3, z_4)$$

and

$$\mathbb{A}(z_2, z_3, z_4) = \mathbb{A}(z_2, z_3, z_1).$$

According to the definition of Cartan's angular invariant, we have

$$\mathbb{A}(z_1, z_2, z_3) = -\mathbb{A}(z_1, z_3, z_2).$$

By [3, Lemma 7.1.10] and Theorem B, it is easy for us to prove that all z_i lie on an $\mathbb{R}$-circle. □

3. The proof of theorem 1.1

We prove this result by contradiction. Suppose that G is non-Fuchsian. Since G is non-elementary, by Proposition 2.1 we can find two loxodromic elements U, $V \in G$ such that A_u, A_v, R_u and R_v lie neither on a chain nor an $\mathbb{R}$-circle and

$$\{A_u, R_u\} \cap \{A_v, R_v\} = \emptyset,$$

where A_w, R_w denote the attracting and repelling fixed points of the loxodromic element $W \in G$, respectively. Without loss of generality, we may assume that

$$U = \begin{pmatrix} r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/r \end{pmatrix}$$

and

$$V = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} \begin{pmatrix} s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/s \end{pmatrix} \begin{pmatrix} \bar{j} & \bar{f} & \bar{c} \\ \bar{h} & \bar{e} & \bar{b} \\ \bar{g} & \bar{d} & \bar{a} \end{pmatrix},$$

where $\left(\begin{smallmatrix} a & b & c \\ d & e & f \\ g & h & j \end{smallmatrix}\right) \in \mathbf{SU}(2,1)$, $ajgc \neq 0$, r, $s > 1$ and $r \neq s$ (if $r = s$, we can use V^2 instead of V). Applying Lemma 2.2, we know that at least one of $\mathbb{X}_j$ $(j = 1, 2, 3)$ is not real, where

$$\mathbb{X}_1 = [A_v, A_u, R_u, R_v],\ \mathbb{X}_2 = [A_v, R_u, A_u, R_v] \text{ and } \mathbb{X}_3 = [A_u, R_u, A_v, R_v].$$

By [6, Proposition 6.4], we have

$$\begin{aligned} \mathrm{tr}(UV) = {} & r + s + r^{-1} + s^{-1} + \mathbb{X}_1(r^{-1} - 1)(s^{-1} - 1) + \overline{\mathbb{X}}_1(r - 1)(s - 1) \\ & + \mathbb{X}_2(r - 1)(s^{-1} - 1) + \overline{\mathbb{X}}_2(r^{-1} - 1)(s - 1) - 1 \end{aligned}$$

and

$$\begin{aligned}\operatorname{tr}[U,V] = 3 &- \Re[(\mathbb{X}_1+\mathbb{X}_2)(r-1)(r^{-1}-1)(s-1)(s^{-1}-1)] \\ &+ [1-2\Re(\mathbb{X}_1+\mathbb{X}_2)][(r-1)^2(s-1)^2+(r^{-1}-1)^2(s^{-1}-1)^2] \\ &+ |\mathbb{X}_1(r-1)(s-1)+\overline{\mathbb{X}}_1(r^{-1}-1)(s^{-1}-1) \\ &+ \mathbb{X}_2(r^{-1}-1)(s-1)+\overline{\mathbb{X}}_2(r-1)(s^{-1}-1)|^2 \\ &+ (|\mathbb{X}_2|^2-|\mathbb{X}_1|^2\mathbb{X}_3)(r^2-2r+2r^{-1}-r^{-2})(s^2-2s+2s^{-1}-s^{-2}).\end{aligned}$$

Now, we divide our proof into four cases.

Case I. $\mathbb{X}_3$ is not real.

By computation, we have

$$\Im(\operatorname{tr}[U,V]) = |\mathbb{X}_1|^2(r-r^{-1})(r+r^{-1}-2)(s-s^{-1})(s+s^{-1}-2)\Im(\mathbb{X}_3),$$

which implies that $\operatorname{tr}[U,V]$ is not real.

Case II. $\mathbb{X}_1$ is real and $\mathbb{X}_2$ is not real.

