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A CHARACTERIZATION OF GRAPHS IN WHICH SOME MINIMUM DOMINATING SET COVERS ALL THE EDGES

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1. INTRODUCTION

If x is a vertex in a connected simple, undirected graph $G = (V, E)$ and k is a positive integer then the open k-neighborhood of x is the set $N_k(x) = \{y \in V \mid d(x, y) = k\}$. Here $d(x, y)$ denotes the length of a shortest path in G joining x and y. When $k = 1$ we have the usual open neighborhood $N(x) = \{y \in V \mid xy \in E\}$. If S is a subset of vertices in G, then $N(S)$ will denote $\bigcup\limits_{x \in S} N(x)$ and $N[S] = N(S) \cup S$. We will write $N[x]$ in place of the more cumbersome $N[x]$ for the closed neighborhood of x. $|A|$ will represent the cardinality of the set A, and so G is a graph of order $|V|$ and size $|E|$.

A set D of vertices is called a *dominating set* of G if $N[D] = V$. If, in addition, G has no dominating set of cardinality less than $|D|$, then the *domination number* of G is $\gamma(G) = |D|$ and D will be referred to as a γ-set of G. When every edge in G is incident with some vertex in $A \subseteq V$, then A is called a *vertex cover* of G. The *vertex cover number* of G, denoted by $\alpha(G)$, is the minimum cardinality among all vertex covers of G. Any vertex cover of this order will be called an α-set of G.

If $A \subseteq V$ and the induced subgraph $\langle A \rangle$ has no edges, then A is an independent set. The *vertex independence number* of G is the cardinality $\beta(G)$ of a largest independent set (a β-set) in G. Gallai established the following relationship between the vertex cover number and the independence number.

Theorem 1.1 [1]. *For every graph G of order n, $\alpha(G) + \beta(G) = n$.*

It is clear that $\gamma(G) \leqslant \alpha(G)$ if G has no isolated vertices, but in an arbitrary graph G neither of these parameters is a particularly good bound for the other as can be seen by considering the n-cycle for large n. In a survey paper in 1981 Laskar and

Walikar [2] posed the problem of finding a characterization of the class of graphs G for which $\gamma(G) = \alpha(G)$. (Note that in [2] a vertex cover is called a transversal of G.)

Our goal here is to give a structural characterization of this class of graphs. For ease of reference in this paper we denote by C the class of all graphs G having $\alpha(G) = \gamma(G)$. One can easily observe that if $G \in C$ is of order n and H is any spanning subgraph of G with no isolated vertices, then $\gamma(G) \leqslant \gamma(H) \leqslant \alpha(H) \leqslant \alpha(G)$ and so H belongs to C as well. In particular, any spanning tree T of G is in C and $\alpha(T) = \alpha(G) = \gamma(G) = \gamma(T)$. Since $\gamma(T)$ can be efficiently computed and since it follows from Theorem 1.1 that $\beta(G) = n - \alpha(G) = n - \alpha(T) = n - \gamma(T)$, it follows that the computation of the domination and independence numbers for members of class C can be done in polynomial time.

Our classification involves the consideration of two types of graphs that can occur in C. In section 2 we will consider graphs which have at least one vertex of degree one and in Section 3 we consider the structure of an arbitrary G in C. We will need the following terminology. A vertex x in G is called a *leaf* if it has degree one, and a vertex y is a *stem* if y is adjacent to a leaf. We let $L = L(G)$ (respectively $S = S(G)$) denote the set of all leaves (respectively stems) of G. Vertex v is called *heavy* if $\deg(v) \geqslant 3$ and $w \in V$ is said to be *neighborhood full* (or just *full*) if every $v \in N[w]$ is heavy. A *basic* 4-cycle in G is one which has some pair of non-adjacent vertices which are not heavy.

2. Graphs with leaves

In this section we will show how one can begin with a graph $G \in C$ for which $L(G) \neq \varphi$ and reduce to a subgraph H of G with $H \in C$ and $L(H) = \varphi$. We begin with a lemma which is used throughout the remainder of the paper. Note that an α-set of such a G is also a γ-set of G.

