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A CHARACTERIZATION OF m-DEPENDENT STATIONARY INFINITELY DIVISIBLE SEQUENCES WITH APPLICATIONS TO WEAK CONVERGENCE

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m-dependent stationary infinitely divisible sequences are characterized as a class of generalized finite moving average sequences via the structure of the associated Lévy measure. This characterization is used to find necessary and sufficient conditions for the weak convergence of centered and normalized partial sums of m-dependent stationary infinitely divisible sequences. Partial sum convergence for stationary infinitely divisible sequences that can be approximated by m-dependent ones is then studied.

1. Introduction. The literature on the Central Limit Theorem (CLT) contains many extensions of the classical i.i.d. results to stationary sequences $\{X_j\}_{j\in\mathbb{Z}}$ such that $\mathrm{Var}(X_0) < \infty$. These extensions show that under various weak dependence assumptions

$$\frac{S_n - ES_n}{\sqrt{\mathrm{Var}(S_n)}} \Rightarrow N(0,1). \tag{1.1}$$

One of the first dependent result of this type is due to Diananda (1955) who showed that m-dependence is sufficient for the Central Limit Theorem to hold. Later work further extends the CLT to stationary mixing sequences, and the reader is referred to Ibragimov and Linnik (1971) and Peligrad (1986) for overviews of these results. Typically, these theorems require $\mathrm{Var}(S_n) \to \infty$, a mixing condition and either a sufficiently fast rate at which the mixing coefficients converge to zero or the existence of a higher order moment.

In recent times heavy-tailed data have been collected from a variety of different sources, and with this in mind, it is imperative to study limit theorems for stationary sequences of random variables with possibly infinite variance. For an i.i.d. sequence $\{X_j\}_{j\in\mathbb{Z}}$, if the sequence of (centered and normalized) partial sums converges, then the limiting distribution is necessarily stable. The stable distributions are indexed by four parameters. In particular, $\alpha \in (0,2]$ is called the index of stability. The distribution of X_0 is said to belong to the domain of attraction of an α-stable distribution if the partial sums, properly centered and normalized, converge to that α-stable random variable.

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When studying partial sum convergence to a stable limit for weakly dependent stationary sequences, the main objective considered to date has been, by fixing an asymptotic independence condition, to find additional requirements on the distribution of the sequence which are sufficient for the stable limit to hold. LePage, Woodroofe and Zinn (1981) used order statistics to give an alternative proof of partial sum convergence for an i.i.d. sequence $\{X_j\}_{j\in\mathbb{Z}}$ such that the distribution of X_0 is in the domain of attraction of a nonnormal stable distribution. This type of argument has been further extended using point processes techniques to obtain corresponding results for weakly dependent sequences. Davis and Resnick (1985) showed partial sum convergence for the moving average of an i.i.d. sequence $\{X_j\}_{j\in\mathbb{Z}}$ such that the distribution of X_0 is in the domain of attraction of a stable random variable. Heinrich (1985, 1987) gave sufficient conditions for the convergence of the partial sums of both m-dependent and ψ-mixing sequences. Heinrich's results include rates of convergence, but his proofs require technical assumptions that are not minimal. Davis (1983) showed partial sum convergence for sequences $\{X_j\}_{j\in\mathbb{Z}}$ satisfying dependence conditions typically used in extreme value theory (distributional mixing and negligible local dependence) and with the distribution of X_0 in the domain of attraction of a nonnormal stable distribution. Jakubowski and Kobus (1989) showed partial sum convergence for m-dependent sequences $\{X_j\}_{j\in\mathbb{Z}}$ such that $(X_0, X_1, \dots, X_m)$ is in the domain of attraction of a multivariate stable distribution. The most general known formulation for m-dependent sequences is as follows:

THEOREM 1.1. *Assume $\{X_j\}_{j\in\mathbb{Z}}$ is a stationary m-dependent sequence, let $S_n = \sum_{j=0}^{n-1} X_j$ and let $0 < \alpha \le 2$ be fixed. Assume S_{m+1} is in the domain of attraction of a nondegenerate α-stable distribution with characteristic function φ_{m+1}, and assume S_m is in the domain of attraction of a nondegenerate α-stable distribution with characteristic function φ_m, with the same normalizing sequence. Then S_n appropriately centered and normalized converges in distribution to X where X is an α-stable random variable with characteristic function*

$$Ee^{itX} = \frac{\varphi_{m+1}(t)}{\varphi_m(t)}.$$

With the additional assumption that $\operatorname{Var} X_0 < \infty$, Theorem 1.1 recovers the result of Diananda (1955). Theorem 1.1 is conjectured and proved in the case $0 < \alpha < 2$ under slightly stronger assumptions in Jakubowski and Kobus (1989). It is proved by Szewczak (1988) when $\alpha = 2$ and by Kobus (1995) when $0 < \alpha < 2$. A one-dependent stationary sequence $\{X_j\}_{j\in\mathbb{Z}}$ such that X_1 is in the domain of attraction of an α-stable random variable, but $X_1 + X_2$ is not in the domain of attraction of any nondegenerate α-stable law is constructed by Jakubowski (1994). For this example, there is no normalizing sequence $\{B_n\}$ such that the partial sums centered and normalized converge to a nondegenerate α-stable random variable.

The present paper continues the study of partial sum convergence for m-dependent stationary sequences, but considers the problem for stationary infinitely divisible (SID) sequences, that is stationary sequences such that all their finite-dimensional marginals are infinitely divisible random vectors. In Section 2 we show that any such m-dependent sequence is equal in distribution to a certain finite generalized moving average sequence. Partial sum convergence is considered in Section 3. While Theorem 1.1 gives sufficient conditions for partial sum convergence of m-dependent stationary sequences, using the characterization theorem given in Section 2, it is easy to find necessary and sufficient conditions for such convergence in the case of m-dependent stationary infinitely divisible sequences. Partial sum convergence for a class of stationary infinitely divisible sequences that can be approximated by m-dependent ones is briefly discussed in Section 3. Section 4 concludes the paper with some final remarks.

We remind the reader that a sequence $\{X_j\}_{j\in\mathbb{Z}}$ is m-dependent if any finite-dimensional marginals $(X_{j_1}, X_{j_2}, \dots, X_{j_n})$ and $(X_{k_1}, X_{k_2}, \dots, X_{k_{n'}})$, such that $k_1 - j_n > m$, are independent. In particular, a (stationary) sequence is 0-dependent if and only if it is formed of independent (identically distributed) random variables.

