A CHARACTERIZATION OF LINEARLY CONTINUOUS FUNCTIONS WHOSE PARTIAL DERIVATIVES ARE MEASURES

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1. In previous papers [9], [10], we introduced a class of functions of several real variables which we designated as linearly continuous. In order to clarify the position of this class of functions, in relation to established classes, it is desirable to first describe matters for functions of one variable.

We shall consider only functions with compact support and shall consider length, area, etc., of a function on a bounded interval containing the support. The functions of one variable whose derivatives are measures are then equivalent to functions of bounded variation. The functions whose derivatives are functions are equivalent to absolutely continuous functions. The set of functions which are continuous and of bounded variation lies between these two classes of functions. These functions are the ones (i) for which the length of the associated curve is equal to the Hausdorff one dimensional measure of the graph, (ii) for which the variation is given by the Banach indicatrix formula, and (iii) for which the derivative is a non-atomic measure. Moreover, the space of these functions is complete with respect to the metric given by

$$d(f, g) = \delta(f, g) + \Delta(\alpha_f, \alpha_g),$$

where δ is the metric for convergence in measure, α_f and α_g are the measures associated with the lengths of curves given by f and g, respectively, and $\Delta(\mu, \nu) = \sup|\mu(E) - \nu(E)|$, for Borel sets E, for any measures μ and ν.

For functions of several variables, the place of functions of bounded variation is assumed by those whose partial derivatives are totally finite measures. This class of functions was introduced by Cesari [5], and was studied, among others, by the author [8], Krickeberg [15], Fleming [7], Michael [16], Serrin [20], and de Vito [23]. (There

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are various other notions of bounded variation for functions of several variables, but they are not considered in this paper.)

Functions with compact support whose partial derivatives are functions take the place of the absolutely continuous functions. They were studied mainly by Calkin [4] and Morrey [17]. One interesting property they possess is that there is always an equivalent function which is absolutely continuous on almost all lines in every direction. Incidentally, this result follows from facts proved in this paper. These functions have proved to be of great importance in partial differential equations and other branches of analysis, starting with the work of Morrey and of Sobolev [21].

Functions in these classes can be discontinuous everywhere [9]. Indeed, it is possible to construct a bounded function f, whose partial derivatives are functions, such that every function equivalent to f is discontinuous everywhere. The continuous functions whose partial derivatives are measures could accordingly hardly assume the same role, in several variables, that the continuous functions of bounded variation have in one variable. A natural candidate for this position is the set of linearly continuous functions whose partial derivatives are measures. For the case of 2 variables, these functions are precisely the ones for which the surface area is equal to the Hausdorff 2 dimensional measure of the graph [9]. Also for the case of 2 variables, the linearly continuous functions whose partial derivatives are measures are the completion of the continuous functions whose partial derivatives are measures with respect to the metric

$$d(f, g) = \delta(f, g) + \Delta(\alpha_f, \alpha_g),$$

where now α_f and α_g are the measures associated with the areas of the surfaces defined by f and g. There are indications that a Banach indicatrix formula holds for this class of functions. Such a formula has recently been obtained by Ziemer (oral communication), for the continuous case. It also seems plausible that the partial derivatives of these functions are characterized as measures which are zero for ridé sets [25], these seemingly being the analogues in several dimensions of the countable sets in 1 dimension.

2. In [10], for dimension 2, we characterized the linearly continuous functions of finite area (i.e. those whose partial derivatives are measures), among all functions of finite area, as those functions f for which, for every $\varepsilon > 0$, there is a continuous g such that if $E = [x: f(x) \neq g(x)]$, then $\alpha_f(E) < \varepsilon$ and $\alpha_g(E) < \varepsilon$. We noted that this implies that linearly continuous functions are coordinate invariant. Indeed, it implies that for every such f there is an equivalent function which is continuous along almost all lines in every direction. We also showed in [10] that if f is merely linearly continuous in one coordinate system, but its partial derivative is not a measure, then it need not be linearly continuous in another coordinate system. We have not been able to obtain an analogous characterization theorem for $n>2$. However, by using an induction argument in [10] we showed that, for every n, every linearly continuous function whose partial derivatives are measures is equivalent to a function which is continuous along almost all lines in every direction. In the present paper, the main result is a characterization theorem for arbitrary n which is similar to the one proved in [10] for $n=2$. In fact, we show that if f is linearly continuous and its partial derivatives are measures then, for every $\varepsilon>0$, there is an approximately continuous g such that if $E=[x: f(x)\neq g(x)]$ then $\alpha_f(E)<\varepsilon$ and $\alpha_g(E)<\varepsilon$, and that the converse also holds. For the latter, we show in particular that every approximately continuous function whose partial derivatives are measures is linearly continuous. Since approximate continuity is invariant under Lipschitzian transformations, it follows that every function of the type considered has an equivalent function with very nice behavior. It accordingly matters little, regarding the implications which may be drawn, whether the approximating functions are continuous or, merely, approximately continuous.

The space of linearly continuous functions whose partial derivatives are measures is complete with respect to the metric mentioned above. We show that the approximately continuous functions are dense in this space. For $n=2$, the continuous functions are dense in the space, but we do not know whether or not this holds for $n>2$.

3. We now give some notations and definitions and a few facts which will be needed in the sequel. We consider a rectangular coordinate system $(x_1, \ldots, x_n)$ in n space. For every $i=1, \ldots, n$, we designate points in $(n-1)$ space with coordinates $(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ as $\bar{x}_i$. Thus a point in n space may be designated as $(x_i, \bar{x}_i)$. We say that a measurable real function f on n space is essentially linearly continuous in this coordinate system if, for every $i=1, \ldots, n$, there is an f_i equivalent to f (i.e., $f_i=f$ except on a set of Lebesgue n measure zero) such that for almost all $\bar{x}_i$, f_i is a continuous function of the one variable x_i. The function f is said to be linearly continuous in the coordinate system $(x_1, \ldots, x_n)$ if, for every $i=1, \ldots, n$, f is a continuous function of x_i for almost all $\bar{x}_i$. We showed in [9] that if f is essentially linearly continuous and its partial derivatives are measures then f is equivalent to a function which is linearly continuous.

We shall consider the class $\mathcal{A}$ of functions whose partial derivatives are measures. These are the functions of finite area. We shall give a brief discussion of these notions. In particular, we describe the partial derivatives in terms of variations of special functions in the equivalence class, the linear Blumberg measurable boundaries. This has the advantage of freeing the definition from an unnatural measurability hypothesis on the linear variations which is customary.

Let f be a measurable function whose support is contained in an n cube Q. We define the area of f in terms of the areas of quasilinear approximations. By a quasilinear function p on Q, we mean a continuous function which is linear on each simplex of a decomposition of Q. The area $L(p, Q)$ of p on Q is the sum of the Lebesgue n measures of the simplexes which form its graph. The area of f on Q is then defined as

$$L(f, Q) = \inf \{\liminf L(p_n, Q)\},$$

where $\{p_n\}$ converges in measure to f and the infimum is taken over all such sequences. Then $L(f, Q)$ is a lower semicontinuous functional with respect to various types of convergence including convergence in measure and convergence in the L_1 metric. It was shown in [5], see also [8] and [15], that $L(f, Q)$ is finite if and only if $f \in \mathcal{A}$. Then, let λ be Lebesgue n measure, and let $\mu_1, \ldots, \mu_n$ be the partial derivative measures of f. The vector valued measure $(\lambda, \mu_1, \ldots, \mu_n)$ has an associated numerical valued measure α_f, defined for Borel sets E by

$$\alpha_f(E) = \sup \sum_{k=1}^{r} \{[\lambda(E_k)]^2 + \sum_{i=1}^{n} [\mu_i(E_k)]^2\}^{\frac{1}{2}},$$

where the supremum is taken over all finite partitions $E_1, \ldots, E_r$ of E into Borel sets.

The area of f has an associated measure $L(f, E)$, defined first for open intervals $R \subset Q$ by means of quasi linear approximations, as above, and then by extending to a measure on the Borel sets. It turns out that $L(f, E) = \alpha_f(E)$ for every E (see [22], [9], [13]).

Of most importance to us will be the following representation of the partial derivatives μ_i, $i = 1, \ldots, n$, of a function $f \in \mathcal{A}$. In this connection, we first consider, as in [8], a measurable function g of one real variable. We define

$$V(g, (a, b)) = \inf v(h, (a, b)),$$

where (a, b) is an open interval, v is the variation of h on (a, b), and the infimum is taken for all h equivalent to g. This infimum is realized by the upper measurable boundary u of g, in the sense of Blumberg [2], which we define shortly. Thus

$$V(g, (a, b)) = v(u, (a, b)).$$

The function u is defined as follows: For each y, let $E_y = [\xi : g(\xi) > y]$. For each x, let

$$u(x) = \inf [y: \text{density of } E_y \text{ at } x \text{ is } 0].$$

Now, for each y, $u(x) \geqslant y$ if and only if the density of $E_{y-1/n}$ at x is not zero for every n. It follows from the Lebesgue density theorem that u is measurable. The proof that $V(g, (a, b)) = v(u, (a, b))$ is easy, and will not be given here.

Now, let f be a measurable function of n variables. For any direction θ, we define the measurable boundary of f in direction θ by letting, for each y, $E_y = [\xi: f(\xi) > y]$. For each x, let

$$u(x) = \inf\,[y\text{: linear density of } E_y \text{ in direction } \theta \text{ at } x \text{ is } 0].$$

The measurability of u follows from the fact [19], that for a measurable set S in n space, the linear density of S exists and is one almost everywhere in S.