Lemma 2.1. *If $G \in C$ and A is any α-set of G which contains adjacent vertices x and y, then x and y are both stems of G.*

Proof. Assume $xy \in E$ and that both x and y belong to some α-set A of $G \in C$. Suppose y is not a stem of G. Clearly y is not a leaf in G or else $A - \{y\}$ is a vertex cover for G. Let $u \in N(y) - \{x\}$. By assumption u is not a leaf of G and so for every edge uv with $v \neq y$, either $u \in A$ or $v \in A$. That is, every vertex of $N[y]$ is dominated by $A - \{y\}$, and so $A - \{y\}$ dominates G. This contradiction establishes the lemma. □

If G has leaves and $d(x, s) = 2$ for a stem s of G where x belongs to $G - N[S]$, then x will be called a *connector* in $G - N[S]$.

Lemma 2.2. *Suppose G belongs to class C and $L(G) \neq \varphi$. Each component of $G - N[S]$ is either an isolated vertex or a member of C. Furthermore, if H_1, H_2, ..., H_m are the nontrivial components of $G - N[S]$, then each H_i is in C and $H = H_1 \cup H_2 \cup \ldots \cup H_m$ has an α-set containing all the connectors of H.*

Proof. Let A be an α-set for G. We may assume that A contains all the stems of G. For any $1 \leqslant i \leqslant m$, let $B_i = A \cap V(H_i)$. Since A is a vertex cover it follows that every edge $xy \in V(H_i)$ is covered by B_i. If H_i has a dominating set D_i with $|D_i| < |B_i|$, then since $S \subseteq A$ it follows that $A' = D_i \cup (A - B_i)$ dominates G. But $|A'| < |A| = \alpha(G) = \gamma(G)$. Hence $|D_i| = |B_i|$ and thus $H_i \in C$. It follows immediately that $B_1 \cup B_2 \cup \ldots \cup B_m$ is an α-set of H and in fact contains all the connectors of H. □

If some nontrivial component H_i of $G - N[S]$ has a leaf x, then it must be the case that x is a connector in $G - N[S]$. Thus it will follow directly from the next lemma that $G - N[S]$ is leafless.

Lemma 2.3. *There does not exist a graph G in C having at least one leaf and also having an α-set which contains all the leaves of G.*

Proof. Assume that G is such a graph in C with an α-set D containing all the leaves of G. First observe that G cannot have two adjacent stems x and y with corresponding leaves v and w (for since v and w belong to D, the edge xy is not covered by D). Hence by Lemma 2.1 G cannot have two adjacent vertices which belong to any α-set. But now consider any leaf v of G adjacent to its stem x. Since $v \in D$ and D is a vertex cover of G it follows that $N(x) \subseteq D$. However, $(D-\{v\}) \cup \{x\}$ is also an α-set of G containing adjacent vertices which is a contradiction. □

Corollary 2.4. *If $\alpha(G) = \gamma(G)$ and $S = S(G) \neq \varphi$, then $G - N[S]$ is leafless.*

Corollary 2.5. *If a tree T belongs to class C and $S = S(T)$, then each component of $T - N[S]$ is an isolated vertex.*

We are now able to give a characterization of the family of trees in C. In addition to the set of leaves, L, and the set of stems, S, of a tree we will need to refer to the following sets of vertices: $P = N(S) - (S \cup L)$ and $R = V(T) - (L \cup S \cup P)$.

Theorem 2.6. *A tree T of order at least two has $\gamma(T) = \alpha(T)$ if and only if*
(i) *no two vertices of P are adjacent;*
(ii) *no two vertices of R are adjacent; and*
(iii) *each vertex of P has at most one neighbor in R.*

Proof. To see that these three conditions are sufficient one can check that when (i), (ii) and (iii) hold, $S \cup R$ is a minimum dominating set which is also a vertex cover.