The technique of proof that will be used throughout this paper involves analyzing the Lévy measure of infinitely divisible vectors and sequences. Recall that a vector $\mathbf{X}$ is infinitely divisible if its characteristic function admits the representation

$$\ln Ee^{i\langle \mathbf{t},\mathbf{X}\rangle} = i\langle \mathbf{b},\mathbf{t}\rangle - \tfrac{1}{2}\langle \Sigma\mathbf{t},\mathbf{t}\rangle + \int_{\mathbb{R}^d\setminus\{\mathbf{0}\}} \bigl(e^{i\langle \mathbf{t},\mathbf{x}\rangle} - 1 - i\langle \mathbf{t},\mathbf{x}\rangle \mathbf{1}_{\|\mathbf{x}\|\le 1}\bigr) Q(d\mathbf{x}),$$

where $\mathbf{b}$ is a d-dimensional vector, Σ is a $d\times d$ positive definite matrix, and Q is a Borel measure on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d\setminus\{\mathbf{0}\}} \min(1, \|\mathbf{x}\|^2)\ Q(d\mathbf{x}) < \infty.$$

This representation is unique, $(\mathbf{b}, \Sigma, Q)$ is called the characterizing triplet of $\mathbf{X}$, and Q is its Lévy measure. Furthermore, if a triplet $(\mathbf{b}, \Sigma, Q)$ satisfies the restrictions given above, then it is the characterizing triplet of some infinitely divisible vector [see, e.g., Sato (1999)].

For $b \in \mathbb{R}$; $\Sigma : \mathbb{R} \to \mathbb{R}$, a positive definite function; and Q, a Borel measure on $\mathbb{R}^{\mathbb{Z}}$; we say that (b, Σ, Q) is the characterizing triplet of a stationary infinitely divisible sequence $\{X_j\}_{j\in\mathbb{Z}}$ if, for any $\Lambda \subset \mathbb{Z}$,

$$\bigl((b, b, \dots, b), \{\Sigma(i-j)\}_{i,j\in\Lambda}, Q|_{\mathbb{R}^\Lambda}\bigr)$$

is the characterizing triplet of $\{X_j\}_{j\in\Lambda}$. It is implicit that each of the one-dimensional marginals of Q are equivalent measures on $\mathbb{R}$ integrating $\min(1, x^2)$. Q is then called the Lévy measure of $\{X_j\}_{j\in\mathbb{Z}}$. Maruyama (1970) proves the existence of a characterizing triplet for all stationary infinitely divisible sequences.

2. Characterization of *m*-dependent SID sequences. Throughout this section, by a finite moving average sequence, we mean a sequence of the form $\{\sum_{k=0}^{m} a_k\ \eta_{j-k}\}_{j\in\mathbb{Z}}$, where $\{\eta_j\}_{j\in\mathbb{Z}}$ is any i.i.d. sequence and $(a_0, a_1, \ldots, a_m)$ is a vector of reals. It is then clear that a finite moving average sequence is stationary and m-dependent. Moreover, as detailed below, the Riesz factorization lemma [see Pólya and Szegő (1978)] implies that all m-dependent stationary Gaussian sequences are (up to a location parameter) finite moving average sequences. This well known result is presented next for the sake of completeness.

PROPOSITION 2.1. *If $\{X_j\}_{j\in\mathbb{Z}}$ is a stationary m-dependent Gaussian sequence with $EX_0 = \mu$, then there exists $\{a_k\}_{k=0}^{m}$ such that*

$$\{X_j\}_{j\in\mathbb{Z}} \stackrel{\mathcal{L}}{=} \left\{\mu + \sum_{k=0}^{m} a_k \eta_{j-k}\right\}_{j\in\mathbb{Z}}$$

where $\{\eta_j\}_{j\in\mathbb{Z}}$ are i.i.d. standard normal random variables.

PROOF. Let $R(j) = \mathrm{Cov}(X_0, X_j)$, for $j \in \mathbb{Z}$. $R(j)$ is a positive definite function on $\mathbb{Z}$ and $R(j) = 0$ for $|j| > m$. The Riesz factorization lemma implies that there exists $\{a_k\}_{k=0}^{m}$ such that

$$\sum_{j=-m}^{m} R(j)e^{ij\theta} = \left|\sum_{j=0}^{m} a_j e^{ij\theta}\right|^2.$$

By expanding the square and grouping the coefficients of $e^{ij\theta}$, one sees that

$$R(j) = \sum_{k=0}^{m-j} a_k a_{k+j} \qquad \text{for } 0 \le j \le m. \tag{2.1}$$

Since the RHS of (2.1) is also the covariance function of the finite moving average sequence

$$\left\{\sum_{k=0}^{m} a_k \eta_{j-k}\right\}_{j\in\mathbb{Z},}$$

the proposition is verified. □

Although all m-dependent stationary Gaussian sequences are finite moving average sequences, it is next shown that there are m-dependent stationary infinitely divisible sequences that do not have the same distribution as any finite moving average of i.i.d. infinitely divisible random variables.

EXAMPLE 2.1. Let $\widehat{Q}$ be the Lévy measure of a two-dimensional infinitely divisible vector $\xi = (\xi^0, \xi^1)$ and suppose that there exists an open set $B \subset \mathbb{R}^2$ such that $\widehat{Q}$ is nonzero on B. Consider the sequence

$$\{X_j\}_{j\in\mathbb{Z}} \stackrel{\mathcal{L}}{=} \{\xi_j^0 + \xi_{j-1}^1\}_{j\in\mathbb{Z}}$$

where $\{\xi_j\}_{j\in\mathbb{Z}}$ are independent copies of ξ. Note that $\{X_j\}_{j\in\mathbb{Z}}$ is a one-dependent stationary sequence with infinitely divisible marginals, and the Lévy measure of (X_0, X_1) is equal to

$$(\widehat{Q}_1 \times \delta_{\{0\}}) + \widehat{Q} + (\delta_{\{0\}} \times \widehat{Q}_0),$$

where $\widehat{Q}_0$ and $\widehat{Q}_1$ denote the one-dimensional marginal distributions of $\widehat{Q}$. Therefore, the Lévy measure of (X_0, X_1) is nonzero on B. Next, if

$$\{X_j\}_{j\in\mathbb{Z}} \stackrel{\mathcal{L}}{=} \{a_0\eta_j + a_1\eta_{j-1}\}_{j\in\mathbb{Z}}$$

for some i.i.d. sequence of infinitely divisible random variables $\{\eta_j\}_{j\in\mathbb{Z}}$ and $a_0, a_1 \in \mathbb{R}$, then

$$(X_0, X_1) \stackrel{\mathcal{L}}{=} (0, a_0\eta_1) + (a_0\eta_0, a_1\eta_0) + (a_1\eta_{-1}, 0).$$

Since the three pairs are independent, the Lévy measure of (X_0, X_1) is equal to the sum of the Lévy measures of the three vectors. The Lévy measure of the first vector is supported on one axis and the Lévy measure of the third vector is supported on the other axis. The Lévy measure of the middle vector is supported on the line $a_1x_0 + a_0x_1 = 0$ in $\mathbb{R}^2$. Therefore, the Lévy measure of (X_0, X_1) is supported on three lines in $\mathbb{R}^2$ and is zero at some point in B. It must be that $\{X_j\}_{j\in\mathbb{Z}}$ does not have the same distribution as any finite moving average sequence.

In an effort to characterize m-dependent stationary infinitely divisible sequences, we introduce the following definition.