For every $i=1, \ldots, n$, let u_i be the measurable boundary of f in direction x_i. Let $Q = Q_i \times (a_i, b_i)$. For each $m=1, 2, \ldots$, let

$$a_i < \xi_1 < \xi_2 < \ldots < \xi_r < b_i$$

be such that $\max(\xi_1 - a_i, \xi_2 - \xi_1, \ldots, b_i - \xi_r) < 1/m$ and $u(\bar{x}_i, \xi_j)$ is measurable in $\bar{x}_i$ for every ξ_j, $j=1, \ldots, r$. Then the function

$$\sum_{j=1}^{r-1} |u(\bar{x}_i, \xi_{j+1}) - u(\bar{x}_i, \xi_j)|$$

is measurable, and it follows that

$$v(u, \bar{x}_i, (a_i, b_i)) = V(f, \bar{x}_i, (a_i, b_i))$$

is measurable. The function $f \in \mathcal{A}$ whenever $V(f, \bar{x}_i, (a_i, b_i))$ is summable, $i=1, \ldots, n$. For an open interval $R = R_i \times (a, b)$, $R \subset Q$, where R_i is the projection normal to the x_i direction, we let

$$\phi(f, \bar{x}_i, (a, b)) = \lim_{x_i \to b-} u_i(x_i, \bar{x}_i) - \lim_{x_i \to a+} u_i(x_i, \bar{x}_i).$$

It follows from the dominated convergence theorem that

$$\mu_i(R) = \int_R \phi(f, \bar{x}_i, (a, b))$$

exists as does

$$|\mu_i|(R) = \int_R V(f, \bar{x}_i, (a, b)).$$

The set function μ_i generates an outer measure, $i=1, \ldots, n$, for which the measurable sets include all Borel sets. The same may be said of the associated α_f. Alternately, we may start with the set function $L(f, R)$ to obtain the outer measure α_f. This set function is known to be (e.g. [6]) such that

(i) if the distance between E and F is positive,

$$\alpha_f(E \cup F) = \alpha_f(E) + \alpha_f(F), \text{ and}$$

(ii) if E is measurable and $\varepsilon>0$ there is a compact F and an open G such that $F\subset E\subset G$ and $m(G)<m(F)+\varepsilon$, and

(iii) compact sets have finite measure.

We are accordingly able to apply to the measure α_f, $f\in\mathcal{A}$, a Vitali Covering Theorem of Besicovitch [1], in a version due to A. P. Morse [18].

THEOREM. *If an outer measure m, defined on the subsets of n space, satisfies the conditions itemized above, then for any set S, if S is covered by a family $\mathfrak{F}$ of closed n cubes (with faces parallel to the coordinate planes) in such a way that, for every $\varepsilon>0$ and $x\in S$ there is a cube in $\mathfrak{F}$ with center x of diagonal less than ε, then there is a countable set of cubes in $\mathfrak{F}$, which are pairwise disjoint, such that almost all of S is contained in their union.*

We also mention the measure β_f which is obtained from the vector valued measure $(\mu_1, ..., \mu_n)$ in the same way that α_f is obtained from $(\lambda, \mu_1, ..., \mu_n)$. The inequality $\alpha_{f+g}(Q)\leqslant \alpha_f(Q)+\beta_g(Q)$ is often useful.

Finally, for every $f\in L_1$, an important set of smoothing functions is the set of integral means

$$f_m(x)=\frac{1}{s_m}\int_{\sigma(x,\,1/m)} f(x+\xi),$$

where $\sigma(x, 1/m)$ is the n ball of center x radius $1/m$ and s_m is its volume. The functions f_m are continuous and converge to f in the L_1 norm as well as almost everywhere. Moreover, if $f\in\mathcal{A}$, then

$$\alpha_f(Q)=\lim_m \alpha_{f_m}(Q).$$

If $f\in\mathcal{A}$ is linearly continuous with respect to a coordinate system and R is an open interval in this coordinate system, it was shown in [10] that

$$\alpha_f(R)=\lim \alpha_{f_n}(R),$$

and if H is a hyperplane and f is linearly continuous then $\alpha_f(H)=0$, for the case $n=2$.

Moreover, if $f\in\mathcal{A}$ is linearly continuous then for almost all lines l, parallel to coordinate axes, we have $\{f_m\}$ converging uniformly on l to f.

4. We fix a coordinate system and say that $f\in\mathcal{L}$ if it is in $\mathcal{A}$ and is linearly continuous in the coordinate system. We suppose that the support of f is interior to a cube

$$Q=\mathop{\times}_{i=1}^{n}[a_i, b_i].$$

In the next few sections, we show that if $f \in \mathfrak{L}$ then, for every $\varepsilon > 0$, there is an approximately continuous g such that, if E is the set for which $f \neq g$, then $\alpha_f(E) < \varepsilon$ and $\alpha_g(E) < \varepsilon$.

As a first step, we note that we may assume f bounded. For every k, the function f^k defined by

$$f^k(x) = \begin{cases} k & \text{if } f(x) \geqslant k \\ f(x) & \text{if } -k < f(x) < k \\ -k & \text{if } f(x) \leqslant -k \end{cases}$$

is linearly continuous. Moreover, it follows directly from the definition of $L(f, Q)$ that $\alpha_{f^k}(Q) \leqslant \alpha_f(Q)$. Hence $f^k \in \mathfrak{L}$.

Let Q_i be a face of Q normal to the x_i direction, $i = 1, \ldots, n$. Let $\delta > 0$. For every $i = 1, \ldots, n$, since f is bounded on almost all lines in every coordinate direction, there is a natural number N_i, and a measurable set $E_i \subset Q_i$ such that $m(Q_i - E_i) < \delta$ and $f^k(x) = f(x)$, on the entire line $l(\bar{x}_i) = [x = (x_i, \bar{x}_i) : x_i \in [a_i, b_i]]$, for every $\bar{x}_i \in E_i$ and $k > N_i$. Then, for $k > \max(N_1, \ldots, N_n)$, $f^k(x) = f(x)$ on the entire set

$$E_\delta = \bigcup_{i=1}^{n} (E_i \times [a_i, b_i]).$$

Choose $\delta > 0$ so small that $\alpha_f(Q - E_\delta) < \varepsilon$. (We indicate below why this is possible). By [9], since f^k and f are both in $\mathfrak{L}$ and $f^k(x) = f(x)$ on E_δ, it follows that $\alpha_f(E_\delta) = \alpha_{f^k}(E_\delta)$. We then have $\alpha_{f^k}(Q - E_\delta) < \alpha_f(Q - E_\delta) < \varepsilon$, proving the desired result.

We next describe a decomposition of $f \in \mathfrak{L}$, given for the 2 dimensional case in [9], which we need in the sequel. Let g be equivalent to f and such that it is continuous in x_i for almost all $\bar{x}_i$, $i = 1, \ldots, n$. Then g is a continuous function, of bounded variation, in x_i for almost all $\bar{x}_i$, $i = 1, \ldots, n$. Choose x_i^0 so that $g(x_i^0, \bar{x}_i)$ is measurable, and summable, in $\bar{x}_i$. We then obtain, as in [9], a decomposition $g = g^+ - g^-$, where g^+ and g^- are measurable, indeed summable, are monotonically non-decreasing in x_i, for almost all $\bar{x}_i$, and $g^+(x_i^0, \bar{x}_i) = g(x_i^0, \bar{x}_i)$ for all $\bar{x}_i \in Q_i$.

Now, let g_m be the mth integral mean of g, $m = 1, 2, \ldots$. As in [9], for almost all $\bar{x}_i$, the integral means $(g^+)_m$ and $(g^-)_m$ converge uniformly in x_i to g^+ and g^-, respectively, for almost all $\bar{x}_i$. Since, $g_m = (g^+)_m - (g^-)_m$, it follows that $\{g_m\}$ converges uniformly in x_i to g, for almost all $\bar{x}_i$, $i = 1, \ldots, n$.

Since the g_m, $m = 1, 2, \ldots$, are continuous, it follows that, for every $\delta > 0$, there is a set

$$E_\delta = \bigcup_{i=1}^{n} (E_i \times [a_i, b_i])$$

on which g is uniformly continuous and on which $f_m = g_m$, $m = 1, 2, \ldots$, converges uniformly to g, with

$$m(Q_i - E_i) < \delta, \quad i = 1, \ldots, n.$$

It is also worthwhile for us to indicate why the fact used above that, for all $\delta > 0$ small enough, $\alpha_f(Q - E_\delta) < \varepsilon$ holds. For every $i = 1, \ldots, n$,

$$|\mu_i|(Q) = \int_{Q_i} V(f, \bar{x}_i, (a_i, b_i)).$$

Let $\varepsilon > 0$. There is a $\delta > 0$, $\delta < \varepsilon/(n+1)$, such that $m(Q_i - E_i) < \delta$ implies

$$|\mu_i|(E_i \times (a_i, b_i)) = \int_{E_i} V(f, \bar{x}_i, (a_i, b_i)) > |\mu_i|(Q) - \frac{\varepsilon}{n+1}.$$

For $E_\delta = \bigcup_{i=1}^n (E_i \times (a_i, b_i))$, we have

$$\begin{aligned}\alpha_f(Q - E_\delta) &\leq \sum_{i=1}^n |\mu_i|(Q - E_\delta) + m(Q - E_\delta) \\ &\leq \sum_{i=1}^n |\mu_i|\{(Q_i - E_i) \times (a_i, b_i)\} + m(Q - E_\delta) \\ &= \sum_{i=1}^n |\mu_i|(Q) - \sum_{i=1}^n |\mu_i|(E_i \times (a_i, b_i)) + m(Q - E_\delta) < \varepsilon.\end{aligned}$$

Remark. In the remainder of this paper, we shall be concerned with the density of a set at a point. This density will always be the ordinary density using cubes containing the point with faces parallel to the coordinate faces, except in one place where balls are used. If the density is 1 in either case, then it is 1 in the other case.