Now assume $\alpha(T) = \gamma(T)$ for some nontrivial tree T, and let D be an α-set of T. We may assume that $S \subseteq D$. If $vw \in E(T)$ and v and w both belong to P then since D covers the edges of T at least one of v and w, say v, must belong to D. But v is adjacent to a stem $x \in D$ and so by Lemma 2.1 v must also be a stem. This contradiction establishes (i). Condition (ii) follows immediately from Corollary 2.5. To verify (iii) assume that some $w \in P$ is adjacent to two distinct members, say y and z, of R. Since $P \cap D = \varphi$ it must be the case that $y, z \in D$. By Corollary 2.5 y and z are isolated in $T - N[S]$ and so $D' = (D - \{y, z\}) \cup \{w\}$ is a dominating set of T smaller than D. Hence (iii) holds in T. □

3. The leafless case

In this section we consider the class of graphs in C which havé no leaves. According to Corollary 2.4 this collection will include the nontrivial components of $G - N[S]$ for any G in C which has leaves. Our first theorem and corollary here show that every such graph must be bipartite.

Theorem 3.1. *If G is in C and G has an odd cycle, then G must have two adjacent stems.*

Proof. Let $B = (v_1, v_2, \ldots, v_{2k+1})$ be an odd cycle in G. If A is any α-set of G, then to cover the edges of B it must be the case that for some i, $v_i \in A$ and $v_{i+1} \in A$ (subscripts computed modulo $2k + 1$). It follows by Lemma 2.1 that both v_i and v_{i+1} must be stems. □

Corollary 3.2. *If G is a leafless graph in C, then G is bipartite.*

We now proceed to determine the structure of a leafless, connected, bipartite graph G with $\alpha(G) = \gamma(G)$. If X and Y are the two color classes of G it is clear that either X or Y is a vertex cover of G, although it is not obvious that either is a minimum vertex cover. In fact, we have the following characterization.

Lemma 3.3. *Suppose G is a connected bipartite graph with no leaves. One of the color classes of G is a γ-set if and only if G is in the class C.*

Proof. Let X and Y be the color classes of G. As noted above the only if part of the lemma is immediate. Now assume that $G \in C$, and let A be any α-set of G. We assume without loss of generality that $A \cap X \neq \varphi$. Let $x \in A \cap X$. Since G is leafless, $N(x) \cap A = \varphi$. But then $N_2(x) \subseteq A$ since A is a vertex cover. Similarly $N_3(x) \cap A = \varphi$. By repeating this and using the assumption that G is connected we conclude that $X = A$. □

While Lemma 3.3 does give us a characterization of sorts for the collection of connected, bipartite, leafless graphs in C it is a weak characterization in the sense that its use does not appear to make it easier to recognize such graphs. The following sequence of results, culminating in Theorem 3.6, will reveal much more of the structure of such a graph.

Lemma 3.4. *Suppose G is a connected, bipartite, leafless graph and $\alpha(G) = \gamma(G)$. Every edge of G is on a 4-cycle.*

Proof. Let xy be an arbitrary edge of G. By Lemma 3.3 one of the color classes of G is a γ-set of G. We assume that $X = \{x\} \cup \big(N_2(x) \cup N_4(x) \cup \dots\big)$ is a γ-set. Vertex y is not a stem and $\varphi \neq N(y) - \{x\} \subseteq N_2(x)$. Let $w \in N(y) - \{x\}$. Since G is bipartite w has no neighbors in $N_2(x)$. If $N(w) - \{y\} \subseteq N_3(x)$, then $A = (X - \{x, w\}) \cup \{y\}$ dominates G contradicting $\gamma(G) = |X|$. It now follows that there exists $z \in N(x) - \{y\}$ so that x, y, w, z is a 4-cycle in G. □

This result can be strengthened as follows.

Lemma 3.5. *Suppose G is a connected, bipartite, leafless graph and $G \in C$. Let x be any vertex belonging to the color class X of G where X is a γ-set of G. Every $w \in N_2(x)$ belongs to a basic 4-cycle that includes x, say $xywz$, where y and z are both of degree two.*

Proof. Let $x \in X$ and $w \in N_2(x)$ be as in the statement of the lemma. If every common neighbor of x and w is heavy, then $(X - \{w, x\}) \cup \{y\}$ dominates G where y is any common neighbor of x and w, and this is a contradiction. But if $z \in N(x) \cap N(w)$ is the only common neighbor of x and w with degree two, then $(X - \{x, w\}) \cup \{z\}$ dominates G which is also a contradiction. Therefore w and x belong to a basic 4-cycle, say $xywz$, where y and z are of degree two. □