DEFINITION 2.1. A sequence $\{X_j\}_{j\in\mathbb{Z}}$ is called a finite generalized moving average sequence (of length $m+1$) if

$$\{X_j\}_{j\in\mathbb{Z}} \stackrel{\mathcal{L}}{=} \left\{ \sum_{k=0}^{m} \xi_{j-k}^k \right\}_{j\in\mathbb{Z}}, \tag{2.2}$$

where $\{\xi_j = (\xi_j^0, \xi_j^1, \dots, \xi_j^m)\}_{j\in\mathbb{Z}}$ is some i.i.d. sequence of $(m+1)$-dimensional vectors. The sequence $\{X_j\}_{j\in\mathbb{Z}}$ is then said to be generated by ξ.

If ξ is a symmetric α-stable vector, then the sequence generated by ξ according to (2.2) is a particular case of the generalized moving average sequence introduced by Surgailis, Rosiński, Mandrekar and Cambanis (1993). Also, it is clear that finite generalized moving average sequences, as defined here, are m-dependent and stationary. In addition, it is next shown that all m-dependent stationary Gaussian sequences are generated by some $(m+1)$-dimensional Gaussian vector.

PROPOSITION 2.2. *Let $\{X_j\}_{j\in\mathbb{Z}}$ be an m-dependent stationary Gaussian sequence with parameters $\mu = EX_0$ and covariance sequence $R(j) = \mathrm{Cov}(X_0, X_j)$. Then, $\{X_j\}_{j\in\mathbb{Z}}$ is a finite generalized moving average sequence generated by an $(m+1)$-dimensional Gaussian vector $\xi = (\xi^0, \dots, \xi^m)$ with parameters*

$$E\xi^k = \frac{\mu}{m+1} \qquad \text{for } 0 \le k \le m \tag{2.3a}$$

and

$$\mathrm{Cov}(\xi^k, \xi^l) = a_k a_l \qquad \text{for } 0 \le k, l \le m, \tag{2.3b}$$

where $\{a_k\}_{k=0}^m$ is given in Proposition 2.1.

The proposition is clear since the generalized finite moving average sequence has the same mean and covariance function as the finite moving average sequence defined in Proposition 2.1.

One of the main results of this paper is that all m-dependent stationary infinitely divisible sequences are generated by an $(m+1)$-dimensional infinitely divisible vector.

THEOREM 2.1. *$\{X_j\}_{j\in\mathbb{Z}}$ is an m-dependent stationary infinitely divisible sequence if and only if $\{X_j\}_{j\in\mathbb{Z}}$ is a finite generalized moving average sequence generated by an $(m+1)$-dimensional infinitely divisible vector $\xi = (\xi^0, \xi^1, \dots, \xi^m)$.*

We showed in Proposition 2.2 that if the m-dependent stationary sequence is Gaussian, then the generating vector can also be taken Gaussian. There are other classes of m-dependent stationary infinitely divisible sequences where the generating vector ξ can be taken from the corresponding class of the infinitely divisible vectors. Some of these additional cases are given in Corollary 2.1.

COROLLARY 2.1. *Let $X = \{X_j\}_{j\in\mathbb{Z}}$ be an m-dependent stationary infinitely divisible sequence. All finite-dimensional marginals of X are α-stable (or compound Poisson) if and only if the generating vector ξ has distribution in the respective class. All one-dimensional marginals of X are Poisson if and only if the generating vector ξ is infinitely divisible and all the one-dimensional marginals of ξ are Poisson.*

The proofs of Theorem 2.1 and its corollary are presented in Section 2.2. Preliminary results on m-dependent stationary infinitely divisible sequences are given in Section 2.1. However, we first point out that Theorem 2.1 is not necessarily true (for $m > 0$) if the hypothesis of infinite divisibility is omitted.

EXAMPLE 2.2. Let $\{Y_j\}_{j\epsilon\mathbb{Z}}$ be an independent and identically distributed sequence of Bernoulli random variables with

$$P(Y_0 = 1) = P(Y_0 = 0) = \tfrac{1}{2},$$

and let $X_j = Y_{j-1}Y_j$. Then $\{X_j\}_{j\in\mathbb{Z}}$ is a stationary one-dependent sequence but is not a generalized finite moving average sequence of length 2. Indeed, for this $\{X_j\}_{j\in\mathbb{Z}}$,

$$[Ee^{it(X_0+X_1)}]^2 - [Ee^{itX_0}][Ee^{it(X_0+X_1+X_2)}] = \tfrac{1}{64}(1-e^{it})^3. \tag{2.4}$$

Now, suppose there exists an i.i.d. sequence $\{\xi_j = (\xi_j^0, \xi_j^1)\}_{j\in\mathbb{Z}}$ such that

$$\{X_j\}_{j\in\mathbb{Z}} \stackrel{\mathcal{L}}{=} \{\xi_j^0 + \xi_{j-1}^1\}_{j\in\mathbb{Z}}.$$

Let $\varphi(s,t)$ be the characteristic function of ξ_j. Then

$$\begin{aligned} Ee^{i(t_0X_0+t_1X_1+t_2X_2)} &= Ee^{i(t_0(\xi_0^0+\xi_{-1}^1)+t_1(\xi_1^0+\xi_0^1)+t_2(\xi_2^0+\xi_1^1))} \\ &= Ee^{i(t_0\xi_{-1}^1)}Ee^{i(t_0\xi_0^0+t_1\xi_0^1)}Ee^{i(t_1\xi_1^0+t_2\xi_1^1)}Ee^{i(t_2\xi_2^0)} \\ &= \varphi(0,t_0)\varphi(t_0,t_1)\varphi(t_1,t_2)\varphi(t_2,0). \end{aligned}$$

Thus $Ee^{itX_0} = \varphi(0,t)\varphi(t,0)$, $Ee^{it(X_0+X_1)} = \varphi(0,t)\varphi(t,t)\varphi(t,0)$ and $Ee^{it(X_0+X_1+X_2)} = \varphi(0,t)(\varphi(t,t))^2\varphi(t,0)$. Therefore, for all $t \in \mathbb{R}$,

$$[Ee^{it(X_0+X_1)}]^2 - [Ee^{itX_0}][Ee^{it(X_0+X_1+X_2)}] = 0$$

which contradicts (2.4). It must be that no such i.i.d. sequence $\{\xi_j\}_{j\in\mathbb{Z}}$ exists.

2.1. *Preliminary results.* It is well known that if $\{X_j\}_{j\in\mathbb{Z}}$ is independent and infinitely divisible, then its Lévy measure is supported on the axes in $\mathbb{R}^{\mathbb{Z}}$ [see, e.g., Sato (1999), page 67]. Lemma 2.1, given below, is the corresponding result for m-dependent infinitely divisible sequences. The proof of Lemma 2.1 uses the result for independent sequences. Actually, the proof only uses pairwise independence of random variables more than m apart; however, for infinitely divisible sequences, m-dependence and pairwise independence are equivalent [this is given as an exercise in Sato (1999), page 67]. Lemma 2.2 shows that the distribution of an m-dependent stationary infinitely divisible sequence is uniquely determined by the distribution of an $(m+1)$-dimensional marginal. After presenting these two lemmas and their proofs, we conclude the section by introducing some useful notation.