5. Let $f \in \mathfrak{L}$ and $\varepsilon > 0$. We shall obtain a certain 0 dimensional closed set V such that $\alpha_f(V) > \alpha_f(Q) - \varepsilon$.

First, there is a $\delta > 0$ such that if $E_i \subset Q_i$ with $m(E_i) > m(Q_i) - \delta$, $i = 1, \ldots, n$, then

$$\alpha_f(Q - \bigcup_{i=1}^n (E_i \times (a_i, b_i)) < \varepsilon.$$

Let E_i, $i = 1, \ldots, n$, be such that $m(E_i) > m(Q_i) - \delta$, and such that the integral means $\{f_m\}$ of f converge uniformly to f on

$$E = \bigcup_{i=1}^n (E_i \times (a_i, b_i)).$$

For each $i = 1, \ldots, n$, let $S_i \subset E_i$ be the set of points in E_i at which the ordinary $(n-1)$ density of E_i, using cubes in Q_i, is 1. Then, if

$$S = \bigcup_{i=1}^{n} (S_i \times (a_i, b_i)),$$

it follows that $\alpha_f(Q-S) < \varepsilon$. We next let $T_i \subset S_i$ be closed, $i = 1, \ldots, n$, and such that $m(Q_i - T_i) < \delta$. Then, if

$$T = \bigcup_{i=1}^{n} (T_i \times (a_i, b_i)),$$

we have $\alpha_f(Q-T) < \varepsilon$. Moreover, T is a closed set (relative to Q).

We next apply the Besicovitch–Morse version of the Vitali Covering Theorem to the set T to obtain a system of closed cubes. All cubes considered will have their faces parallel to the faces of Q. For each $x \in T$, interior to Q, let $\mathcal{R}_x$ be a collection of closed cubes, each in the interior of Q, with x as center, and with diagonal less than $\frac{1}{2}$, such that for each $\eta > 0$ there is a cube in $\mathcal{R}_x$ with diagonal less than η. Let

$$\mathcal{R} = \bigcup [\mathcal{R}_x : x \in (\operatorname{int} Q) \cap T].$$

There is a finite set $$R_1, R_2, \ldots, R_{n_1},$$

of pairwise disjoint closed cubes in the collection $\mathcal{R}$ such that

$$\alpha_f\{Q - \bigcup_{j_1=1}^{n_1} (R_{j_1} \cap T)\} < \varepsilon.$$

For every $j_1 = 1, \ldots, n_1$, and $x \in (\operatorname{int} R_j) \cap T$, let $\mathcal{R}_x^{j_1}$ be a collection of closed cubes, each in the interior of R_{j_1}, with x as center, and with diagonal less than $1/2^2$, such that for each $\eta > 0$, there is a cube in $\mathcal{R}_x^{j_1}$ with diagonal less than η.

Let $$\mathcal{R}^{j_1} = \bigcup [\mathcal{R}_x^{j_1} : x \in (\operatorname{int} R_{j_1}) \cap T].$$

There is a finite set $$R_{j_1 1}, \ldots, R_{j_1 n_{j_1}}, \quad j_1 = 1, \ldots, n_1,$$

of pairwise disjoint closed cubes in the collection $\mathcal{R}^{j_1}$ such that

$$\alpha_f\{Q - \bigcup_{j_1=1}^{n_1} \bigcup_{j_2=1}^{n_{j_1}} (R_{j_1 j_2} \cap T)\} < \varepsilon.$$

By continuing in this way, and applying the Besicovitch–Morse covering theorem an infinite number of times we obtain a system

$$\begin{aligned} R_{j_1 \ldots j_k}, \quad & k = 1, 2, \ldots, j_1 = 1, \ldots, n_1, \\ & \text{for each } j_1, j_2 = 1, \ldots, n_{j_1}, \\ & \text{for each } j_1 j_2, j_3 = 1, \ldots, n_{j_1 j_2}, \\ & \ldots \\ & \text{for each } j_1 \ldots j_{k-1}, j_k = 1, \ldots, n_{j_1 \ldots j_{k-1}}, \\ & \ldots \end{aligned}$$

of closed cubes. Each $R_{j_1 \dots j_{k-1} j_k}$ is contained in the interior of $R_{j_1 \dots j_{k-1}}$ and is of diagonal less than $1/2^k$. The cubes $R_{j_1 \dots j_k}$ are pairwise disjoint, for every $k = 1, 2, \dots$. Moreover, for every $k = 1, 2, \dots$, we have

$$\alpha_f(Q - \bigcup (R_{j_1 \dots j_k} \cap T)) < \varepsilon,$$

where the k-tuples $j_1 \dots j_k$ vary over the finite set of possibilities.

The cubes $R_{i_1 \dots i_k}$ chosen at the kth stage will be designated as of rank k. Then every cube of rank $k+1$ is contained in the interior of a cube of rank k.

For every $k = 1, 2, \dots$, let

$$V_k = \bigcup (R_{j_1 \dots j_k} \cap T)$$

and let

$$V = \bigcap_{k=1}^{\infty} V_k.$$

Since each $R_{j_1 \dots j_k} \cap T$ is non-empty, and T is closed, it follows that $V \subset T$. We show that

$$V = \bigcap_{k=1}^{n} (\bigcup R_{j_1 \dots j_k}).$$

Clearly, $V \subset \bigcap_{k=1}^{\infty} (\bigcup R_{j_1 \dots j_k})$.

Conversely, if $x \in \bigcap_{k=1}^{\infty} (\bigcup R_{j_1 \dots j_k})$, then x is a limit point of T. Since T is closed, $x \in T$, so that $x \in \bigcup (R_{j_1 \dots j_k} \cap T) = V_k$, for every $k = 1, 2, \dots$. Hence $x \in V$.

Finally, since $\alpha_f(Q - V_k) < \varepsilon$, for every $k = 1, 2, \dots$, it follows that $\alpha_f(Q - V) \leqslant \varepsilon$.

In the above construction, all cubes involved may be chosen so that $\{f_m\}$ converges uniformly to f on almost all line segments on each of their boundary faces which are in any coordinate direction. We make certain that they are so chosen.

6. The density of S is 1 at every point of T, hence of V. The boundaries of the $R_{j_1 \dots j_k}$ are pairwise disjoint. We consider a closed frame about the boundary of each $R_{j_1 \dots j_k}$ so that the frames are pairwise disjoint and their union has density 0 at each point of V. We accomplish this in the following way.

For a cube R, with boundary ∂R, the frame of width h about ∂R is the closed set between 2 closed cubes concentric with R, the larger one having edges which exceed those of R by h, and the smaller one having edges which are exceeded by those of R by h. Since the boundaries of the $R_{i_1 \dots i_k}$ are pairwise disjoint we may put frames $G_{i_1 \dots i_k}$ about the sets $\partial R_{i_1 \dots i_k}$ in such a way that the frames are pairwise disjoint and also meet no $\partial R_{i_1 \dots i_k i_{k+1}}$. Thus we may choose the $G_{i_1 \dots i_k}$ so that, even for different k, they are pairwise disjoint. A frame about the boundary of a cube of rank k will be called a frame of rank k.

For every $k=1, 2, \ldots$, there is an M_k such that if R is any cube such that at least one set $R \cap R_{i_1 \ldots i_k i_{k+1}}$ is non-empty and the set $R \cap G_{i_1 \ldots i_k}$ is non-empty, then the measure of the portion of R which is inside $G_{i_1 \ldots i_k}$ but outside all the $G_{i_1 \ldots i_k i_{k+1}}$ exceeds M_k.

Let $\{\varepsilon_k\}$ be a sequence of positive numbers such that $\sum_{k=1}^{\infty} \varepsilon_k < \infty$. Shrink each frame $G_{i_1 \ldots i_k}$ to a frame $F_{i_1 \ldots i_k}$ about $\partial R_{i_1 \ldots i_k}$ so that, for every $k=1, 2, \ldots$, the sum of the measures of the $F_{i_1 \ldots i_k}$ are less than $\varepsilon_k M_k$. Now, let $x \in V$ and let R be a cube containing x. Let r be the smallest rank of frames $G_{i_1 \ldots i_r}$ which meet R. Then,

$$m(R) > M_r + M_{r+1} + \ldots.$$

But,
$$m(R \cap (\bigcup F_{i_1 \ldots i_k})\} < \varepsilon_r M_r + \varepsilon_{r+1} M_{r+1} + \ldots,$$

so that
$$\frac{m\{R \cap (\bigcup F_{i_1 \ldots i_k})\}}{m(R)} < \sum_{k=r}^{\infty} \varepsilon_r.$$

Since r goes to infinity as the diagonal of R goes to 0, it follows that $\bigcup F_{i_1 \ldots i_k}$ has density 0 at x. In other words, the density of $S - \bigcup F_{i_1 \ldots i_k}$ is 1 at every $x \in V$.