Theorem 3.6. *Let G be a connected, bipartite, leafless graph. G belongs to the class C if and only if the following two conditions hold in G:*

(i) *Every edge of G belongs to a 4-cycle; and*

(ii) *For every pair x, y of adjacent heavy vertices exactly one of them, say y, is neighborhood full and has the additional property that every pair u, $v \in N(y)$ are nonadjacent members of a basic 4-cycle.*

Proof. Assume G is connected, leafless and bipartite with color classes X and Y. If $\alpha(G) = \gamma(G)$, then Lemma 3.4 guarantees that (i) holds. Let x and y be adjacent heavy vertices and assume without loss of generality that X is a γ-set of G containing vertex x. By Lemma 3.5 every $w \in N_2(x) \cap N(y)$ belongs to a basic 4-cycle with x so that the two vertices on the cycle other than x and w have degree two. Thus x is not full. Let $z \in N(y) - \{x, w\}$. Then $z \in X \cap N_2(w)$ and Lemma 3.5 will place z and w on a basic 4-cycle. It now follows that $\deg(w) \geqslant 3$ and so y is a full vertex. Another application of Lemma 3.5 to an arbitrary pair u, $v \in N(y)$ shows that (ii) is necessary as well.

To prove the sufficiency of (i) and (ii) we first establish the following two statements: (I) All full vertices of G belong to the same color class of G; (II) All heavy vertices which are not full belong to the same color class of G.

Suppose that (I) does not hold. Let P: $y_1, v_1, v_2, \ldots, v_n, y_2$ be a shortest path in G between 2 full vertices with $y_1 \in X$ and $y_2 \in Y$. Then v_1 and v_n are heavy but not full and they belong to opposite color classes. Choose v_i, $v_j \in P$ with $i < j$ to be the pair of heavy vertices from P which are closest to each other and from opposite color classes. Then v_i and v_j are not adjacent by our hypothesis (ii) and so each of $v_{i+1}, \ldots, v_{j-1}$ is of degree two. However, each of these degree two vertices is on a 4-cycle by (i), and so it must be the case that $j = i + 3$ and v_i, v_{i+1}, v_{i+2}, v_j is a 4-cycle. This implies $v_i v_j \in E(G)$ which contradicts our choice of the path P. Hence (I) holds. (II) is proved in a similar manner.

Assume then that X is the color class containing the set of heavy, but not full, vertices. Let D be a γ-set of G for which $|D \cap X|$ is a maximum. Suppose first that there exists a full vertex y in D. Let $N(y) = \{x_1, x_2, \ldots, x_n\}$. If there is a unique $1 \leqslant j \leqslant n$ so that $x_j \notin D$, then let $D' = (D - \{y\}) \cup \{x_j\}$. If $x_i \notin D$ and $x_j \notin D$ $(i \neq j)$, then x_i, a, x_j, b forms a basic 4-cycle [by (ii)] for some vertices a and b of degree two and a, $b \in D$. In this case let $D' = (D - \{a, b\}) \cup \{x_i, x_j\}$. In either case D' is a γ-set but $|D' \cap X| > |D \cap X|$. Therefore D contains no full vertices.

If $D = X$, then since X is a vertex cover it follows that $G \in C$. Suppose then that there exists $v \in X - D$. If $\deg(v) = 2$, then by (i) there is a 4-cycle v, w, x, y and at least one of w and y belongs to D. Note that $\deg(y) = 2 = \deg(w)$. If w and y both belong to D, let $D' = (D - \{w, y\}) \cup \{v, x\}$. If only one of w or y belongs to D, then let $D' = (D - \{w, y\}) \cup \{v\}$. In both cases D' is a γ-set of G and again $|D' \cap X| > |D \cap X|$ contradicts the choice of D. Thus all degree two vertices in X belong to D, and any $v \in X - D$ is heavy. Since no full neighbor of v is in D there must be $z \in N(v) \cap D$ and $\deg(z) = 2$. Let $N(z) = \{u, v\}$. The edge zu belongs to a 4-cycle which must be v, z, u, y for some y. If $\deg(y) = 2$, then let $D' = (D - \{z\}) \cup \{v\}$. If $\deg(y) \geqslant 3$, then u is heavy and by (ii) there is a degree two vertex $w \in Y$ with v, z, u, w a basic 4-cycle. But the same choice for D' as above is a γ-set with $|D' \cap X| > |D \cap X|$.