In this case,

$$\Im(\operatorname{tr}(UV)) = (r^{-1}-s^{-1})(r-1)(s-1)\Im(\mathbb{X}_2).$$

Since $r, s > 1$ and $r \neq s$, $\Im(\operatorname{tr}(UV)) \neq 0$. Therefore $\operatorname{tr}(UV)$ is not real.

Case III. $\mathbb{X}_2$ is real and $\mathbb{X}_1$ is not real.

Then

$$\Im(\operatorname{tr}(UV)) = (r^{-1}s^{-1}-1)(r-1)(s-1)\Im(\mathbb{X}_1).$$

It follows that $\operatorname{tr}(UV)$ is not real.

Case IV. Neither $\mathbb{X}_1$ nor $\mathbb{X}_2$ are real.

If $\Im[\overline{\mathbb{X}}_1(r-1)+\overline{\mathbb{X}}_2(r^{-1}-1)] = 0$, then $\Im(\mathbb{X}_2) = r\Im(\mathbb{X}_1)$. So

$$\Im(\operatorname{tr}(UV)) = (r-1)(s-1)r^{-1}s^{-1}(1-r^2)\Im(\mathbb{X}_1) \neq 0.$$

Hence $\operatorname{tr}(UV)$ is not real.

If $\Im[\overline{\mathbb{X}}_1(r-1)+\overline{\mathbb{X}}_2(r^{-1}-1)] \neq 0$, according to the definition of the cross-ratio variety, we know that $\mathbb{X}_j$ $(j=1,2,3)$ is independent of the value of s and r. Then there must exist a sufficiently large integer m such that

$$\begin{aligned}&\Im[\mathbb{X}_1(r^{-1}-1)(s^{-m}-1)+\mathbb{X}_2(r-1)(s^{-m}-1)] \\ &\qquad + \Im[\overline{\mathbb{X}}_1(r-1)(s^m-1)+\overline{\mathbb{X}}_2(r^{-1}-1)(s^m-1)] \neq 0.\end{aligned}$$

This implies that $\operatorname{tr}(UV^m)$ is not real. □

4. Three examples

Example 4.1. Let

$$G_1 = \left\langle A = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ i & 0 & 1 \end{pmatrix} \right\rangle.$$

Then G_1 is $\mathbb{C}$-Fuchsian and each element in G_1 has real trace.

P r o o f. It is obvious that G_1 is a $\mathbb{C}$-Fuchsian group which keeps the complex line $L_0 = \{(z_1, z_2) \in \mathbf{H}^{\mathbf{2}}_{\mathbb{C}}\colon z_2 = 0\}$ invariant. We only need to show that every element in G_1 has real trace. Let M be an element having the following form

$$M = \begin{pmatrix} a & 0 & ib \\ 0 & 1 & 0 \\ ic & 0 & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{Z}$. Since the generators of G_1 and their inverses have this form it is clear that this form is preserved under matrix multiplication. This implies that each element in G_1 has real trace. □

Example 4.2. Let

$$G_2 = \mathbf{SO}(2, 1; \mathbb{Z}).$$

Then G_2 is $\mathbb{R}$-Fuchsian and each element in G_2 has real trace.

It is known that the converse to Maskit's theorem is clearly true (the trace of every element in a Fuchsian subgroup of $\mathbf{SL}(2, \mathbb{C})$ is real), the converse to Theorem 1.1 is true for $\mathbb{R}$-Fuchsian groups, but false for $\mathbb{C}$-Fuchsian groups. The following is a $\mathbb{C}$-Fuchsian group but does not comprise only matrices with real trace.

Example 4.3.

$$G_3 = \left\langle A = \begin{pmatrix} -i & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -i \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ i & 0 & 1 \end{pmatrix} \right\rangle.$$

Acknowledgement. Most of this work was completed while the first author was a visiting research student at the University of Durham. He is grateful to Professor John R. Parker for his valuable help. All of the authors heartily thank the referee for careful reading of this paper as well as for many useful comments and suggestions. In particular, the proof of Proposition 2.1 was given by him or her.

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