7. We need the following lemma which was proved for $n=2$ in [10].

LEMMA 1. *If $f \in \mathfrak{L}$ and I is an open interval then*

$$\lim_m \alpha_{f_m}(I) = \alpha_f(I).$$

Proof. We first show that if H is any hyperplane parallel to a coordinate hyperplane then $\alpha_f(H)=0$. Let H be normal to the x_i direction. Then, for every $j \neq i$, $|\mu_j|(H)=0$, since $|\mu_j|$ is defined as the integral of a variation function with respect to $(n-1)$ dimensional Lebesgue measure, and $|\mu_j|(H)$ is this integral on a set of $(n-1)$ measure zero. Moreover, $|\mu_i|(H)=0$ since, for almost all points $(x_i^0, \bar{x}_i)$ in H, f is continuous in the x_i direction. Then, using the Lebesgue dominated convergence theorem, we obtain

$$|\mu_i|(H) = \lim_{\varepsilon \to 0} \int d\bar{x}_i \cdot V(f, \bar{x}_i, (x_i^0 - \varepsilon, x_i^0 + \varepsilon)) = 0.$$

It follows that
$$\alpha_f(H) \leqslant m(H) + \sum_{i=1}^{n} |\mu_i|(H) = 0.$$

It is known (e.g. [8]) that for every $f \in \mathcal{A}$ and every interval J, whose closure is contained in I, that $\lim_m' \alpha_{f_m}(I) \geqslant \alpha_f(J)$, and that, for every open interval K, which contains the closure of I, $\lim_m \alpha_{f_m}(I) \leqslant \alpha_f(K)$. Since for $f \in \mathfrak{L}$, $\alpha_f(\partial I)=0$, for every $\eta > 0$ there are J and K as above such that $\alpha_f(J) > \alpha_f(K) - \eta$. We then have

$$|\lim_m \alpha_{f_m}(I) - \alpha_f(I)| < \eta,$$

for every $\eta > 0$, proving the lemma.

It is easy to give an example of an $f \in \mathcal{A}$ for which the lemma fails to hold.

8. We now consider the restriction of the function f to the set V defined in § 5. We obtain an extension g of this function to Q which is approximately continuous at every $x \in V$ and is continuous at every $x \in Q - V$, except at points on the boundaries of the cubes $R_{i_1 \dots i_k}$. Let $Q_k = \bigcup R_{i_1 \dots i_k}$, $k = 1, 2, \dots$. Then Q_k is the union of a finite set of pairwise disjoint closed cubes. Suppose $|f(x)| < M$, for every $x \in Q$, and let A_k be the $(n-1)$-dimensional Lebesgue area of ∂Q_k, $k = 1, 2, \dots$. We consider sequences $\{\eta_k\}$, $\{\zeta_k\}$, and $\{\xi_k\}$ of positive numbers such that

$$\sum_{k=1}^{\infty} \xi_k < \varepsilon, \quad \sum_{k=1}^{\infty} \eta_k A_k < \varepsilon \quad \text{and} \quad \sum_{k=1}^{\infty} 2\,\zeta_k M < \varepsilon.$$

We first define g on $Q -$ int Q_1 as an integral mean f_{m_1} of f, where m_1 is chosen so that

a$_1$) $|f_{m_1}(x) - f(x)| < 1$ for every $x \in S$,
b$_1$) $|f_{m_1}(x) - f_m(x)| < \eta_1$ on all of ∂Q_1,

except for a subset of $(n-1)$ dimensional measure less than ζ_1, for each $m > m_1$, and

c$_1$) $\alpha_{f_{m_1}}(Q - \text{int}\, Q_1) < \alpha_f(Q - \text{int}\, Q_1) + \xi_1$.

That m_1 can be chosen so that a$_1$) holds follows from the uniform convergence of $\{f_m\}$ to f on S, that it may be chosen so that b$_1$) holds follows from the choice of the $R_{i_1 \dots i_k}$ so that $\{f_m\}$ converges uniformly on almost all lines on their boundaries which are in coordinate directions. For c$_1$), we appeal to Lemma 1.

We next define g on $Q_1 - \text{int}\, Q_2$ as an integral mean f_{m_2} of f, where $m_2 > m_1$ is chosen so that

a$_2$) $|f_{m_2}(x) - f(x)| < \frac{1}{2}$ for every $x \in S$,
b$_1$) $|f_{m_2}(x) - f_m(x)| < \eta_2$ on all of ∂Q_2,

except for a subset of $(n-1)$ dimensional measure less than ζ_2, for each $m > m_2$, and

c$_2$) $\alpha_{f_{m_2}}(Q_1 - \text{int}\, Q_2) < \alpha_f(Q_1 - \text{int}\, Q_2) + \xi_2$.

We continue by induction. Having defined $m_1 < m_2 < \dots < m_{k-1}$, and defined g on $Q_{i-1} - \text{int}\, Q_i$ as f_{m_i}, $i = 1, \dots, k-1$, we define g on $Q_k - \text{int}\, Q_{k-1}$ as f_{m_k}, where $m_k > m_{k-1}$ is chosen so that

$\mathrm{a}_k)$ $|f_{m_k}(x)-f(x)|<1/k$, for every $x\in S$,

$\mathrm{b}_k)$ $|f_{m_k}(x)-f_m(x)|<\eta_k$ on all of ∂Q_k, except for a subset of $(n-1)$ dimensional measure less than ζ_k for each $m>m_k$, and

$\mathrm{c}_k)$ $\alpha_{f_{m_k}}(Q_{k-1}-\operatorname{int} Q_k)<\alpha_f(Q_{k-1}-\operatorname{int} Q_k)+\xi_k$.

In this way, g is defined on all of Q, and is equal to f on V, since

$$Q-V=\bigcup_{k=1}^{\infty}(Q_{k-1}-\operatorname{int} Q_k),$$

where $Q=Q_0$. The function g is double valued on ∂Q_k for every k. We select either of the two continuous branches. The set for which $f(x)\neq g(x)$ is a subset of $Q-V$. Now

$$\alpha_g(Q-V)=\sum_{k=1}^{\infty}\alpha_{f_{m_k}}(Q_{k-1}-\operatorname{int} Q_k)+\sum_{k=1}^{\infty}\alpha_g(\partial Q_k)$$

$$\leqslant\sum_{k=1}^{\infty}\alpha_f(Q_{k-1}-\operatorname{int} Q_k)+\sum_{k=1}^{\infty}\xi_k+\sum_{k=1}^{\infty}\alpha_g(\partial Q_k).$$

But

$$\sum_{k=1}^{\infty}\alpha_f(Q_{k-1}-\operatorname{int} Q_k)\leqslant\alpha_f(Q-V)<\varepsilon,$$

and $\sum_{k=1}^{\infty}\xi_k<\varepsilon$. Moreover,

$$\sum_{k=1}^{\infty}\alpha_g(\partial Q_k)=\sum_{k=1}^{\infty}\beta_g(\partial Q_k)\leqslant\sum_{k=1}^{\infty}\eta_k A_k+\sum_{k=1}^{\infty}2M\zeta_k\leqslant 2\varepsilon,$$

since $|f_{m_{k-1}}(x)-f_{m_k}(x)|<\eta_k$ for all $x\in\partial Q_k$ except for a subset of measure less than ζ_k, on which $|f_{m_{k-1}}(x)-f_{m_k}(x)|<2M$. This shows that $\alpha_g(Q-V)<4\varepsilon$.

The function g we have constructed is approximately continuous at every $x\in V$. For this we first note that S has density 1 at x and f is continuous at x relative to S. So, let $\varepsilon>0$ and let $\delta>0$ be such that if $y\in S$ and the distance from y to x is less than δ, then $|f(y)-f(x)|<\varepsilon/2$. Moreover, let $1/k<\varepsilon/2$. There is an $R_{j_1\ldots j_k}$ such that $x\in\operatorname{int} R_{j_1\ldots j_k}$. Then $y\in S\cap\operatorname{int} R_{j_1\ldots j_k}$ implies

$$|g(y)-g(x)|\leqslant|g(y)-f(y)|+|f(y)-f(x)|<\frac{1}{k}+\frac{\varepsilon}{2}<\varepsilon.$$

Thus, g is approximately continuous at x.

It is clear, by the construction, that g is continuous at every point in $Q-V$, except at points in ∂Q_k, $k=1,2,\ldots$. On ∂Q_k, we may define g as either the continuous function $f_{m_{k-1}}$ or the continuous function f_{m_k}. The contribution of ∂Q_k to α_g is the n dimensional measure of the set between these continuous surfaces.

9. We now show how the discontinuities on the sets ∂Q_k, $k=1, 2, \ldots$, may be removed. We have a system $F_{i_1 \ldots i_k}$ of pairwise disjoint frames about the boundaries $\partial R_{i_1 \ldots i_k}$ so that the density of $\bigcup F_{i_1 \ldots i_k}$ is 0 at each $x \in V$. Then, any modification of g within the frames will not change the approximate continuity of g at the points of V. We let $R = R_{i_1 \ldots i_k}$ be an arbitrary cube in our system, and let F be the corresponding frame about ∂R. The technique we use is to transfer the discontinuities of g from the surface ∂R to the surfaces of two other cubes, but with the saltus of the function greatly reduced. By repeating the operation, we obtain a sequence of functions which converges uniformly to a continuous function h in F, which agrees with g on the boundary of F, and is such that $\alpha_h(F)$ is not much greater than $\alpha_g(F)$. We proceed with the details.

