A characterization of horizontal visibility graphs and combinatorics on words

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An Horizontal Visibility Graph (for short, HVG) is defined in association with an ordered set of non-negative reals. HVGs realize a methodology in the analysis of time series, their degree distribution being a good discriminator between randomness and chaos [B. Luque, *et al.*, *Phys. Rev. E* **80** (2009), 046103]. We prove that a graph is an HVG if and only if outerplanar and has a Hamilton path. Therefore, an HVG is a noncrossing graph, as defined in algebraic combinatorics [P. Flajolet and M. Noy, *Discrete Math.*, **204** (1999) 203-229]. Our characterization of HVGs implies a linear time recognition algorithm. Treating ordered sets as words, we characterize subfamilies of HVGs highlighting various connections with combinatorial statistics and introducing the notion of a visible pair. With this technique we determine asymptotically the average number of edges of HVGs.

I. INTRODUCTION

A *graph* is an ordered pair $G = (V, E)$, where V is a set of elements called *vertices* and $E \subseteq V \times V - \{\{i, i\}: i \in V\}$ is set of unordered pairs of vertices called *edges*. Let $X = \left(x_i \in \mathbb{R}^{\geq 0} : i = 1, 2, ..., n\right)$ be an ordered set (or, equivalently, a *sequence*) of non-negative real numbers. The *horizontal visibility graph* (for short, *HVG*) [12] of X is the graph $G = (V, E)$, with $V = X$ and an edge $x_i x_j$ if $x_i, x_j > x_k$ for every $i < k < j$. Clearly $x_i x_j \in E$ whenever $j = i + 1$. If the elements of X are given with finite precision, we can always take $X = (x_i \in \mathbb{N} : i = 1, 2, ..., n)$. The HVG of X is also denoted by HVG(X).

The term HVG also describes some diagrams employed in floorplanning and channel routing of integrated circuits [6]; visibility graphs obtained from polygons constitute an area of extensive study in computational geometry (see, *e.g.*, Ch. 15 in [1]). Inspired by the geometric notion, Lacasa *et al.* [7] introduced visibility graphs and shortly afterwards their variations, HVGs [12]. Further research on such graphs is reported in [8, 9]. The principal idea of these papers is to translate dynamical properties of time series into structural features of graphs. It was shown in [12] that the degree distribution of HVGs has a role in discriminating between random and chaotic series obtained from discrete dynamical systems. In [9], the authors use an algorithm for HVGs to characterize and distinguish between correlated stochastic, uncorrelated and chaotic processes.

The purpose of the present paper is to study combinatorial properties of HVGs. We characterize the family of HVGs by proving that a graph is an HVG if and only if is outerplanar and has a Hamilton path. This result allows us to use, in the study of HVGs, results obtained for outerplanar graphs. Recognize properties of a sequence X via the properties of its HVG is a wide direction of analysis. Treating ordered sets as words, we characterize subfamilies of HVGs highlighting various connections with combinatorial statistics.

II. OUTERPLANARITY

A *drawing* of a graph G is a function f that maps each vertex $v \in V(G)$ to a point $f(v) \in \mathbb{R}^2$ and each edge $uv \in E(G)$ to a curve whose endpoints are $f(u)$ and $f(v)$. A graph is *planar* if it has a drawing so that any pair of edges can only intersect at their endpoints. A *face* is a bounded region of a planar graph. The *infinite face* is the outer one. A planar graph is *outerplanar* if it has a drawing such that each vertex is incident to the infinite face. A linear time recognition algorithm for outerplanar graphs has been given by Mitchell [14]. Outerplanar graphs are used in the design of integrated circuits when the terminals are on the periphery of the chip [10]. In this section, we give a characterization of HVGs with respect to outerplanarity. The characterization is a consequence of the following two results:

Lemma 1 *Every HVG is outerplanar and has a Hamilton path.*

Proof. Let $G = \mathrm{HVG}(X)$, where $X = (x_1, \ldots, x_n)$. By definition, G has a Hamilton path $P = x_1 x_2 \ldots x_n$. Observe that there is no a pair $x_i x_j$, $x_s x_t$ of edges such that $i < s < j < t$. Indeed, the existence of such a pair implies that $x_j > x_s$ (due to $x_i x_j \in G$) and $x_j < x_s$ (due to $x_s x_t \in G$), a contradiction. Thus, we can obtain an outerplanar embedding of G as follows: set as an interval of a horizontal straight line with points being vertices of P and depict each edge of G outside P as an arc above the interval for P such that if $x_i x_j, x_s x_t \in E(G) \setminus E(P)$

and $s < i < j < t$ then the arc corresponding to x_ix_j is situated below the arc corresponding to x_sx_t. Hence, we have proved that G is outerplanar. ■

Lemma 2 *If a graph H is outerplanar and has a Hamilton path, then H is an HVG.*

Proof. Let H have a Hamilton path $v_1v_2 \dots v_n$. As H is outerplanar, all vertices of the Hamilton path belong to the infinite face in an outerplanar embedding of H into plane. Observe that H has no crossing edges, i.e., pairs of edges v_iv_j, v_sv_t such that $i < s < j < t$. We say that an edge v_iv_j, $i < j$, of H is of *nesticity* 0 if $j = i + 1$. We say that an edge v_iv_j, $i < j$, of H is of *nesticity* k if k is the minimum nonnegative integer such that if $v_sv_t \in E(H)$ and $i < s < t < j$ then v_sv_t is of nesticity p, where $p < k$. Nesticity of each edge is well-defined as there are no crossing edges. The *nesticity* of a vertex v_i is the maximum nesticity of an edge incident to v_i.

Now we will construct an ordered set $X = (x_1, \dots, x_n)$ such that $H = \mathrm{HVG}(X)$ using the following algorithm which consists of two stages. In the first stage, we determine the nesticity of each edge of H. To initialize, set the nesticity of each edge x_ix_{i+1} to 0. Now for each q from 2 to $n-1$ consider all edges of H of the form x_ix_{i+q} and set the nesticity of x_ix_{i+q} to $p+1$, where p is the maximum nesticity of an edge x_jx_k with $i \le j \le k \le i+q$ and $x_jx_k \neq x_ix_{i+q}$. In the second stage, compute the nesticity of each vertex v_i of H by considering all edges incident to v_i and set x_i to the nesticity of v_i.

It is easy to see, by induction on the nesticity of edges, that indeed $H = \mathrm{HVG}(X)$ (we use the fact that H has no crossing edges). ■

Theorem 3 *A graph is an HVG if and only if it is outerplanar and has a Hamilton path.*

Since recognizing an outerplanar graph and determining if it has a Hamilton path are tasks that can be done in linear time (see Mitchell [14] and Lingas [11], respectively), we have following result:

Corollary 4 *We can determine in linear time (with respect to the number of vertices) if a graph is an HVG.*

We conclude this section with a further characterization. A *noncrossing graph* [5] with n vertices is a graph drawn on n points numbered in counter-clockwise order on a circle such that the edges lie entirely within the circle and do not cross each other. It is useful to point out that these objects are HVGs because of Theorem 3:

Corollary 5 *An HVG is a noncrossing graph.*

III. UNIMODAL HVGS

We characterize here the HVGs with minimum number of edges. The *degree* of a vertex i is $d(i) := |\{j : \{i,j\} \in E\}|$. The *degree sequence* is the unordered multiset of the degrees. HVGs are not characterized by their degree sequence. In other words, there are non-isomorphic HVGs with the same degree sequence. For example, given $X = (x_i : 1, 2, ..., 5)$, let $x_1 = x_3$, $x_2 < x_1$ and $x_2 = x_4 = x_5$. The degree sequence of $\mathrm{HVG}(X)$ is $\{3, 2, 3, 2, 2\}$. The same degree sequence is associated to the graph $\mathrm{HVG}(X')$, where $x'_1 = x'_4$, $x'_2 < x'_1$ and $x'_2 = x'_3 = x'_5$. Let $\delta(G)$ and $\Delta(G)$ be the *minimum degree* and *maximum degree* of a graph G, respectively. Theorem 3 implies the following:

Proposition 6 *If G is an HVG then $\delta(G) = 1$ or 2 and $\Delta(G) \le n-1$. If $\Delta(G) = n-1$, then there is only one vertex of maximum degree.*

A real-valued function f is *unimodal* if there exists a value m such that $f(x)$ is monotonically increasing for $x \le m$ and monotonically decreasing for $x \ge m$. Thus, $f(m)$ is the maximum of f and its only local maximum. If X is unimodal then $\mathrm{HVG}(X)$ is said to be *unimodal.*

Proposition 7 *An HVG is unimodal if and only if it is a path.*

We associate a word of length $n-1$, called a *difference*, to a set with n elements. The letters of the word are from the alphabet $\{0, +, -\}$. Each letter corresponds to a pair of adjacent elements. If $x_i = x_{i+1}$ the letter corresponding to the pair (x_i, x_{i+1}) is 0. When $x_i < x_{i+1}$ (resp. $x_i < x_{i+1}$) then the letter for the pair is + (resp. −). For example, $X = (1, 4, 8, 8, 2, 4)$ gives the difference $+ + 0 - +$. Let us observe that the difference D of a unimodal HVG graph does contain the *pattern* $(-, +)$, *i.e.*, it does not contain a subword $-P+$, where P is an arbitrary subword of D.

Proposition 8 *The number of HVGs isomorphic to the n-path and associated to different ordered sets of cardinality n (without taking into account the actual values of the single elements) is exactly $a(n) = 2^{n-1}(n+2)$.*

Proof. We prove that the number of differences of length n without the pattern $(-, +)$ is exactly $a(n)$. Clearly, $a_0 = 1$ and $a_1 = 3$. Since each such a word $x = x_1x_2 \cdots x_n$ of length n can be written as $x = +x'$, $x = 0x'$, or $x = -x''$ then $a_n = 2a_{n-1} + b_n$, where b_n denotes the number words in B_n of length n on the alphabet $\{0, +, -\}$ that do not contain the pattern $(-, +)$ and its leftmost letter is −. Since any word in B_n has leftmost letter − and there is no pattern $(-, +)$ in the word, any letter which is not the leftmost has two possibilities: 0 or −. This implies that $b_n = 2^{n-1}$ and, hence, a_n satisfies the recurrence relation $a_n = 2a_{n-1} + 2^{n-1}$ with initial conditions $a_0 = 1$ and $a_1 = 3$. Solving this recurrence relation gives the formula for a_n. ■

The difference does not uniquely specify the HVG. In fact, there are nonisomorphic HVGs with the same difference. For example, the sets $(5, 4, 3, 5)$ and $(5, 4, 3, 4)$ have the same difference $- - +$, but the HVGs associated to these sets have five and four edges, respectively.

IV. MAXIMAL HVGS

The *triangulation of a* (convex) *polygon* is a planar graph obtained by partitioning the polygon into disjoint triangles such that the vertices of the triangles are chosen from the vertices of the polygon. An outerplanar graph is *maximal outerplanar* if it is not possible to add an edge such that the resulting graph is still outerplanar. A maximal outerplanar graph can be viewed as a triangulation of a polygon. By Theorem 3,

Corollary 9 *The maximum number of edges in an HVG on n vertices is $2n-3$ and the graph is maximal outerplanar.*

In computational geometry, the *visibility graph* [15] of a polygon with n angles is obtained by constructing a graph on n vertices, each vertex of the graph representing an angle of the polygon, and each edge of the graph joining only those pairs of vertices that represent visible pairs of angles in the polygon. A polygon in the plane is called *monotone* with respect to a straight line L, if every of the lines orthogonal to L intersects the polygon at most twice. By a result of ElGindy [4] and Theorem 3, we have an observation relating HVGs to a special classes of visibility graphs:

Corollary 10 *An HVG on n vertices and $2n-3$ edges is the visibility graph of a monotone polygon.*

A characterization of HVGs with maximal number of edges is given with Corollary 19. In fact, the characterization is easy once established a connection between HVGs and combinatorics on words.

