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Ladislav Nebeský

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A CHARACTERIZATION OF GEODETIC GRAPHS

LADISLAV NEBESKÝ, Praha

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By a graph we mean a finite undirected graph with no multiple edge or loop (i.e. a graph in the sense of the book [1], for example).

Let G be a connected graph with a vertex set $V(G)$ and an edge set $E(G)$. A sequence $\langle u_0, \dots, u_m \rangle$ is called a u_0-u_m path (of length m) in G if $m \geqslant 0$, $u_0, \dots, u_m$ are mutually distinct vertices of G and $u_i u_{i+1} \in E(G)$ for each integer i, $0 \leqslant i < m$. We denote by $\mathscr{P}_G$ the set of all paths in G and by $\mathscr{P}_G(r,s)$ the set of all r-s paths in G for any $r, s \in V(G)$. The distance function d_G of G is defined as follows:

$$d_G(t,u) = \min(j; \quad \text{there exists a } t\text{-}u \text{ path of length } j \text{ in } G),$$

for any $t, u \in V(G)$. Next, we denote by $\mathscr{D}_G(v,w)$ the set of all v-w paths of length $d_G(v,w)$ for any $v, w \in V(G)$. Finally, denote

$$\mathscr{D}_G = \bigcup_{x,y \in V(G)} \mathscr{D}_G(x,y).$$

A connected graph G is called *geodetic* if $|\mathscr{D}_G(u,v)| = 1$ for every ordered pair of vertices u and v of G. The following theorem gives a characterization of geodetic graphs:

Theorem. *A connected graph G is geodetic if and only if there exists $\mathscr{A} \subseteq \mathscr{P}_G$ such that $\mathscr{A}$ fulfils the following Axioms* I–V *(for arbitrary u, v, $u_0, \dots, u_m$, $v_0, \dots, v_n \in V(G)$, where $m \geqslant 2$ and $n \geqslant 1$):*

I $|\mathscr{A} \cap \mathscr{P}_G(u,v)| = 1$;

II *if* $uv \in E(G)$, *then* $\langle u, v \rangle \in \mathscr{A}$;

III *if* $\langle u_0, \dots, u_m \rangle \in \mathscr{A}$, *then* $\langle u_m, \dots, u_0 \rangle \in \mathscr{A}$;

IV *if* $\langle u_0, \dots, u_m \rangle \in \mathscr{A}$ *and* $m \geqslant 3$, *then* $\langle u_0, \dots, u_{m-1} \rangle \in \mathscr{A}$;

V *if* $\langle u_0,\dots,u_m\rangle$, $\langle v_0,\dots,v_n\rangle$, $\langle u_m,v_n\rangle \in \mathscr{A}$, $u_0 = v_0$ *and* $u_1 \neq v_1$, *then* $\langle u_1,\dots,u_m,v_n\rangle \in \mathscr{A}$.

This theorem (more exactly: a theorem very similar to it) was proved by the present author in [2] but its proof was rather complicated and long: the theorem was derived from another (much more general) theorem proved there.

In the present note a simple proof of our theorem will be given. We will obtain the theorem as a consequence of the following lemma:

Lemma. *Let G be a connected graph, and let $\mathscr{A} \subseteq \mathscr{P}_G$. If $\mathscr{A}$ fulfils Axioms* I–V, *then $\mathscr{A} = \mathscr{D}_G$.*

P r o o f. Let $\mathscr{A}$ fulfil Axioms I–V. Consider arbitrary $r,s \in V(G)$. According to Axiom I, $|\mathscr{A} \cap \mathscr{P}_G(r,s)| = 1$. Let $\alpha(r,s)$ denote the only element of $\mathscr{A} \cap \mathscr{P}_G(r,s)$.

Consider arbitrary $u,v \in V(G)$. Obviously, $\mathscr{D}_G(u,v) \neq \emptyset$. We want to prove that

$$\beta = \alpha(u,v) \qquad \text{for each } \beta \in \mathscr{D}_G(u,v). \tag{1}$$

We proceed by induction on $d_G(u,v)$. If $d_G(u,v) = 0$, then (1) follows from the fact that $|\mathscr{P}_G(u,v)| = 1$. If $d_G(u,v) = 1$, then (1) follows from Axiom II. Let $d_G(u,v) = n \geqslant 2$. Suppose the assertion is true for all pairs of vertices whose distance is less than n. Consider an arbitrary $\beta \in \mathcal{D}_G(u,v)$. There exist $x_0,\dots,x_m$, $y_0,\dots,y_n \in V(G)$ such that $m \geqslant n$, $x_0 = u = y_n$, $x_m = v = y_0$,

$$\alpha(u,v) = \langle x_0,\dots,x_m\rangle \quad \text{and} \quad \beta = \langle y_n,\dots,y_0\rangle.$$