Let $\eta > 0$ and let $\sum_{k=1}^{\infty} \zeta_k < \infty$ and $\sum_{k=1}^{\infty} \xi_k < \eta$. Now, let A and B be cubes concentric with R, with ∂A and ∂B in the interior of F and with

$$A \subset \operatorname{int} R \quad \text{and} \quad R \subset \operatorname{int} B.$$

Then let $\{A_m\}$ be a decreasing sequence of cubes, and $\{B_m\}$ an increasing sequence of cubes, all concentric with R, such that

$$\operatorname{int} R \supset A_1, \quad \operatorname{int} B_1 \supset R$$

$$\operatorname{int} A_m \supset A, \quad \operatorname{int} B \supset B_m, \quad m=1, 2, \ldots,$$

and

$$A_{m+1} \subset \operatorname{int} A_m, \quad B_m \subset \operatorname{int} B_{m+1}, \quad m=1, 2, \ldots.$$

Let H_m be the frame bounded by ∂A_m and ∂B_m, $m=1, 2, \ldots$, and let H be the frame bounded by ∂A and ∂B. Then $\{H_m\}$ is an increasing sequence of frames about ∂R, all contained in the frame H, which in turn is contained in the interior of the frame F.

We define a sequence $\{g^m\}$ of functions on F. Let $g^1 = g$ on $F - H_1$. On H_1, let $g^1 = g_{k_1}$, where k_1 is chosen so large that

α_1) $|g(x) - g_{k_1}(x)| < \zeta_1$ on the boundary of H_1,

β) $\alpha_{g^1}(\operatorname{int} H_1) < \alpha_g(\operatorname{int} H_1) + \xi_1$,

γ_1) $\alpha_{g^1}(\partial H_1) < \alpha_g(\partial H_1) + \xi_1$,

and for every $x \in F$,

δ_1) $|g^1(x) - g(x)| < 2K$, where K is the maximum of the saltus of g on ∂R.

In particular, we have $\alpha_{g^1}(F) < \alpha_g(F) + 2\xi_1$, and the saltus of g^1 does not exceed ζ_1 at any point of F.

Let $g^2 = g^1$ on $F - H_2$. On H^2, let $g^2 = g^1_{k_2}$, where k_2 is chosen so large that

α_2) $|g^1(x) - g^1_{k_2}(x)| < \zeta_2$ on the boundary of H_2,

$\beta_2)\quad \alpha_{g^2}(\operatorname{int} H_2) < \alpha_{g^1}(\operatorname{int} H_2) + \xi_2,$

$\gamma_2)\quad \alpha_{g^2}(\partial H_2) < \alpha_{g^1}(\partial H_2) + \xi_2,$

and, for every $x \in F$,

$\delta_2)\quad |g^2(x) - g^1(x)| < 2\zeta_1.$

In particular, we have $\alpha_{g^2}(F) < \alpha_{g^1}(F) + 2\zeta_2$, and the saltus of g^2 does not exceed ζ_2 at any point of F.

By continuing, we obtain a sequence $\{g^m\}$ of functions, all continuous and agreeing with each other on $F - H$, such that

(i) g^m is discontinuous only on ∂H_m,
(ii) $|g^{m+1}(x) - g^m(x)| < 2\zeta_m$, for every $x \in F$,
(iii) $\alpha_{g^m}(F) < \alpha_{g^{m-1}}(F) + 2\zeta_m$, and
(iv) the saltus of g^m does not exceed ξ_m at any point of F.

Since $\sum_{m=1}^{\infty} \zeta_m < \infty$, the sequence $\{g^m\}$ converges uniformly to a function h. We note that h is continuous on F, since for every $x \in F$, and every m, the saltus of h at x is less than

$$t_m = \zeta_m + \sum_{r=m}^{\infty} 2\,\zeta_m.$$

Since $\{t_m\}$ is a null sequence, the saltus of h at x is zero.

Moreover, by lower semicontinuity,

$$\alpha_h(F) \leqslant \liminf_m \alpha_{g^m}(F) < \alpha_g(F) + 2\sum_{m=1}^{\infty} \xi_m < \alpha_g(F) + 2\eta.$$

Having established this, we now order and relable the countable set of frames $F_{j_1 \dots j_k}$ as

$$F_1,\ F_2,\ \dots,\ F_m,\ \dots.$$

Let $\eta_m > 0$ be such that $\sum_{m=1}^{\infty} \eta_m < \varepsilon$. For each $m = 1, 2, \dots$, modify g in the frame F_m, as in the above construction, so that

$$\alpha_h(F_m) < \alpha_g(F_m) + \eta_m.$$

The resulting function h defined on Q is equal to f on V, is approximately continuous at every $x \in V$, is continuous at every $x \in Q - V$, and

$$\alpha_h(Q) < \alpha_f(Q) + 5\varepsilon.$$

This completes the proof of the direct part of our main theorem. Specifically, we have proved:

For every $f \in \mathcal{L}$ and $\varepsilon > 0$ there is an approximately continuous g such that if $E = [x: f(x) \neq g(x)]$ then $\alpha_f(E) < \varepsilon$ and $\alpha_g(E) < \varepsilon$.

10. We now turn to a proof of the converse. In this connection, the main fact is that if $f \in \mathcal{A}$ is approximately continuous then $f \in \mathcal{L}$. We consider $f \in \mathcal{A}$ with support in the interior of a cube Q. For a given $i = 1, \ldots, n$, we consider the linear Blumberg measurable boundary u, as defined in § 3, in direction x_i. Let $Q = Q_i \times (a, b)$. For each $\bar{x}_i \in Q_i$ and $x_i \in (a, b)$, let $V(x) = V(x_1, \bar{x}_i)$ be the variation of $u(x_i, \bar{x}_i)$ on the interval (a, x_i). Then V is a measurable function on Q. To show this, we proceed as before and let $a_1 < a_2 < \ldots < a_k$ be such that $\max(a_1 - a, a_2 - a_1, \ldots, b - a_k) < 1/m$, and the $u(a_j, \bar{x}_i)$ are measurable for $j = 1, \ldots, k$. Then let

$$v_m(x_i, \bar{x}_i) = \sum_{a_j < x_i} |u(a_j, \bar{x}_i) - u(a_{j-1}, \bar{x}_i)|.$$

The functions v_m are measurable and

$$V(x) = \lim_m v_m(x).$$

Suppose now that $f \in \mathcal{A}$ but $f \notin \mathcal{L}$. Then there is an $i = 1, \ldots, n$ for which the associated $V(x)$ is discontinuous at a set of points in Q_i which is not of measure 0. For any pair of reals s and t, $s < t$, let $T = [x: V(x) < s]$ and $U = [x: V(x) > t]$. Then T and U are measurable sets. We associate functions ϕ and ψ, defined on Q_i, with the sets T and U. For each $\bar{x}_i \in Q_i$, let

$$\phi(\bar{x}_i) = \sup [x_i: (x_i, \bar{x}_i) \in T] \quad \text{and} \quad \psi(\bar{x}_i) = \inf [x_i: (x_i, \bar{x}_i) \in U].$$

Then $\phi(\bar{x}_i) \leqslant \psi(\bar{x}_i)$, for every $\bar{x}_i \in Q_i$. We note that the functions ϕ and ψ are measurable. Since their measurability does not seem to be standard knowledge, we indicate a proof.

The measurable set T is the union of line segments $l(\bar{x}_i)$, with end points $(a, \bar{x}_i)$ and $(\phi(x_i), \bar{x}_i)$. Let $\eta > 0$. There are intervals

$$I_1, \ldots, I_m; J_1, J_2, \ldots, K_1, K_2, \ldots,$$

such that

$$\sum_{j=1}^{\infty} m(J_j) < \eta \quad \text{and} \quad \sum_{j=1}^{\infty} m(K_j) < \eta,$$

with

$$T \subset \Big(\bigcup_{j=1}^{m} I_j\Big) \cup \Big(\bigcup_{j=1}^{\infty} J_j\Big) \quad \text{and} \quad T \supset \Big(\bigcup_{j=1}^{m} I_j\Big) - \Big(\bigcup_{j=1}^{\infty} K_j\Big),$$

(e.g. [12]). Associate with each interval $I = I_j \times (c, d)$, the function ζ defined by $\zeta(\bar{x}_i) = 0$, $\bar{x}_i \notin I_j$ and $\zeta(\bar{x}_i) = d - c$, $\bar{x}_i \in I_j$. Now, let r_j be the function associated with I_j, $j = 1, \ldots, m$, s_j the function associated with J_j, $j = 1, 2, \ldots$, and t_j the function associated with K_j, $j = 1, 2, \ldots$, in the above manner. Then, let

$$\phi_1 = \sum_{j=1}^{m} r_j + \sum_{j=1}^{\infty} s_j, \quad \text{and} \quad \phi_2 = \sum_{j=1}^{m} r_j - \sum_{j=1}^{\infty} t_j.$$

The functions ϕ_1 and ϕ_2 are measurable and summable, $\phi_1(\bar{x}_i) \leqslant \phi(\bar{x}_i) \leqslant \phi_2(\bar{x}_i)$, for every $\bar{x}_i \in Q_i$, and $\int_{Q_i} \phi_2 < \int_{Q_i} \phi_1 + 2\eta$. Since this holds for every $\eta > 0$, ϕ is measurable. Similarly, ψ is measurable.