LEMMA 2.1. *Let $\{X_j\}_{j\in\mathbb{Z}}$ be an m-dependent infinitely divisible sequence* (*not necessarily stationary*). *Let Q be the Lévy measure of $\{X_j\}_{j\in\mathbb{Z}}$. Then*

$$\{(\mathbb{R}\setminus\{0\})^A \times \{0\}^{\mathbb{Z}\setminus A}\}_{A\in\mathbb{A}_m}$$

is a disjoint partition of the support of Q, where $\mathbb{A}_m$ denotes the collection of subsets of $\mathbb{Z}$ such that the distance between any two elements is no more than m.

PROOF. The sets are pairwise disjoint. It remains to show that $Q(S^c) = 0$, where

$$S = \bigcup_{A \in \mathbb{A}_m} (\mathbb{R} \setminus \{0\})^A \times \{0\}^{\mathbb{Z} \setminus A}.$$

Let

$$T_{a,b} = \{x \in \mathbb{R}^{\mathbb{Z}} \mid x_a \neq 0, x_b \neq 0\}.$$

When $|a - b| > m$,

$$Q(T_{a,b}) = Q|_{\mathbb{R}^{\{a,b\}}}\big((\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})\big) = 0,$$

since X_a and X_b are independent, and thus the Lévy measure of (X_a, X_b) is supported on the axes. Now

$$S^c = \bigcup_{|a-b|>m} T_{a,b}.$$

Thus

$$Q(S^c) \leq \sum_{|a-b|>m} Q(T_{a,b}) = 0.$$

□

LEMMA 2.2. *Let $\{X_j\}_{j\in\mathbb{Z}}$ and $\{Y_j\}_{j\in\mathbb{Z}}$ be two m-dependent stationary infinitely divisible sequences. If*

$$\{X_j\}_{j=0}^m \stackrel{\mathcal{L}}{=} \{Y_j\}_{j=0}^m,$$

then

$$\{X_j\}_{j\in\mathbb{Z}} \stackrel{\mathcal{L}}{=} \{Y_j\}_{j\in\mathbb{Z}}.$$

PROOF. If $\{X_j\}_{j\in\mathbb{Z}}$ is an m-dependent Gaussian sequence, then the distribution of both $\{X_j\}_{j\in\mathbb{Z}}$ and $\{X_j\}_{j=0}^m$ are uniquely determined by EX_0 and $\mathrm{Cov}(X_0, X_j)$ for $0 \leq j \leq m$. Thus, the result is clear in the Gaussian case. Therefore, assume that $\{X_j\}_{j\in\mathbb{Z}}$ and $\{Y_j\}_{j\in\mathbb{Z}}$ are infinitely divisible without Gaussian component, and let A_m be as defined in Lemma 2.1.

Let Q_X be the Lévy measure of $\{X_j\}_{j\in\mathbb{Z}}$, and let Q_Y be the Lévy measure of $\{Y_j\}_{j\in\mathbb{Z}}$. For each $A \in \mathbb{A}_m$, let $Q_X(A)$ and $Q_Y(A)$ denote the restrictions of Q_X and Q_Y to $(\mathbb{R} \setminus \{0\})^A \times \{0\}^{\mathbb{Z}\setminus A}$, respectively. By Lemma 2.1, the Lévy measure of $\{X_j\}_{j\in A}$ is

$$Q_X|_{\mathbb{R}^A} = \sum_{\substack{B \in \mathbb{A}_m, \\ A \subset B}} Q_X(B)|_{\mathbb{R}^A}.$$

The Lévy measure of $\{Y_j\}_{j\in A}$ is similarly given, and since $\{X_j\}_{j\in\mathbb{Z}}$ and $\{Y_j\}_{j\in\mathbb{Z}}$ are stationary with $\{X_j\}_{j=0}^m \stackrel{\mathcal{L}}{=} \{Y_j\}_{j=0}^m$, the Lévy measure of $\{X_j\}_{j\in A}$ restricted to $(\mathbb{R} \setminus \{0\})^A$ is equal to the Lévy measure of $\{Y_j\}_{j\in A}$ restricted to $(\mathbb{R} \setminus \{0\})^A$. Symbolically,

$$\sum_{\substack{B\in\mathbb{A}_m,\\ A\subset B}} Q_X(B)|_{\mathbb{R}^A} = \sum_{\substack{B\in\mathbb{A}_m,\\ A\subset B}} Q_Y(B)|_{\mathbb{R}^A}. \tag{2.5}$$

Note that the B's in the summation index are either $B = A$ or $|B| > |A|$ where $|\cdot|$ denotes cardinality. Thus, for A with $|A| = m + 1$, the above equation reduces to

$$Q_X(A)|_{\mathbb{R}^A} = Q_Y(A)|_{\mathbb{R}^A}. \tag{2.6}$$

Furthermore, if (2.6) holds for $k < |A| \le m$, then (2.5) reduces to (2.6) for $|A| = k$. Thus, it is inductively shown that (2.6) holds for all $A \in \mathbb{A}_m$. Finally, it is clear that $Q_X(A) = Q_X(A)|_{\mathbb{R}^A} \times \{0\}^{\mathbb{Z}\setminus A}$ and $Q_Y(A) = Q_Y(A)|_{\mathbb{R}^A} \times \{0\}^{\mathbb{Z}\setminus A}$. Therefore, (2.6) implies that $Q_X(A) = Q_Y(A)$, for all $A \in \mathbb{A}_m$; and thus $Q_X = Q_Y$. $\square$

We now introduce some notation that will be used in the proof of Theorem 2.1. In the following J is an interval subset of $\mathbb{Z}$ and $\mathscr{P}$ denotes the class of Borel measures on $\mathbb{R}^J$.

DEFINITION 2.2. For $k \in \mathbb{Z}$, let $T^k : \mathbb{R}^J \to \mathbb{R}^J$ be defined by

$$[T^k(\mathbf{x})]_j = \begin{cases} x_{j-k}, & \text{if } j-k \in J,\\ 0, & \text{otherwise,}\end{cases}$$

for each $j \in J$ and $\mathbf{x} = \{x_j\}_{j\in J}$, where $x_j \in \mathbb{R}$.

If $J = \mathbb{Z}$, then T and T^{-1} are inverse functions and $T^{k+l} = T^k \circ T^l$.

DEFINITION 2.3. For $k \in \mathbb{R}$, let $\tau^k : \mathscr{P} \to \mathscr{P}$ be defined by

$$\tau^k P(A) = P(\{\mathbf{x}|T^k(\mathbf{x}) \in A\}),$$

for each measure $P \in \mathscr{P}$ and $A \in \mathscr{B}(\mathbb{R}^J)$.

2.2. *Proof of the characterization theorem.* In this section, we prove Theorem 2.1 and Corollary 2.1. We start with the proof of Theorem 2.1.