This final contradiction forces $D = X$ and so X is a γ-set which is also a vertex cover, and $G \in C$. □

Figure 1 shows several connected, bipartite, leafless graphs belonging to the class C. In each the solid vertices are those of the color class which forms a γ-set for that particular component. Of course in some cases either color class will suffice.

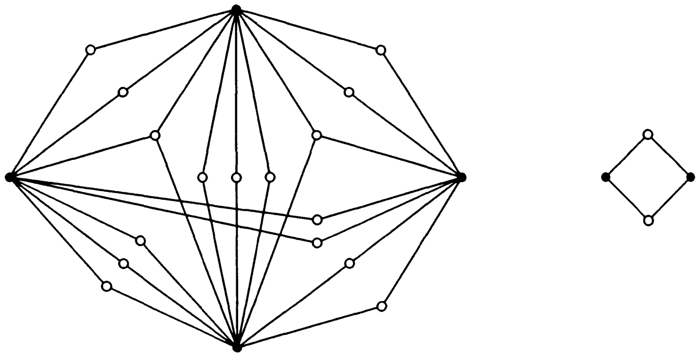

Figure 1. H_1 and H_2 satisfying Theorem 3.6.

We now complete our characterization of graphs G with $\alpha(G) = \gamma(G)$ by detailing the manner in which a collection of components each of which is a bipartite leafless graph from C (as described in Theorem 3.6) or an isolated vertex can be "hooked together" to form a general graph G with $\alpha(G) = \gamma(G)$. As in Section 2 we let $P = N(S) - (S \cup L)$ where S is the set of stems in G. In addition, if $H_1, H_2, \ldots, H_m$ are the nontrivial components (necessarily leafless and bipartite) in $G - N[S]$, then X_i will denote a color class of H_i with $|X_i| = \gamma(H_i)$. The proof of one of the implications of the following theorem is a direct consequence of Theorem 3.6. The other consists of checking that $R \cup S \cup X_1 \cup \ldots \cup X_m$ is a γ-set which is also a vertex cover. The proof is omitted.

Theorem 3.7. *Let G be a connected graph with at least one leaf. Then $\alpha(G) = \gamma(G)$ if and only if $G - N[S]$ consists of nontrivial leafless, bipartite components H_1, $H_2, \ldots, H_m$ along with a set of isolated vertices R so that the following conditions hold.*

(1) *For each i, one of the color classes of H_i, say X_i, is a γ-set for H_i.*

(2) *P is an independent set in G.*

(3) *The subgraph induced by S in G is arbitrary.*

(4) *The set of edges in G joining vertices in P and vertices in S is arbitrary.*

(5) *$N(P) - S \subseteq R \cup X_1 \cup \ldots \cup X_m$ so that for every $v \in P$, if $u_1, u_2 \in N(v) - S$, then u_1 and u_2 belong to a common X_i and there is a path of length two in H_i joining u_1 and u_2.*

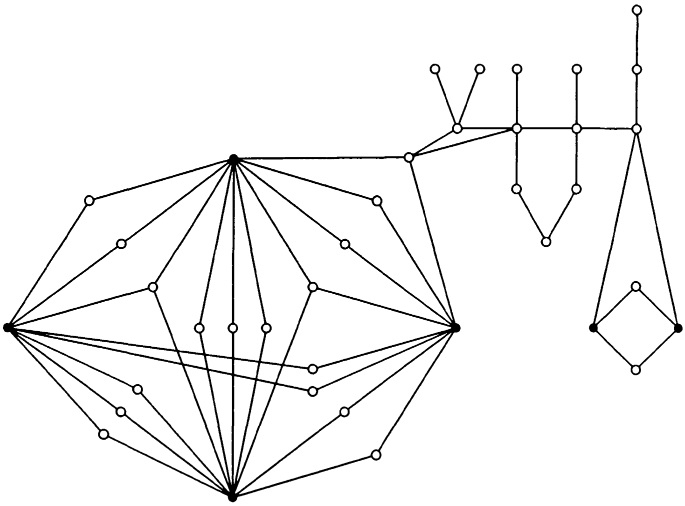

Figure 2. Graph in class C.