The set for which $\phi(\bar{x}_i) = \psi(\bar{x}_i)$ is measurable. Since $V(x) = V(x_i, \bar{x}_i)$ is monotonically non-decreasing in x_i, for every $\bar{x}_i$, and is discontinuous for some x_i for each $\bar{x}_i$ in a set which is not of measure 0, there is a pair s, t, with $s < t$, for which the corresponding functions ϕ and ψ are such that the set

$$A = [\bar{x}_i : \phi(\bar{x}_i) = \psi(\bar{x}_i)]$$

has positive measure. We shall henceforth assume this holds and consider the function ϕ defined on the set A.

11. We need a fact about approximate differentiability of measurable functions. For the one variable case, such a result has been obtained by Burkill and Haslam-Jones [3], and for the two variables case, the kind of result we need follows from a theorem of Ward [24]. It seems that the methods of neither author extend readily to n dimensions. We are able to give a simple proof of the fact we need.

Let f be a bounded measurable function defined on a measurable set A of positive measure. There is an increasing sequence $\{A_m\}$ of closed sets such that f is continuous on each A_m and $m(\bigcup_{n=1}^{\infty} A_n) = m(A)$. Let $k > 0$, and for each $x \in A$, let

$$B_x^{(k)} = \left[y : \frac{f(y) - f(x)}{|y - x|} < k \right].$$

Let $\sigma(x, r)$ be the notation for an open ball of center x and radius r. Let $0 < \phi < 1$. Let $r_0 > 0$. Let $G \subset A$ be the set of points $x \in A$ such that, for some $r < r_0$, the relative measure of $B_x^{(k)}$ in $\sigma(x, r)$ exceeds ϕ. Let G_m be the set of points $x \in A_m$ such that for some $r < r_0$ the relative measure of $B_x^{(k)} \cap A_m$ in $\sigma(x, r)$ exceeds ϕ. Then

$$G = \left(\bigcup_{m=1}^{\infty} G_m \right) \cup Z,$$

where Z has measure 0. Each G_m is easily seen to be open relative to A_m, so that G is measurable.

This remark implies that if, for each $x \in A$,

$$A_x = \left[y : \frac{f(y) - f(x)}{|y - x|} \geqslant k \right],$$

then the set of points $x \in A$ at which the density of A_x is 1 is measurable, and for every $0<\theta<1$ and $r_0>0$ the set of points $x \in A$ for which, for all $r<r_0$, the relative measure of A_x in $\sigma(x, r)$ is at least θ is measurable.

We now let $\theta<1$ be so large that there is a $\phi(\theta)$, independent of r, such that if $x \in \sigma(y, r)$, $y \in \sigma(x, r)$, $S \subset \sigma(x, r)$ has relative measure in $\sigma(x, r)$ at least θ, and $T \subset \sigma(y, r)$ has relative measure in $\sigma(y, r)$ at least θ, then the diameter of $S \cap T$ exceeds $\phi(\theta) r$.

Suppose the set of points $x \in A$ at which the density of A_x is 1 has positive measure. Then, there is an $r_0>0$, such that the set E of points x such that, for every $r<r_0$ the relative measure of A_x in $\sigma(x, r)$ is at least θ has positive measure. Let $x \in A$ be a point at which the density of E is 1. There is an $r<r_0$ such that the relative measure of E in $\sigma(x, r)$ exceeds θ. Let $y \in E \cap \sigma(x, r)$ be such that

$$f(y) > \sup\,[f(z): z \in E \cap \sigma(x, r)] - \frac{k\phi(\theta) r}{2}.$$

Now, A_y has relative measure at least θ in $\sigma(y, r)$. The diameter of the set

$$A_y \cap \sigma(y, r) \cap E \cap \sigma(x, r)$$

accordingly exceeds $\phi(\theta) r$. There is thus a u in this set such that $|y-u| > \phi(\theta) r/2$. Then

$$f(u) - f(y) \geqslant k\,|u-y| > \frac{k\phi(\theta) r}{2}.$$

But then $f(u) > \sup\,[f(z): z \in E \cap \sigma(x, r)]$. Since $u \in E \cap \sigma(x, r)$, this contradiction shows that the set of points $x \in A$ at which the density of A_x is 1 has measure 0. By considering the function $-f$ we obtain a similar result regarding sets at which $(f(y)-f(x))/|y-x| < -k$. Finally, the boundedness restriction is redundant. We thus have the

LEMMA 2. *If f is a measurable function of n variables on a measurable set A, then for every $k>0$, for almost all $x \in A$, the upper densities at x, of the sets for which $(f(y)-f(x))/|y-x| > -k$ and $(f(y)-f(x))/|y-x| < k$, are positive.*

Remark. In contrast with Lemma 2 it is possible for the approximate limit of $|f(y)-f(x)|/|y-x|$ to be infinite almost everywhere.

12. We now show that $f \in \mathcal{A}$ and $f \notin \mathcal{L}$ implies that there is an x at which f is not approximately continuous. We start with the measurable function ϕ defined on the set $A \subset Q_i$ of positive measure. For every $\bar{x}_i \in A$,

$$\Big|\lim_{x_i \to \phi(\bar{x}_i)+} u(x_i, \bar{x}_i) - \lim_{x_i \to \phi(\bar{x}_i)-} u(x_i, \bar{x}_i)\Big| \geqslant t - s.$$

We may assume then that, for every $\bar{x}_i$, in a subset of A of positive outer measure,

$$\lim_{x_i \to \phi(\bar{x}_i)+} u(x_i, \bar{x}_i) \geqslant \lim_{x_i \to \phi(\bar{x}_i)-} u(x_i, \bar{x}_i) + (t-s).$$

Letting $q = \frac{1}{2}(t-s)$, there is then a subset $B \subset A$ of positive outer measure, $p > 0$, and r real, such that, for every $\bar{x}_i \in B$, we have

$$u(x_i, \bar{x}_i) < r, \quad \text{for} \quad x_i \in (\phi(\bar{x}_i) - p), \phi(\bar{x}_i))$$

and

$$u(x_i, \bar{x}_i) > r + q, \quad \text{for} \quad x_i \in (\phi(\bar{x}_i), \phi(\bar{x}_i) + p).$$

We now choose $k > 0$ so small that, for every cube $R = \times_{j=1}^{n} [x_j - h, x_j + h]$, $h > 0$, the conical surface

$$y_i = x_i + k\{\sum_{j \neq i} (x_j - y_j)^2\}^{\frac{1}{2}},$$

with vertex x, meets the lateral faces of R at ordinates $y_i < x_i + h/2$, i.e., at a distance more than $h/2$ from the upper face of R. The choice $k = 1/2n$ will accomplish this.

Let $\bar{x}_i \in B$ be a point of approximate continuity of ϕ, relative to A, using $(n-1)$ cubes in Q_i, and a point of outer density 1 of B, and such that the upper densities of the sets

$$D_{\bar{x}_i} = \left[\bar{y}_i : \frac{\phi(\bar{y}_i) - \phi(\bar{x}_i)}{|\bar{y}_i - \bar{x}_i|} < k\right],$$

and

$$D'_{\bar{x}_i} = \left[\bar{y}_i : \frac{\phi(\bar{y}_i) - \phi(\bar{x}_i)}{|\bar{y}_i - \bar{x}_j|} > -k\right]$$

at $\bar{x}_i$ are positive, say greater than $3w > 0$. There is an $h_0 > 0$, $h_0 < p/2$, such that for any $(n-1)$ cube $I \subset Q_i$, with center $\bar{x}_i$ and edge less than h_0, the relative measure in I of the set C for which $|\phi(\bar{y}_i) - \phi(\bar{x}_i)| < p/2$ exceeds $1-w$, and the relative outer measure of B in I exceeds $1-w$. There is a sequence $\{I_m\}$ of $(n-1)$ cubes, with edges smaller than h_0, and converging to zero, in each of which the relative measure of $D_{\bar{x}_i}$ exceeds $3w$, and another such sequence $\{I'_m\}$, in each of which the relative measure of $D'_{\bar{x}_i}$ exceeds $3w$. The relative outer measure of $B \cap C \cap D_{\bar{x}_i}$ exceeds w in each I_m, and the relative outer measure of $B \cap C \cap D'_{\bar{x}_i}$ exceeds w in each I'_m.

For each I_m, we consider the n cube K_m with center $(\phi(\bar{x}_i), \bar{x}_i)$ and projection I_m in Q_i, and for each I_m, we consider the n cube K'_m with center $(\phi(\bar{x}_i), \bar{x}_i)$ and projection I'_m in Q_i.

Let $S = B \cap C \cap D_{\bar{x}_i}$ and $T = B \cap C \cap D'_{\bar{x}_i}$; also let h_m be the edge of K_m and h'_m the edge of K'_m, $m = 1, 2, \ldots$. For each $\bar{y}_i \in I_m \cap S$, we have

$$\phi(\bar{x}_i) - \frac{p}{2} < \phi(\bar{y}_i) < \phi(\bar{x}_j) + \frac{h_m}{4}.$$

Now, since $\bar{y}_i \in B$, $u(y_i, \bar{y}_i) > r + q$ for all $y \in (\phi(\bar{y}_i), \phi(\bar{y}_i) + p)$. It follows that $u(y_i, \bar{y}_i) > r + q$, for all $y_i \in (\phi(\bar{x}_i) + h/4, \phi(\bar{x}_i) + h/2)$. The relative outer measure in I_m of the set for which $u(x) > r + q$ then exceeds $\frac{1}{4}w$, $m = 1, 2, \ldots$.