PROOF OF THEOREM 2.1. It is easy to check that if ξ is an infinitely divisible vector and $\{X_j\}_{j\in\mathbb{Z}}$ is generated by ξ according to (2.2), then $\{X_j\}_{j\in\mathbb{Z}}$ is necessarily m-dependent, stationary, and infinitely divisible. Conversely, Proposition 2.2 implies that all m-dependent stationary Gaussian sequences are generated by some Gaussian vector ξ. Now consider a general m-dependent stationary infinitely divisible sequence $\{X_j\}_{j\in\mathbb{Z}}$ with characterizing triplet (b, Σ, Q). We will show that this sequence is generated by $\xi = \xi_G + \xi_P$ where ξ_G generates the Gaussian part [the infinitely divisible sequence with characterizing triplet $(b, \Sigma, 0)$], and ξ_P is the infinitely divisible vector with characterizing triplet $(0, 0, \widehat{Q}|_{\mathbb{R}^{\{0,\dots,m\}}})$, where $\widehat{Q}$ is the Lévy measure of $\{X_j\}_{j=0}^{2m}$ restricted to

$$\mathbb{R}^{\{0,\dots,m-1\}} \times (\mathbb{R} \setminus \{0\})^{\{m\}} \times \{0\}^{\{m+1,\dots,2m\}}.$$

To complete the proof of Theorem 2.1, we only need to prove the following:

LEMMA 2.3. *The sequence generated by ξ_P has Lévy measure Q.*

PROOF. As a consequence of Lemma 2.2, it is sufficient to show that

$$Q|_{\mathbb{R}^{\{0,1,\dots,2m\}}} = \sum_{k=-m}^{2m} \tau^k \widehat{Q}, \tag{2.7}$$

where the RHS is the Lévy measure of a $(2m+1)$-dimensional marginal of ξ_P. □

The proof that (2.7) holds is most clear when $m = 1$. Lemma 2.1 shows that the support of $Q|_{\mathbb{R}^{\{0,1,2\}}}$ can be partitioned into five subsets. Each subset is rectangular with all cross sections equal to $\{0\}$ or $\mathbb{R} \setminus \{0\}$. $P(0)$, $P(1)$, $P(2)$, $R(1)$ and $R(2)$ are the Lévy measure of $\{X_0, X_1, X_2\}$ restricted to each of these subsets. This division is illustrated in Figure 1.

Since the sequence is stationary, the Lévy measures of $\{X_0, X_1\}$ and $\{X_1, X_2\}$ are the same. These two Lévy measures are the projections of $Q|_{\mathbb{R}^{\{0,1,2\}}}$ onto $\mathbb{R}^{\{0,1\}}$ and $\mathbb{R}^{\{1,2\}}$ respectively. Figure 2 shows where each of the sub-measures of $Q|_{\mathbb{R}^{\{0,1,2\}}}$ is projected to. Thus,

$$R(2)|_{\mathbb{R}^{\{1,2\}}} = R(1)|_{\mathbb{R}^{\{0,1\}}} = [\tau R(1)]|_{\mathbb{R}^{\{1,2\}}},$$

where the first equality is clear (see Figure 2) and the second equality follows from the definition of τ. Since $R(2)$ and $\tau R(1)$ are supported on $\{0\}^{\{0\}} \times \mathbb{R}^{\{1,2\}}$, it follows that

$$R(2) = \tau R(1). \tag{2.8}$$

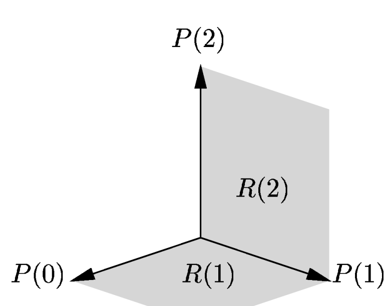

FIG. 1. *Lévy measure of* $\{X_0, X_1, X_2\}$.

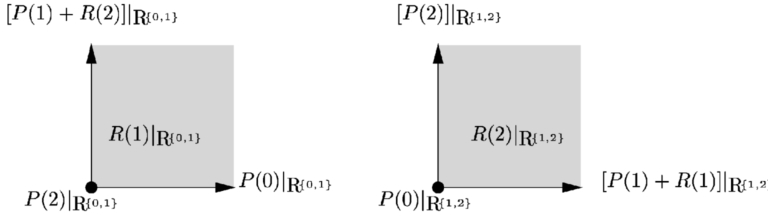

FIG. 2. *Lévy measures of* $\{X_0, X_1\}$ *and* $\{X_1, X_2\}$.

Similarly,

$$
\begin{aligned}
P(2)|_{\mathbb{R}^{\{1,2\}}} &= [P(1)+R(2)]|_{\mathbb{R}^{\{0,1\}}} \\
&= [\tau P(1)+\tau R(2)]|_{\mathbb{R}^{\{1,2\}}} \\
&= [\tau P(1)+\tau^2 R(1)]|_{\mathbb{R}^{\{1,2\}}},
\end{aligned}
$$

where the last equality follows from (2.8). Since $P(2)$, $\tau P(1)$ and $\tau^2 R(1)$ are supported on $\{0\}^{\{0\}} \times \mathbb{R}^{\{1,2\}}$, it follows that

$$P(2) = \tau P(1) + \tau^2 R(1). \tag{2.9}$$

Finally,

$$
\begin{aligned}
P(0)|_{\mathbb{R}^{\{0,1\}}} &= [P(1)+R(1)]|_{\mathbb{R}^{\{1,2\}}} \\
&= \left[\tau^{-1} P(1)+\tau^{-1} R(1)\right]\big|_{\mathbb{R}^{\{0,1\}}},
\end{aligned}
$$

and since $P(0)$, $\tau^{-1}P(1)$, and $\tau^{-1}R(1)$ are supported on $\mathbb{R}^{\{0,1\}} \times \{0\}^{\{2\}}$, it follows that

$$P(0) = \tau^{-1} P(1) + \tau^{-1} R(1). \tag{2.10}$$

In conclusion,

$$
\begin{aligned}
Q|_{\mathbb{R}^{\{0,1,2\}}} &= P(0)+P(1)+P(2)+R(1)+R(2) \\
&= \sum_{k=-1}^{1} \tau^k P(1) + \sum_{k=-1}^{2} \tau^k R(1) \\
&= \sum_{-1}^{2} \tau^k \widehat{Q},
\end{aligned}
$$

where the second equality follows from (2.8)–(2.10). The last equality holds since $\widehat{Q} = P(1) + R(1)$. Thus (2.7) is verified for $m = 1$.

For arbitrary values of m, (2.7) is verified by the same method as in the case $m = 1$. Lemma 2.1 shows that the support of $Q|_{\mathbb{R}^{\{0,1,\dots,2m\}}}$ is partitioned into rectangular subsets, the cross sections of which are either $\{0\}$ or $\mathbb{R}\setminus\{0\}$. The subsets are indexed in such a way that it is clear which subsets are shifts of each other and so that $\widehat{Q}$ can be easily identified. These facts are formally presented in the following.