V. WORDS

We denote by $[k]^n$ the set of all words of length n over an alphabet $[k]=\{1,2,\dots,k\}$. Each word $x=x_1x_2\cdots x_n$ defines an ordered set X by $X=\{x_1,x_2,\dots,x_n\}$ and conversely. In order to describe the edges of an HVG with respect to words, we need the following definition:

Definition 11 *Let $x=x_1x_2\cdots x_n$ be any word in $[k]^n$. We say that the pair (x_i,x_j) with $i+1\le j$ is* visible *if $x_{i+1},x_{i+2},\dots,x_{j-1}<\min\{x_i,x_j\}$. Clearly, (x_i,x_{i+1}) is a visible pair. We denote the number of visible pairs in x by $vis(x)$.*

For instance, if $x=21232143112112$ is a word in $[4]^{14}$ then (x_1,x_3), (x_4,x_7), (x_5,x_7), (x_8,x_{11}) and (x_{11},x_{14}) are the visible pairs of x, in addition to the 13 edges of P_{14}. Thus $vis(x)=18$.

By the definition of HVG and visibility of pairs in words we can state the following result.

Theorem 12 *Let $X=\{x_1,x_2,\dots,x_n\}$ be an ordered set of n elements and let $k=\max X$. The HVG of X, $G=$HVG(X), can be represented uniquely as a word $x=x_1x_2\cdots x_n$ with a set E of visible pairs, where each edge $\{i,j\}$ in the graph G corresponds to visible pair (x_i,x_j).*

Let $A=(x_i,x_j)$ and $B=(x_{i'},x_{j'})$ be two visible pairs in a word $x=x_1x_2\cdots x_n\in[k]^n$ with $i\le i'$. Clearly, $i<j$ and $i'<j'$. From the above definition, we obtain that either $i\le i'\le j'\le j$ or $i<j\le i'<j'$. In such cases, we say that the pair A *covers* the pair B and that A and B are *disjoint* pairs, respectively.

Fact 13 *If x is the word realizing an HVG then in any two disjoint pairs $A,B\in x$ either A covers B or B covers A.*

How many words are there in $[k]^n$ with a fixed number of visible pairs?

Definition 14 *We denote the generating function for the number of words $x\in[k]^n$ according to the number of visible pairs in x by $F_k(x,q)$, that is,*

$$F_k(x,q)=\sum_{n\ge0}x^n\sum_{x\in[k]^n}q^{vis(x)}.$$

Similarly, we denote the generating function for the number of words $x\in[k-1]^n$ according to the number of visible pairs in xk (respectively, kx) by $L_k(x,q)$ (respectively, $R_k(x,q)$). Also, we denote the generating function for the number of words $x\in[k-1]^n$ according to the number of visible pairs in kxk, which are not equal to (k,k), by $M_k(x,q)$.

Note that each word x in $[k]^n$ can be decomposed either as

- $x\in[k-1]^n$, that is, x does not contain the letter k,
- $x=x^{(1)}kx^{(2)}\cdots kx^{(m+1)}$, where $x^{(j)}$ is a word in $[k-1]^{i_j}$ with $i_1+\cdots+i_{m+1}+m=n$.

Thus, rewriting these rules in terms of generating functions we obtain the following lemma.

Lemma 15 *For all $k\ge1$,*

$$F_k(x,q)=F_{k-1}(x,q)+\frac{xR_k(x,q)L_k(x,q)}{1-xqM_k(x,q)},$$

and for all $k\ge2$,

$$M_k(x,q)=M_{k-1}(x,q)+\frac{xq^2M_{k-1}(x,q)}{1-xqM_{k-1}(x,q)}$$

$$R_k(x,q)=\frac{R_{k-1}(x,q)}{1-xqM_{k-1}(x,q)}$$

$$L_k(x,q)=R_k(x,q).$$

Also $F_0(x,q)=M_1(x,q)=R_1(x,q)=L_1(x,q)=1$.