Figure 2 shows one way in which H_1, H_2 and several isolates can be combined to give a graph in C.

4. Bipartite graphs with one color class a γ-set

We observe that Theorem 3.6 gives a characterization of those connected, bipartite leafless graphs which have a color class serving as a minimum dominating set. Any nontrivial component of $G - N[S]$ for a graph G satisfying $\alpha(G) = \gamma(G)$ must be of this type. Next consider a bipartite graph H with $\alpha(H) = \gamma(H)$ where neither of its color classes is a γ-set. By Lemma 3.3, such a graph H must have leaves. (See Figure 3 for an example.)

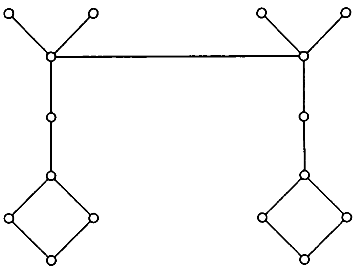

Figure 3. An example.

We now complete a characterization of the class of connected bipartite graphs which have one color class being a γ-set as follows.

Theorem 4.1. *Let G be a connected bipartite graph with at least one leaf. One of the color classes of G, say A, is a γ-set for G if and only if conditions* (1)–(5) *of Theorem* 3.7 *hold (with* (3) *and* (4) *modified only to require G to be bipartite) and so that* (6) *no member of P belongs to A and* (7) *if x is a stem adjacent to two or more leaves then x belongs to A.*

Proof. For each i let $V(H_i) = X_i \cup Y_i$. If A is a γ-set for G then it follows that $\alpha(G) = \gamma(G)$ since A is certainly a vertex cover. By Theorem 3.7 $R \cup S \cup X_1 \cup \ldots \cup X_m$ is also a γ-set for G where $X = X_1 \cup X_2 \cup \ldots \cup X_m$ contains all the connectors of the bipartite components $H_1, H_2, \ldots, H_m$.

First we show that no member of P can belong to A. By Theorem 3.7 we note that each vertex in P is either (i) adjacent to a single member of R or (ii) adjacent to vertices in a single X_i or (iii) adjacent only to vertices in S.

Consider any $v \in P$ which is adjacent to some $r \in R$. Let P_r be those vertices in P adjacent to r. If $v \in A$, then $P_r \subseteq A$ and any stem adjacent to a vertex in P_r belongs to $V - A$. But then $(A \cup \{r\}) - P_r$ dominates G and is smaller than A since $|P_r| \geqslant 2$. This contradiction shows that such a v is not in A.

Next consider any $v \in P$ which is adjacent to some subset of vertices in X_i. Let $P(X_i)$ be the set of all vertices in P which are adjacent to at least one vertex in X_i. If $v \in A$, then $Y_i \subseteq A$ and $P(X_i) \subseteq A$. But now $(A \cup X_i) - \big(Y_i \cup P(X_i)\big)$ dominates G and is smaller than A (since X_i is a γ-set for H_i and so $|X_i| \leqslant |Y_i|$). Thus it again follows that no such v can belong to A.

Finally consider the set of vertices $P(S)$ in P which are adjacent only to stems of G. If some $v \in P(S)$ also belongs to A, then let $S_v = \{x \in S \mid xv \text{ is an edge}\}$ and let $L_v = \{w \in L \mid wx \text{ is an edge for some } x \in S_v\}$. Note that $L_v \subseteq A$ and that $(A \cup S_v) - (L_v \cup \{v\})$ is a smaller dominating set than A, a contradiction.

Hence P has no members in A, and so any stem adjacent to a vertex in P must be in A. Furthermore, any stem x which is adjacent to two or more leaves must be in A or else A is not a minimum dominating set. Thus, the seven conditions have been shown to be necessary.

If we assume (1) through (7) hold, then the stems adjacent to more than one leaf or adjacent to a member of P as well as the vertices of $R \cup X_1 \cup \ldots \cup X_m$ all belong to the same color class, say A. If this does not force all stems to be in A, then only stems adjacent to a single leaf remain to be considered, and for each such stem-leaf pair we can choose whichever is in A and it follows that A is a γ-set for G. □

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