Let $\mathbb{X}$ be the collection of $\mathbf{x}=\{x_j\}_{j=0}^{2m}$ such that x_j is 0 or 1, for $0\le j<m$, $x_m=1$, and $x_j=0$, for $m<j\le 2m$. Define

$$L(\mathbf{x})=m-\min\{j\mid x_j=1\}.$$

For $\mathbf{x}\in\mathbb{X}$ and $k\in\{L(\mathbf{x})-m,\dots,m\}$, let $A(\mathbf{x},k)$ be the rectangular subset of $\mathbb{R}^{\{0,\dots,2m\}}$ such that the projection of $A(\mathbf{x},k)$ onto $\mathbb{R}^{\{j\}}$ is $\mathbb{R}\setminus\{0\}$ if $[T^k(\mathbf{x})]_j=1$ and $\{0\}$ if $[T^k(\mathbf{x})]_j=0$, where T^k is given in Definition 2.2. $\{A(\mathbf{x},k)\}_{\mathbf{x}\in\mathbb{X},k\in\{L(\mathbf{x})-m,\dots,m\}}$ is a disjoint partition of the support of $Q|_{\mathbb{R}^{\{0,1,\dots,2m\}}}$, and $\{A(\mathbf{x},0)\}_{\mathbf{x}\in\mathbb{X}}$ is a disjoint partition of the support of $\widehat{Q}$. Let $Q(\mathbf{x},k)$ be the restriction of $Q|_{\mathbb{R}^{\{0,1,\dots,2m\}}}$ to $A(\mathbf{x},k)$. Using this notation, in order to show (2.7), one must show that

$$\sum_{\mathbf{x}\in\mathbb{X}}\sum_{k=L(x)-m}^{m}Q(\mathbf{x},k)=\sum_{\mathbf{x}\in\mathbb{X}}\sum_{k=-m}^{2m}\tau^kQ(\mathbf{x},0). \tag{2.11}$$

The above equality holds if and only if for each $A(\mathbf{x},k)$, the measure on the LHS restricted to $A(\mathbf{x},k)$ is equal to the measure on the RHS restricted to $A(\mathbf{x},k)$. Therefore, (2.11) holds if and only if

$$Q(\mathbf{x},k)=\sum_{\substack{(\mathbf{x}',k')\in\mathbb{X}\times\mathbb{Z},\\ T^{k'}(\mathbf{x}')=T^k(\mathbf{x})}}\tau^{k'}Q(\mathbf{x}',0),$$

for $\mathbf{x}\in\mathbb{X}$ and $L(\mathbf{x})-m\le k\le m$. More precisely stated, equality holds in (2.11) if and only if for each $\mathbf{x}\in\mathbb{X}$,

$$Q(\mathbf{x},k)=\begin{cases}\tau^kQ(\mathbf{x},0), & \text{for } 1\le k\le L(\mathbf{x}),\\[1ex] \tau^kQ(\mathbf{x},0)+\displaystyle\sum_{j=1}^{k-L(\mathbf{x})}\Bigg(\sum_{\substack{\mathbf{x}'\in\mathbb{X},\\ T^{j+m}(\mathbf{x}')=T^k(\mathbf{x})}}\tau^{m+j}Q(\mathbf{x}',0)\Bigg), & \\ & \text{for } L(\mathbf{x})<k\le m,\end{cases} \tag{2.12}$$

and

$$Q(\mathbf{x},k)=\sum_{\substack{\mathbf{x}'\in\mathbb{X},\\ T^k(\mathbf{x}')=T^k(\mathbf{x})}}\tau^kQ(\mathbf{x}',0)\qquad\text{for } L(\mathbf{x})-m\le k\le -1. \tag{2.13}$$

Since the sequence is stationary, the Lévy measures of $(X_k,\dots,X_{k+m})$ are equal for $0\le k\le m$. These Lévy measures are the respective projections of $Q|_{\mathbb{R}^{\{0,1,\dots,2m\}}}$ onto $\mathbb{R}^{\{k,\dots,k+m\}}$.

$\mathbb{R}^{\{k,\dots,k+m\}}$ is partitioned into rectangular subsets with cross-sections either $\{0\}$ or $\mathbb{R}\setminus\{0\}$. For $\mathbf{x}\in\mathbb{X}$ and $l\in\{L(\mathbf{x})-m,\dots,0\}$, let $B_k(\mathbf{x},l)$ be a rectangular subset of $\mathbb{R}^{\{k,\dots,k+m\}}$ such that the projection of $B_k(\mathbf{x},l)$ onto $\mathbb{R}^{\{k+j\}}$ is $\mathbb{R}\setminus\{0\}$ if $[T^l(\mathbf{x})]_j=1$ and is $\{0\}$ if $[T^l(\mathbf{x})]_j=0$. Then, $\{B_k(\mathbf{x},l)\}_{\mathbf{x}}\in\mathbb{X}, l\in\{L(\mathbf{x})-m,\dots,0\}$ is a disjoint partition of $\mathbb{R}^{\{k,\dots,k+m\}}\setminus\{0\}$.

Note that $B_k(\mathbf{x},0)\subset\mathbb{R}^{\{k,\dots,k+m-1\}}\times(\mathbb{R}\setminus\{0\})^{\{k+m\}}$. When $Q|_{\mathbb{R}^{\{0,1,\dots,2m\}}}$ is projected onto $\mathbb{R}^{\{k,\dots,k+m\}}$, the projection of the following is supported on $B_k(\mathbf{x},0)$:

$$\sum_{j=0}^{m-L(\mathbf{x})}\left(\sum_{\substack{\mathbf{x}'\in\mathbb{X},\\ T^{m+j}(\mathbf{x}')=T^m(\mathbf{x})}} Q(\mathbf{x},k+j)\right)\qquad \text{if } 0\le k\le L(\mathbf{x}) \tag{2.14a}$$

and

$$\sum_{j=0}^{m-k}\left(\sum_{\substack{\mathbf{x}'\in\mathbb{X},\\ T^{m+j}(\mathbf{x}')=T^m(\mathbf{x})}} Q(\mathbf{x},k+j)\right)\qquad \text{if } L(\mathbf{x})\le k\le m. \tag{2.14b}$$

Also, $B_k(\mathbf{x},L(\mathbf{x})-m)\subset(\mathbb{R}\setminus\{0\})^{\{k\}}\times\mathbb{R}^{\{k+1,\dots,k+m\}}$. When $Q|_{\mathbb{R}^{\{0,1,\dots,2m\}}}$ is projected onto $\mathbb{R}^{\{k,\dots,k+m\}}$, the projection of the following is supported on $B_k(\mathbf{x},L(\mathbf{x})-m)$:

$$\sum_{\substack{\mathbf{x}'\in\mathbb{X},\\ L(\mathbf{x})\le L(\mathbf{x}')\le L(\mathbf{x})+k,\\ T^{L(\mathbf{x})-m}(\mathbf{x}')=T^{L(\mathbf{x})-m}(\mathbf{x})}} Q\big(\mathbf{x}',L(\mathbf{x})-m+k\big)\qquad \text{if } 0\le k\le m-L(\mathbf{x}) \tag{2.15a}$$

and

$$\sum_{\substack{\mathbf{x}'\in\mathbb{X},\\ L(\mathbf{x})\le L(\mathbf{x}')\le m,\\ T^{L(\mathbf{x})-m}(\mathbf{x}')=T^{L(\mathbf{x})-m}(\mathbf{x})}} Q\big(\mathbf{x}',L(\mathbf{x})-m+k\big)\qquad \text{if } m-L(\mathbf{x})\le k\le m. \tag{2.15b}$$