The above lemma together with induction give the following result.

Theorem 16 *Let $k \geq 1$. The generating function $F_k(x,q)$ is given by*

$$F_k(x,q) = 1 + \sum_{j=1}^{k} \frac{1 - xqM_j(x,q)}{\prod_{i=1}^{j}(1 - xqM_i(x,q))^2},$$

where $M_k(x,q)$ satisfies the recurrence relation

$$M_k(x,q) = M_{k-1}(x,q) + \frac{xq^2M_{k-1}(x,q)}{1 - xqM_{k-1}(x,q)},$$

with initial condition $M_1(x,q) = 1$.

For instance, the theorem gives

$$F_1(x,q) = 1 + \frac{x}{1 - xq}$$

and

$$F_2(x,q) = 1 + \frac{x((1-xq)^2 + 1 - x^2q^3)}{(1-xq)((1-xq)^2 - x^2q^3)}.$$

Note that it appears to be hard to derive an explicit formula for the generating functions $M_k(x,q)$ and $F_k(x,q)$. But we can use the result for studying the total number of visible pairs in all words in $[k]^n$.

Theorem 17 *The generating function for the total number of visible pairs in all words in $[k]^n$ ($k \geq 1$) is*

$$\begin{aligned} F_k'(x) &= \sum_{n\geq 0} x^n \sum_{x\in[k]^n} vis(x) \\ &= 2x^2 \sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1-(i-1)x + x\sum_{\ell=1}^{i-1} \frac{2-(2\ell-1)x}{1-(\ell-1)x}}{(1-(i-1)x)(1-ix)}}{(1-(j-1)x)(1-jx)} \\ &\quad - x^2 \sum_{j=1}^{k} \frac{1-(j-1)x + x\sum_{i=1}^{j-1}\frac{2-(2i-1)x}{1-(i-1)x}}{(1-(j-1)x)^2(1-jx)^2}. \end{aligned}$$

Proof. We define

$$F_k'(x) = \frac{d}{dq} F_k(x,q) \mid_{q=1}$$

and

$$M_k'(x) = \frac{d}{dq} M_k(x,q) \mid_{q=1}.$$

Theorem 16 with induction on k implies that $F_k(x,1) = \frac{1}{1-kx}$ and $M_k(x,1) = \frac{1}{1-(k-1)x}$. By differentiating respect to q and by using the expressions $F_k(x,1)$ and $M_k(x,1)$, Theorem 16 gives

$$\begin{aligned} F_k'(x) &= -x^2 \sum_{j=1}^{k} \frac{M_j + M_j'}{\prod_{i=1}^{j}(1 - xM_i)^2} \\ &\quad + 2x^2 \sum_{j=1}^{k} \frac{1 - xM_j}{\prod_{i=1}^{j}(1-xM_i)^2} \sum_{i=1}^{j} \frac{M_i + M_i'}{1 - xM_i}, \end{aligned}$$

where $M_j = \frac{1}{1-(j-1)x}$ and M_j' is given by the recurrence relation

$$\begin{aligned} M_k' &= \frac{(1-(k-2)x)^2}{(1-(k-1)x)^2} M_{k-1}' \\ &\quad + \frac{x(2-(3k-3)x)}{(1-(k-2)x)(1-(k-1)x)}, \end{aligned}$$

with the initial condition $M_1' = 0$. Hence, by induction on k we obtain that

$$M_k' = \frac{x}{(1-(k-1)x)^2} \sum_{j=1}^{k-1} \frac{2-(2j-1)x}{1-(j-1)x}.$$

Plugging this into the equation of F_k' we have the result. For instance, ∎

- $F_1'(x) = \frac{x^2}{(1-x)^2}$,
- $F_2'(x) = \frac{(4-3x)x^2}{(1-x)(1-2x)^2}$,
- $F_3'(x) = \frac{(10x^2-22x+9)x^2}{(1-x)(1-2x)(1-3x)^2}$.