Similarly, for every $\bar{y}_i \in I'_m \cap T$, we have

$$\phi(\bar{x}_i) + \frac{p}{2} > \phi(\bar{y}_i) > \phi(\bar{x}_i) - \frac{h_m}{4}.$$

Also, $u(y_i, \bar{y}_i) < r$ for all $y_i \in (\phi(\bar{y}_i) - p, \phi(\bar{y}_i))$, so that $u(y_i, \bar{y}_i) < r$, for all $y_i \in (\phi(\bar{x}_i) - h_m/2, \phi(\bar{x}_i) - h_m/4)$. The relative outer measure in I'_m of the set for which $u(x) < r$ then exceeds $\frac{1}{4}w$, $m = 1, 2, \ldots$. Thus, since u is equivalent to f, f is not approximately continuous at $(\phi(\bar{x}_i), \bar{x}_i)$. We thus have

LEMMA 3. *If $f \in \mathcal{A}$ is approximately continuous then $f \in \mathcal{L}$.*

13. In order to complete the proof of our theorem, it remains only to show that if $f \in \mathcal{A}$ and $f \notin \mathcal{L}$, then there is a $k > 0$ such that, for every $g \in \mathcal{L}$, if $E = [x : f(x) \neq g(x)]$, then $\alpha_g(E) > k$. Just as in the above discussion, there is a measurable $A \subset Q_i$, of positive measure, a measurable function ϕ on A, and $q > 0$ such that, for every $\bar{x}_i \in A$,

$$\Big| \lim_{x_i \to \phi(\bar{x}_i)+} u(x_i, \bar{x}_j) - \lim_{x_i \to \phi(\bar{x}_i)-} u(x_i, \bar{x}_i) \Big| > q.$$

There is then a $B \subset A$ of positive outer measure $\mu > 0$ and a $p > 0$ such that the saltus of u is less than $q/4$ on each of the intervals

$$(\phi(\bar{x}_i) - p, \phi(\bar{x}_i)) \quad \text{and} \quad (\phi(\bar{x}_i), \phi(\bar{x}_i) + p)$$

for every $\bar{x}_i \in B$. Let

$$k = \min\left(\mu p, \frac{\mu q}{4}\right).$$

Suppose $g \in \mathcal{L}$, and let $E = [x : f(x) \neq g(x)]$. Let $C \subset B$ be the subset for which at least one of $E \supset (\phi(\bar{x}_i) - p, \phi(\bar{x}_i))$ or $E \supset (\phi(\bar{x}_i), \phi(\bar{x}_i) + p)$ holds, and let $D = B - C$. The outer measure of either C or D is at least $\frac{1}{2} m_e(B) = \frac{1}{2}\mu$.

a) Suppose $m_e(C) \geqslant \frac{1}{2}\mu$. Then

$$\alpha_g(E) \geqslant m(E) \geqslant 2p \cdot \tfrac{1}{2}\mu \geqslant k.$$

b) Suppose $m_e(D) \geqslant \frac{1}{2}\mu$.

For each $\bar{x}_i \in D$, there are $x_i \in (\phi(\bar{x}_i) - p, \phi(\bar{x}_i))$ and $y_i \in (\phi(\bar{x}_i), \phi(\bar{x}_i) + p)$, such that $u(x_i, \bar{x}_i) = g(x_i, \bar{x}_i)$ and $u(y_i, \bar{x}_i) = g(y_i, \bar{x}_i)$. There is then a subinterval (a, b) of (x_i, y_i) with $\phi(\bar{x}_i) \in (a, b)$, $u(x_i) \neq g(x_i)$ on (a, b), except possibly at $\phi(\bar{x}_i)$, and $|g(b) - g(a)| > q/2$. Then $V(g, \bar{x}_i, (a, b)) > q/2$ and

$$\alpha_g(E) \geqslant |\mu_i|(E) > \frac{q}{2}\frac{\mu}{2} = \frac{q\mu}{4} \geqslant k.$$

We have thus proved.

Lemma 4. *If $f \in \mathcal{A}$ and $f \notin \mathcal{L}$ then there is a $k > 0$ such that if $g \in \mathcal{L}$ and*

$$E = [x : f(x) \neq g(x)]$$

then $\alpha_g(E) > k$.

Lemmas 3 and 4 prove the converse of the following theorem whose direct part was proved earlier in the paper.

Theorem. *If $f \in \mathcal{A}$ then $f \in \mathcal{L}$ if and only if for every $\varepsilon > 0$ there is an approximately continuous g such that if $E = [x : f(x) \neq g(x)]$ then $\alpha_f(E) < \varepsilon$ and $\alpha_g(E) < \varepsilon$.*

14. It seems appropriate to remark on the case of 2 dimensions. In [10], we obtained a result like the above theorem except that approximate continuity may be replaced by continuity. Thus, for this case, if $f \in \mathcal{A}$, then $f \in \mathcal{L}$ if and only if for every $\varepsilon > 0$ there is a continuous g and an approximately continuous h such that if $E = [x : f(x) \neq g(x)]$ and $F = [x : f(x) \neq h(x)]$ then $\alpha_f(E) < \varepsilon$, $\alpha_f(F) < \varepsilon$, $\alpha_g(E) < \varepsilon$ and $\alpha_h(F) < \varepsilon$. One of these results is stronger in one direction and the other is stronger in the other direction.

It may be instructive to see how the approximation by means of continuous functions may be proved, for $n = 2$, using the methods of this paper. Let $\varepsilon > 0$. We consider the sets Q_k, $k = 1, 2, \ldots$, each the union of finitely many pairwise disjoint closed squares, designated of rank k, such that each square of rank $k+1$ is in the interior of a square of rank k, and the zero dimensional closed set $V = \bigcap_{k=1}^{\infty} Q_k$ satisfies $\alpha_f(V) > \alpha_f(Q) - \varepsilon$. Moreover, f is uniformly continuous on $E = V \cup (\bigcup_{k=1}^{\infty} \partial Q_k)$ with f_n converging uniformly to f on E. Using the technique of this paper, slightly modified so that g does not differ by too much within a frame from the values of f on the boundary of the square of which it is the frame, we may obtain a function g such that

a) $g(x) = f(x)$, $x \in V$,
b) $\alpha_g(Q) < \alpha_f(Q) + \varepsilon$,
c) g is uniformly continuous on E,
d) g is continuous on $Q - V$.

We indicate how g may be modified to a continuous function h on Q such that $h(x) = f(x)$, $x \in V$, and $\alpha_h(Q) \leqslant \alpha_g(Q)$.

Letting $Q = Q_0$, on each set $Q_{i-1} - \text{int}\, Q_i$, $i = 1, 2, \ldots$, define

$$\psi_i = \max\,[g(x) : x \in \partial Q_{i-1} \cup \partial Q_i], \quad \phi_i = \min\,[g(x) : x \in \partial Q_{i-1} \cup \partial Q_i].$$

Then define h as
$$h(x) = \begin{cases} \psi_i & \text{if } g(x) \geqslant \psi_i, \\ g(x) & \text{if } \phi_i < g(x) < \psi_i, \\ \phi_i & \text{if } g(x) \leqslant \phi_i. \end{cases}$$

Then $\alpha_h(Q_{i-1} - \text{int}\, Q_i) \leqslant \alpha_g(Q_{i-1} - \text{int}\, Q_i)$, the function h is continuous and our result follows.

15. The theorem of this paper, and in particular Lemma 3, reveal a surprising connection between linear continuity and approximate continuity, the latter being an n dimensional notion. It may be of some interest to note that no such relation holds for other forms of continuity.

For this purpose, we give an example of a function f defined on 3 space such that $f \in \mathcal{A}$, f is approximately continuous, and every g equivalent to f is not continuous, as a function of two variables, on any plane parallel to a given coordinate plane.

Let F be a function of 2 variables, defined on the closed square $[0, 1] \times [0, 1]$, which is zero, except on a sequence $\{\sigma_n\}$ of disjoint closed disks converging to $(\frac{1}{2}, \frac{1}{2})$. The $\{\sigma_n\}$ are such that the sum of their circumferences converges, and the density of the set $\bigcup \sigma_n$ is zero at $(\frac{1}{2}, \frac{1}{2})$. We define F so that its graph on σ_n is a right circular cone of altitude 1. We define f on the unit cube by means of

$$f(x, y, z) = F(x, y).$$

The function f is continuous at all (x, y, z) for which $(x, y) \neq (\frac{1}{2}, \frac{1}{2})$ and is approximately continuous at the points $(\frac{1}{2}, \frac{1}{2}, z)$. Finally, if g is equivalent to f, then for almost all $z \in [0, 1]$, f is not continuous at $(\frac{1}{2}, \frac{1}{2}, z)$ as a function of (x, y).