Expressions (2.14) and (2.15) guarantee that (2.12) and (2.13) hold as shown below in Lemmas 2.4 and 2.5, respectively. □

LEMMA 2.4. *For each* $\mathbf{x}\in\mathbb{X}$, *the measures* $\{Q(\mathbf{x},k)\}_{1\le k\le m}$ *satisfy the recursion equation*

$$Q(\mathbf{x},k)=\begin{cases}\tau Q(\mathbf{x},k-1), & \text{for } 1\le k\le L(\mathbf{x}),\\ \tau Q(\mathbf{x},k-1)+\tau Q(\mathbf{x}',m), & \text{for } L(\mathbf{x})<k\le m,\end{cases} \tag{2.16}$$

where $\mathbf{x}'$ *is such that* $T^{m+1}(\mathbf{x}')=T^k(\mathbf{x})$. *Furthermore*, (2.12) *solves this recursion equation.*

PROOF. Lemma 2.4 is proven inductively. First, we show that (2.16) is satisfied for $\mathbf{x}$ such that $L(\mathbf{x}) = m$. When $Q|_{\mathbb{R}^{\{0,1,\dots,2m\}}}$ is projected onto $\mathbb{R}^{\{k,\dots,k+m\}}$ only the projection of $Q(\mathbf{x},k)$ is supported on $B_k(\mathbf{x},0)$, for $0 \le k \le m$ [see (2.14)]. Since the sequence is stationary,

$$Q(\mathbf{x},k)|_{\mathbb{R}^{\{k,\dots,k+m\}}} = Q(\mathbf{x},k-1)|_{\mathbb{R}^{\{k-1,\dots,k-1+m\}}} = [\tau Q(\mathbf{x},k-1)]_{\mathbb{R}^{\{k,\dots,k+m\}}}.$$

The second equality follows from the definition of τ. Since $Q(\mathbf{x},k)$ and $\tau Q(\mathbf{x}, k-1)$ are supported in $\{0\}^{\{0,\dots,k-1\}} \times \mathbb{R}^{\{k,\dots,k+m\}} \times \{0\}^{\{k+m+1,\dots,2m\}}$, it follows that

$$Q(\mathbf{x},k) = \tau Q(\mathbf{x},k-1). \tag{2.17}$$

Next, fix $\mathbf{x}$ arbitrary with $L(\mathbf{x}) < m$. We show that if (2.16) is true for all $\mathbf{x}'$ such that $L(\mathbf{x}') > L(\mathbf{x})$, then it is true for $\mathbf{x}$. The assumption that (2.16) is true for all $\mathbf{x}'$ such that $L(\mathbf{x}') > L(\mathbf{x})$ will be referred to as the induction hypothesis.

For $0 \le k \le L(\mathbf{x})$, when $Q|_{\mathbb{R}^{\{0,1,\dots,2m\}}}$ is projected onto $\mathbb{R}^{\{k,\dots,k+m\}}$, (2.14) gives that the restriction to $B_k(\mathbf{x},0)$ is

$$\sum_{j=0}^{m-L(\mathbf{x})} \left(\sum_{\substack{\mathbf{x}' \in \mathbb{X}, \\ T^{m+j}(\mathbf{x}') = T^m(\mathbf{x})}} Q(\mathbf{x}', k+j)|_{\mathbb{R}^{\{k,\dots,k+m\}}} \right).$$

Since the sequence is stationary, for $1 \le k \le L(\mathbf{x})$

$$\begin{aligned} &\sum_{j=0}^{m-L(\mathbf{x})} \left(\sum_{\substack{\mathbf{x}' \in \mathbb{X}, \\ T^{m+j}(\mathbf{x}') = T^m(\mathbf{x})}} Q(\mathbf{x}', k+j)|_{\mathbb{R}^{\{k,\dots,k+m\}}} \right) \\ &\quad = \sum_{j=0}^{m-L(\mathbf{x})} \left(\sum_{\substack{\mathbf{x}' \in \mathbb{X}, \\ T^{m+j}(\mathbf{x}') = T^m(\mathbf{x})}} Q(\mathbf{x}', k-1+j)|_{\mathbb{R}^{\{k-1,\dots,k-1+m\}}} \right). \end{aligned} \tag{2.18}$$

Since $1 \le k + j \le L(\mathbf{x}) + j = L(\mathbf{x}')$, the induction hypothesis implies that the LHS of (2.18) is

$$\begin{aligned} &Q(\mathbf{x},k)|_{\mathbb{R}^{\{k,\dots,k+m\}}} \\ &\quad + \sum_{j=1}^{m-L(\mathbf{x})} \left(\sum_{\substack{\mathbf{x}' \in \mathbb{X}, \\ T^{m+j}(\mathbf{x}') = T^m(\mathbf{x})}} [\tau Q(\mathbf{x}', k-1+j)]|_{\mathbb{R}^{\{k,\dots,k+m\}}} \right). \end{aligned} \tag{2.19}$$

Moreover, the RHS of (2.18) is equal to

$$\sum_{j=0}^{m-L(\mathbf{x})}\left(\sum_{\substack{\mathbf{x}'\in\mathbb{X},\\ T^{m+j}(\mathbf{x}')=T^m(\mathbf{x})}}[\tau Q(\mathbf{x}',k-1+j)]|_{\mathbb{R}^{\{k,\dots,k+m\}}}\right). \tag{2.20}$$

By canceling out the common terms in (2.19) and (2.20), the equation reduces to

$$Q(\mathbf{x},k)|_{\mathbb{R}^{\{k,\dots,k+m\}}}=[\tau Q(\mathbf{x},k-1)]|_{\mathbb{R}^{\{k,\dots,k+m\}}}.$$

Since $Q(\mathbf{x},k)$ and $\tau Q(\mathbf{x},k-1)$ are each supported on

$$\{0\}^{\{0,\dots,k-1\}}\times\mathbb{R}^{\{k,\dots,k+m\}}\times\{0\}^{\{k+m+1,\dots,2m\}},$$

it follows that

$$Q(\mathbf{x},k)=\tau Q(\mathbf{x},k-1). \tag{2.21}$$

For $L(\mathbf{x})\le k\le m$, when $Q|_{\mathbb{R}^{\{0,1,\dots,2m\}}}$ is projected onto $\mathbb{R}^{\{k,\dots,k+m\}}$, (2.14) gives that the restriction to $B_k(\mathbf{x},0)$ is

$$\sum_{j=0}^{m-k}\left(\sum_{\substack{\mathbf{x}'\in\mathbb{X},\\ T^{m+j}(\mathbf{x}')=T^m(\mathbf{x})}}Q(\mathbf{x}',k+j)|_{\mathbb{R}^{\{k,\dots,k+m\}}}\right).$$

Since the sequence is stationary, for $L(\mathbf{x})<k\le m$,

$$\begin{aligned}&\sum_{j=0}^{m-k}\left(\sum_{\substack{\mathbf{x}'\in\mathbb{X},\\ T^{m+j}(\mathbf{x}')=T^m(\mathbf{x})}}Q(\mathbf{x},k+j)|_{\mathbb{R}^{\{k,\dots,k+m\}}}\right)\\&\quad=\sum_{j=0}^{m-k+1}\left(\sum_{\substack{\mathbf{x}'\in\mathbb{X},\\ T^{m+j}(\mathbf{x}')=T^m(\mathbf{x})}}Q(\mathbf{x},k-1+j)|_{\mathbb{R}^{\{k-1,\dots,k-1+m\}}}\right).\end{aligned} \tag{2.22}$$