Theorem 18 shows that the function $F_k'(x)$ has a pole of degree two at $\frac{1}{k}$. Direct calculations show that

$$\begin{aligned} C &= \lim_{x\to\frac{1}{k}} (1-kx)^2 F_k'(x) \\ &= \lim_{x\to\frac{1}{k}} \frac{x^2\left(1-(k-1)x + x\sum_{i=1}^{k-1}\frac{2-(2i-1)x}{1-(i-1)x}\right)}{(1-(k-1)x)^2} \\ &= \frac{1}{k}\sum_{i=1}^{k} \frac{2k+1-2i}{k+1-i} \\ &= 2 - \frac{\Psi(k+1)+\gamma}{k}, \end{aligned}$$

where $\Psi(x)$ is the digamma function and γ is Euler's constant. Thus, asymptotically, the total number of visible pairs in all words in $[k]^n$ ($k \geq 1$) is given by Cnk^n. This means that the average number of visible pairs in all words in $[k]^n$ is given by Cn.

Corollary 18 *The average number of edges in an HVG is*

$$(2 - (\Psi(k+1)+\gamma)/k)\, n,$$

when X is an ordered subset of n elements from $[k]$ and $n \to \infty$.

The corollary shows that the number of maximal edges is in fact $2n-3$, where X contains n elements (cfr. Section III). This can be obtained easily from our representation as words:

Corollary 19 *Let G be an HVG with $2n-3$ edges. Then*

$$X = \{\ldots, 8, 6, 4, 2, 1, 3, 5, 6, 7, 9, \ldots\}$$

or

$$X = \{\ldots, 9, 7, 5, 3, 1, 2, 4, 6, 8, \ldots\}.$$

Proof. Without loss of generality, we can assume that X has n different numbers. Assume $\pi = \pi_1\pi_2\cdots\pi_n$ is a permutation on $[n]$ with a maximal number of visible pairs. The pair (π_1, π_n) is visible, hence either $\pi_1 + 1 = \pi_n = n$ or $\pi_n + 1 = \pi_1 = n$. Delete n from π and denote the resulting permutation by π' By induction on n, we obtain that π has the form $\pi = \cdots 864213579\cdots$ or $\pi = \cdots 975312468\cdots$, as claimed. ■

VI. PERMUTATIONS

We denote by S_n the set of all permutations on $[n]$. Each permutation $\pi = \pi_1\pi_2\cdots\pi_n$ defines an ordered set X by $X = \{\pi_1, \pi_2, \ldots, \pi_n\}$ and conversely. By definition 11, we can describe the edges of an HVG with respect to permutations (a permutation is in fact a word without repetitions). Define $S_n(q)$ to be the generating function for the number of permutations π on $[n]$ according to the number of visible pairs in π by $S_n(q)$, that is,

$$S_n(q) = \sum_{\pi\in S_n} q^{vis(\pi)}.$$

For example, in S_2 there are two permutations 12 and 21. Thus $S_2(q) = 2q$. In S_3 there are 6 permutations 123, 132, 213, 231, 312 and 321. Thus $S_3(q) = 4q^2 + 2q^3$.

Now, let us find an explicit formula for $S_n(q)$. Define $S_{n,j}$ to be the set of permutations $\pi = \pi_1\pi_2\cdots\pi_n$ in S_n such that $\pi_j = 1$. Let $\pi \in S_n$ and define π' to be the permutation obtained from π by deleting the letter 1 and by decreasing each letter by 1. Let us write an equation for $S_n(q)$. From the definitions, we have that, if $j = 1, n$ then the set permutations $S_{n,j}$ is counted by $qS_{n-1}(q)$, and if $2 \le j \le n-1$ then the set of permutations $S_{n,j}$ is counted by $q^2S_{n-1}(q)$. Thus, for all $n \ge 2$,