First, we will prove that

$$\{x_1,\dots,x_{m-1}\} \cap \{y_1,\dots,y_{n-1}\} \neq \emptyset. \tag{2}$$

Suppose, to the contrary,

$$\{x_1,\dots,x_{m-1}\} \cap \{y_1,\dots,y_{n-1}\} = \emptyset. \tag{$\overline{2}$}$$

Denote

$$\alpha_i = \langle x_i,\dots,x_m = y_0,\dots,y_i\rangle \quad \text{and}$$
$$\beta_i = \langle x_i,\dots,x_0 = y_n,\dots,y_i\rangle$$

for each $i \in \{0,\dots,n\}$. Thus $\alpha_0 = \alpha(u,v)$ and $\beta_0 = \beta$. Recall that $\alpha(u,v),\beta \in \mathscr{P}_G(u,v)$. It follows from $(\overline{2})$ that $\alpha_i,\beta_i \in \mathscr{P}_G$ for each $i \in \{0,\dots,n\}$. If $\alpha_n \in \mathscr{A}$, then combining Axioms III and IV we get $\beta = \alpha(u, v)$, which is a contradiction with $(\overline{2})$. Thus $\alpha_n \notin \mathscr{A}$. Since $\alpha_0 \in \mathscr{A}$, there exists $k \in \{0, \ldots, n-1\}$ such that $\alpha_k \in \mathscr{A}$ but $\alpha_{k+1} \notin \mathscr{A}$. Hence $\alpha_k = \alpha(x_k, y_k)$. Since $\beta_k \in \mathscr{P}_G(x_k, y_k)$, we have $d_G(x_k, y_k) \leqslant n$. If $d_G(x_k, y_k) < n$, then the induction hypothesis implies that $d_G(x_k, y_k) = m$, and thus $m < n$, which is a contradiction. Therefore, $d_G(x_k, y_k) = n$. We get $\beta_k \in \mathscr{D}_G(x_k, y_k)$. This implies that

$$\langle x_k, \ldots, x_0 = y_n, \ldots, y_{k+1} \rangle \in \mathscr{D}_G \quad \text{and} \quad d_G(x_k, y_{k+1}) = n - 1.$$

By the induction hypothesis,

$$\langle x_k, \ldots, x_0 = y_n, \ldots, y_{k+1} \rangle \in \mathscr{A}.$$

Recall that $\alpha_k = \langle x_k, \ldots, x_m = y_0, \ldots, y_k \rangle$, $\langle y_k, y_{k+1} \rangle \in \mathscr{A}$, $x_1 \neq y_{n-1}$ and if $k \geqslant 1$, then $x_{k+1} \neq x_{k-1}$. As follows from Axiom V, $\alpha_{k+1} \in \mathscr{A}$, which is a contradiction. Thus (2) holds.

It follows from (2) that there exist integers g and h, $1 \leqslant g \leqslant m-1$ and $1 \leqslant h \leqslant n-1$, such that $x_g = y_h$. Put $w = x_g = y_h$. Since $\beta \in \mathscr{D}_G$, we get $d_G(u, w) = n - h < n$ and $d_G(w, v) = h < n$. By the induction hypothesis,

$$\langle y_n, \ldots, y_h \rangle = \alpha(u, w) \quad \text{and} \quad \langle y_h, \ldots, y_0 \rangle = \alpha(w, v).$$

Recall that $\alpha(u, v) = \langle x_0, \ldots, x_m \rangle$. Combining Axioms III and IV we get

$$\alpha(u, w) = \langle x_0, \ldots, x_g \rangle \quad \text{and} \quad \alpha(w, v) = \langle x_g, \ldots, x_m \rangle.$$

Hence $\beta = \alpha(u, v)$. We see that (1) holds, which completes the proof of the lemma. □

Proof of the Theorem. Let G be geodetic. Put $\mathscr{A} = \mathscr{D}_G$. It is easy to see that $\mathscr{A}$ fulfils Axioms I–V.

Conversely, suppose there exists $\mathscr{A} \subseteq \mathscr{P}_G$ such that $\mathscr{A}$ fulfils Axioms I–V. According to the lemma, $\mathscr{A} = \mathscr{D}_G$. Axiom I implies that G is geodetic, which completes the proof. □

References

[1] *M. Behzad, G. Chartrand and L. Lesniak-Foster*: Graphs & Digraphs. Prindle, Weber & Schmidt, Boston, 1979.

[2] *L. Nebeský*: A characterization of the set of all shortest paths in a connected graph. Mathematica Bohemica *119* (1994), 15–20.

Author's address: Filozofická fakulta Univerzity Karlovy, nám. J. Palacha 2, 116 38 Praha 1, Czech Republic.