16. We now show that the class $\mathcal{L}$ is invariant under bilipschitzian mappings. By a bilipschitzian mapping we mean a mapping f which is one-one between the open cube Q and a set P in n space, for which there is an L such that for every $x, y \in Q$, $|f(x) - f(y)| < L|x - y|$, and for every $\xi, \eta \in P$, $|f^{-1}(\xi) - f^{-1}(\eta)| < L|\xi - \eta|$. A set $S \subset Q$ is said to be d-open [14], if S is measurable and the density of S is 1 at every point of S. We show that bilipschitzian mappings take d-open sets into d-open sets. First, there is a constant $K > 0$, depending only on n and L such that for every cube $R \subset Q$, with center x, $f(R)$ contains a cube R' and is contained in a cube R'', with centers at $f(x)$, such that

$$m(R') > \frac{1}{K}\, m(R) \quad \text{and} \quad m(R) > \frac{1}{K}\, m(R''),$$

and the same holds for f^{-1}, for the same K.

Let $x \in S$ and let R be a cube with center $f(x)$. Let U be the cube of center x, containing $f^{-1}(R)$, with $m(U) = Km(R)$. Let $\varepsilon > 0$. There is an $\eta > 0$ such that, if $m(R) < \eta$, the relative measure of S in U exceeds $1 - \eta/K^2$. The $U - S$ may be covered by a sequence $\{I_k\}$ of cubes such that

$$\sum_{k=1}^{\infty} m(I_k) < \frac{\eta}{K^2} m(U).$$

Then,

$$m(f(U-S)) < \sum_{k=1}^{\infty} m(f(I_k)) < K \sum_{k=1}^{\infty} m(I_k) < \frac{\eta}{K} m(U) < \eta m(R),$$

and

$$m(f(S) \cap R) > (1-\eta) m(R).$$

It follows that $f(S)$ is d-open. Since a function is approximately continuous if and only if the inverse images of open sets are d-open, it follows that if g is approximately continuous, and f is bilipschitzian then $g \circ f$ is approximately continuous.

Using the constant K above, it is a standard fact that if G is an open set, g is a measurable function, and f is a bilipschitzian mapping, then

$$\left| \int_{f(G)} g \right| \leqslant K \left| \int_G g \circ f \right|.$$

In particular, this holds if g is the area integrand for a lipschitzian function h. Then, for any lipschitzian h and bilipschitzian f, and open set G, we have

$$\alpha_h(f(G)) \leqslant K \alpha_{h \circ f}(G).$$

Now, let G be open, and let $\{G_m\}$ be a sequence of open sets with

$$\bar{G}_m \subset G_{m+1}, \quad m = 1, 2, \ldots,$$

and

$$\bigcup_{m=1}^{\infty} G_m = G.$$

Let h_m be lipschitzian on G_m, $m = 1, 2, \ldots$, with $\{h_m\}$ converging in measure to a function h and $\{\alpha_{h_m \circ f}(G_m)\}$ converging to $\alpha_{h \circ f}(G)$. Then

$$\alpha_h(f(G)) \leqslant \liminf \alpha_{h_m}(f(G_m)) \leqslant K \lim \alpha_{h_m \circ f}(G_m) = K \alpha_{h \circ f}(G).$$

It now follows immediately from our theorem that the following corollary holds.

COROLLARY. *If g is linearly continuous and of finite area, and f is bilipschitzian, then $g \circ f$ is linearly continuous and of finite area.*

As a consequence of this corollary, functions which are in $\mathfrak{L}$ with respect to any rectangular coordinate system are also in $\mathfrak{L}$ with respect to any other rectangular coordinate system.

Let $f \in \mathfrak{L}$ with respect to a coordinate system $(x_1, ..., x_n)$. The function g, equivalent to f, which is linearly continuous in this coordinate system, was obtained in [9] as the limit of the integral means $\{f_m\}$ of f. Since $f \in \mathfrak{L}$ with respect to every rectangular coordinate system, we may now draw the following consequences.

a) The integral means $\{f_m\}$ converge to g on a set S whose complement projects on every hyperplane as $(n-1)$ measure zero.

b) The function g, defined on S, is continuous along almost all lines in every direction.

c) If M is any "smooth" (i.e. C^1) $(n-1)$ dimensional manifold, then g is defined $(n-1)$ almost everywhere on M.

17. In this last section, we discuss the completeness of $\mathfrak{L}$ with respect to the metric

$$d(f, g) = \delta(f, g) + \Delta(\alpha_f, \alpha_g)$$

which was introduced in § 1. We shall also note that the space $\mathfrak{F}$ of functions whose partial derivatives are functions is complete with respect to this metric.

We first consider the one variable case. A function $f \in \mathfrak{L}$ if it has an equivalent continuous function, with compact support, which is of bounded variation. Suppose then that f is of bounded variation and not equivalent to a continuous function. It may be taken so that there is, for example, a $k>0$ such that $f(x+)>f(x-)+k$, and a $\delta>0$ such that the variation of f is less than $k/8$ in each of the intervals $(x-\delta, x)$ and $(x, x+\delta)$. Now, there is an $\eta>0$ such that $\delta(f, g)<\eta$ implies there are $\xi_1 \in (x-\delta, x)$ and $\xi_2 \in (x, x+\delta)$ for which $|f(\xi_1)-g(\xi_1)|<k/8$ and $|f(\xi_2)-g(\xi_2)|<k/8$. Suppose then that g is continuous and that $\delta(f, g)<\eta$. There is then a pair of intervals $I \subset (x-\delta, x)$ and $J \subset (x, x+\delta)$, on which $f \neq g$ and on which the sum of the variations of g exceeds $k/2$. This shows that $\mathfrak{L}$ is complete.

For the n variables case, suppose $f \in \mathcal{A}$ and $f \notin \mathfrak{L}$. Then, for some $i=1, ..., n$, for a set E of points $\bar{x}_i$, of positive outer measure, there is a $k>0$ and $\delta>0$ such that, say, for each $\bar{x}_i \in E$ there is an $x_i(\bar{x}_i)$ such that

$$f(x_i(\bar{x}_i)+, \bar{x}_i) > f(x_i(\bar{x}_i)-, \bar{x}_i) + k,$$

and the variation of $f(x_i, \bar{x}_i)$ in x_i is less than $k/8$ in each of the intervals $(x_i(\bar{x}_i)-\delta, x_i(\bar{x}_i))$ and $(x_i(\bar{x}_i), x_i(\bar{x}_i)+\delta)$. There is an $\eta>0$ such that $\delta(f, g)<\eta$ implies that for every $\bar{x}_i$, not in a set of measure less that $\frac{1}{2}m_e(E)$, for every interval $(x_i, x_i+\delta)$ there is a $\xi_i \in (x_i, x_i+\delta)$

such that $|f(\xi_i)-g(\xi_i)|<k/8$. So, let g be continuous and such that $\delta(f, g)<\eta$. Then there is an $F\subset E$ with $m_e(F)>\frac{1}{2}m_e(E)$, such that for every $\bar{x}_i\in F$ there are intervals

$$I\subset (x_i(\bar{x}_i)-\delta,\, x_i(\bar{x}_i)) \quad \text{and} \quad J\subset (x_i(\bar{x}_i),\, x_i(\bar{x}_i)+\delta),$$

on which $f\neq g$ and on which the sum of the variations of g exceeds $k/2$. It follows that

$$\Delta(\alpha_f, \alpha_g) > \frac{k}{4}\, m_e(E)$$

so that $\mathcal{L}$ is complete.

We turn to the completeness of $\mathcal{F}$, and again consider the one variable case. Let $f\in\mathcal{L}$, with support in (a, b) be non absolutely continuous. There is a $k>0$ and a compact set E of measure zero, such that for any $\delta>0$, there is a disjoint set of intervals

$$[a_1, b_1], \ldots, [a_m, b_m],$$

the sum of whose lengths is less than δ, with

$$\sum_{i=1}^{m} |f(b_i)-f(a_i)| > k.$$

Now, let g be absolutely continuous. There is a $\delta>0$ such that for every S, of measure less than δ, $\alpha_g(S)<k/2$. Then, if $S=\bigcup_{i=1}^{m}[a_i, b_i]$, where $m(S)<\delta$ and $\sum_{i=1}^{m}|f(b_i)-f(a_i)|>k$, we have $\alpha_f(S)>k$. Since $\alpha_f(A)=\alpha_g(A)$, for any A on which $f=g$, it follows that there is a subset $B\subset S$ on which $f\neq g$, with $\alpha_f(B)>\alpha_g(B)+k/2$. Since $\mathcal{L}$ is complete, it follows that $\mathcal{F}$ is complete.

We leave the necessary adaptation for the n variables case to the reader.

A result of J. Michael [16] in a form given by author [11] asserts that for every $f\in\mathcal{A}$, for every $\varepsilon>0$, there is a $g\in C^1$ such that $f(x)=g(x)$, except on a set of measure less than ε, and $|\alpha_f(Q)-\alpha_g(Q)|<\varepsilon$. This result implies that for every $f\in\mathcal{F}$ there is a $g\in C^1$ such that if $E=[x: f(x)\neq g(x)]$, then $\alpha_f(E)<\varepsilon$ and $\alpha_g(E)<\varepsilon$. Indeed, this gives a characterization of $\mathcal{F}$ similar to the one given for $\mathcal{L}$ in this paper, but with continuously differentiable functions in the position of the approximately continuous ones.

Using this fact and the results of this paper, we may make the following assertions, always using the above metric.

a) In one dimension, the space of continuous functions of bounded variations is complete.

b) In two dimensions, $\mathcal{L}$ is the completion of the set of continuous functions in $\mathcal{A}$.

c) In n dimensions, $\mathcal{L}$ is the completion of the set of approximately continuous functions in $\mathcal{A}$.

d) $\mathcal{F}$ is the completion of C^1.

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