$$S_n(q) = 2qS_{n-1}(q) + (n-2)q^2S_{n-1}(q),$$

which implies that

$$S_n(q) = (2q)^{n-1}\prod_{j=2}^{n}\left(1 + \frac{j-2}{2}q\right).$$

Using the fact that the unsigned Stirling numbers $s(n, j)$ of the first kind satisfy the relation

$$(1+x)\cdots(1+(n-1)x) = \sum_{j=0}^{n} s(n, n-j)x^j,$$

we obtain that

$$\begin{aligned} S_n(q) &= (2q)^{n-1}\prod_{j=1}^{n-2}\left(1 + j\frac{q}{2}\right) \\ &= (2q)^{n-1}\sum_{j=0}^{n-1} s(n-1, n-1-j)\frac{q^j}{2^j} \\ &= \sum_{j=0}^{n-1} 2^{n-1-j}s(n-1, n-1-j)q^{n-1+j}. \end{aligned}$$

Hence, we can state the following result:

Theorem 20 *Let $n \ge 2$. The number of permutations π with exactly $n-1+j$, $0 \le j \le n-1$, visible pairs is given by*

$$2^{n-1-j}s(n-1, n-1-j),$$

where $s(n-1, n-1-j)$ is the unsigned Stirling number of the first kind.

Corollary 21 *The average number of edges in an HVG is*

$$2n - \sum_{j=1}^{n}\frac{1}{j},$$

when X is an ordered subset of n different elements.

Proof. By Theorem 20 we have that the average number of edges in an HVG is

$$p_n = \frac{1}{n!}\sum_{j=0}^{n-1}(n-1+j)2^{n-1-j}s(n-1, n-1-j),$$

when X is an ordered subset of n different elements. The expression p_n can be written as

$$p_n = \frac{1}{n!}\sum_{j=0}^{n-1}(2n-2-j)2^js(n-1, j).$$

By the fact that

$$x(x+1)\cdots(x+n-1) = \sum_{j=0}^{n} s(n, j)x^j,$$

we obtain

$$\sum_{j=0}^{n} js(n, j)x^j = x(x+1)\cdots(x+n-1)\sum_{j=0}^{n-1}\frac{1}{x+j},$$

which implies that

$$\sum_{j=0}^{n-1} s(n-1, j)2^j = n!$$

and

$$\sum_{j=0}^{n-1} j2^js(n-1, j) = n!\sum_{j=2}^{n}\frac{1}{j}.$$

Hence,

$$p_n = 2(n-1) - \sum_{j=2}^{n}\frac{1}{j} = 2n - \sum_{j=1}^{n}\frac{1}{j},$$

which completes the proof. ■

VII. CONCLUSIONS

We have characterized the family HVGs in terms of their combinatorial properties. The characterization is useful because it gives an efficient recognition algorithm. Moreover, it places the study of HVGs in a specific mathematical context, related to a well-known class of graphs. As it is originally observed in the literature, the potential importance of HVGs stems in their use for describing properties of dynamical objects like time series. Therefore, the main goal would be to determine the dynamical and structural properties of an ordered set that are readable through the analysis of its HVG. In this perspective, we have shown that combinatorics on words are a useful tool. The connection suggests a number of natural open problems. For example, it may be valuable to study HVGs and statistics on words and forbidden subsequences. The mathematical scope turns out to be wider than the original dynamical systems framework.

It would be interesting to characterize visibility graphs [7, 8], which can be defined as follows. For an ordered set $X = \left(x_i \in \mathbb{R}^{\geq 0} : i = 1, 2, ..., n\right)$, its *visibility graph* $\mathrm{VG}(X)$ has the vertex set X and x_i, x_j $(i < j)$ are adjacent if for each k with $i < k < j$ we have

$$x_k < x_j + (x_i - x_j)\frac{j-k}{j-i}.$$

It is easy to see that for each X, $\mathrm{HVG}(X)$ is a subgraph of $\mathrm{HV}(X)$. Not each visibility graph is outerplanar as $\mathrm{HV}(Y)$, where $Y = (4, 2, 1, 4)$ is K_4, which is not outerplanar.

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