Since $k+j>L(\mathbf{x})+j=L(\mathbf{x}')$, the induction hypothesis implies that the LHS of (2.22) is

$$Q(\mathbf{x},k)+\sum_{j=1}^{m-k}\left(\sum_{\substack{(\mathbf{x}',\mathbf{x}'')\in\mathbb{X}\times\mathbb{X},\\ T^{m+j}(\mathbf{x}')=T^m(\mathbf{x}),\\ T^{m+1}(\mathbf{x}'')=T^{k+j}(\mathbf{x}')}}\big(\tau Q(\mathbf{x}',k-1+j)+\tau Q(\mathbf{x}'',m)\big)\right) \tag{2.23}$$

projected onto $\mathbb{R}^{\{k,\dots,k+m\}}$. The RHS of (2.22) is

$$\sum_{j=0}^{m-k+1}\left(\sum_{\substack{\mathbf{x}'\in\mathbb{X},\\ T^{m+j}(\mathbf{x}')=T^m(\mathbf{x})}}\tau Q(\mathbf{x}',k-1+j)\right)\tag{2.24}$$

projected onto $\mathbb{R}^{\{k,\dots,k+m\}}$. By cancelling out the common terms in (2.23) and (2.24), the equation reduces to

$$\begin{aligned}Q(\mathbf{x},k)&+\sum_{j=1}^{m-k}\left(\sum_{\substack{(\mathbf{x}',\mathbf{x}'')\in\mathbb{X}\times\mathbb{X},\\ T^{m+j}(\mathbf{x}')=T^m(\mathbf{x}),\\ T^{m+1}(\mathbf{x}'')=T^{k+j}(\mathbf{x}')}}\tau Q(\mathbf{x}'',m)\right)\\ &=\tau Q(\mathbf{x},k-1)+\sum_{\substack{\mathbf{x}'\in\mathbb{X},\\ T^{2m-k+1}(\mathbf{x}')=T^m(\mathbf{x})}}\tau Q(\mathbf{x}',m),\end{aligned}\tag{2.25}$$

where each side is projected onto $\mathbb{R}^{\{k,\dots,k+m\}}$. Equation (2.25) is further simplified by carefully comparing the terms in the sums on each side. Recall that $\mathbf{x}$ is a vector of length $2m+1$ of the form

$$\left|\begin{array}{c|c|c}0\dots0 & 1*\cdots*1 & 0\dots0\\ m-L(\mathbf{x}) & L(\mathbf{x})+1 & m\end{array}\right|$$

where $1*\cdots*1$ denotes the pattern of 0's or 1's in $\mathbf{x}$ [if $L(\mathbf{x})=0$, then this pattern is a single 1]. For the terms in the LHS of (2.25), $\mathbf{x}'$ is a vector of the form

$$\left|\begin{array}{c|c|c|c|c}0\dots0 & 1*\cdots*1 & \text{0's or 1's} & 1 & 0\dots0\\ m-L(\mathbf{x})-j & L(\mathbf{x})+1 & j-1 & 1 & m\end{array}\right|$$

and $\mathbf{x}''$ is a vector of the form

$$\left|\begin{array}{c|c|c|c|c|c|c}0\dots0 & 1*\cdots*1 & \text{0's or 1's} & 1 & 0\dots0 & 1 & 0\dots0\\ k-L(\mathbf{x})-1 & L(\mathbf{x})+1 & j-1 & 1 & m-k-j & 1 & m\end{array}\right|.$$

For the terms on the RHS of (2.25), $\mathbf{x}'$ is a vector of the form

$$\left|\begin{array}{c|c|c|c|c}0\dots0 & 1*\cdots*1 & \text{0's or 1's} & 1 & 0\dots0\\ k-L(\mathbf{x})-1 & L(\mathbf{x})+1 & m-k & 1 & m\end{array}\right|.$$

Therefore, the terms on each side are the same except for the additional term on the RHS where $\mathbf{x}'$ is

$$\left|\begin{array}{c|c|c|c|c}0\dots0 & 1*\cdots*1 & 0\dots0 & 1 & 0\dots0\\ k-L(x)-1 & L(\mathbf{x})+1 & m-k & 1 & m\end{array}\right|.$$

Thus (2.25) further simplifies to

$$Q(\mathbf{x},k)|_{\mathbb{R}^{\{k,\dots,k+m\}}}=\left[\tau Q(\mathbf{x},k-1)+\tau Q(\mathbf{x}',m)\right]\big|_{\mathbb{R}^{\{k,\dots,k+m\}}},$$

where $\mathbf{x}'$ is such that $T^{m+1}(\mathbf{x}') = T^k(\mathbf{x})$. Since $Q(\mathbf{x}, k)$, $\tau Q(\mathbf{x}, k-1)$, and $\tau Q(\mathbf{x}', m)$ are each supported on

$$\{0\}^{\{0,\dots,k-1\}} \times \mathbb{R}^{\{k,\dots,k+m\}} \times \{0\}^{\{k+m+1,\dots,2m\}},$$

it follows that

$$Q(\mathbf{x}, k) = \tau Q(\mathbf{x}, k-1) + \tau Q(\mathbf{x}', m). \tag{2.26}$$

Equations (2.17), (2.21) and (2.26) complete the proof of (2.16). It remains to show (by induction) that since $\{Q(\mathbf{x}, k)\}_{1 \le k \le m}$ satisfy (2.16), $\{Q(\mathbf{x}, k)\}_{1 \le k \le m}$ also satisfy (2.12).

For pairs $Q(\mathbf{x}, k)$ with $1 \le k \le L(\mathbf{x})$, the implication is clear. Fix $k > L(\mathbf{x})$ and assume that $Q(\mathbf{x}, k-1)$ and $Q(\mathbf{y}, m)$ satisfy (2.12) for $\mathbf{y} \in \mathbb{X}$ such that $T^{m+1}(\mathbf{y}) = T^k(\mathbf{x})$. Note that $L(\mathbf{y}) = L(\mathbf{x}) + m - k + 1 = m$.

If $k = L(\mathbf{x}) + 1$, then

$$Q(\mathbf{x}, k) = \tau Q(\mathbf{x}, k-1) + \tau Q(\mathbf{y}, m) = \tau^k Q(\mathbf{x}, 0) + \tau^{m+1} Q(\mathbf{y}, 0),$$

which is equivalent to $Q(\mathbf{x}, k)$ satisfying (2.12). The first equality is the recursion equation. The second equality comes from the induction assumption.

If $L(\mathbf{x}) + 1 < k \le m$, then

$$\begin{aligned}
Q(\mathbf{x}, k) &= \tau Q(\mathbf{x}, k-1) + \tau Q(\mathbf{y}, m) \\
&= \tau^k Q(\mathbf{x}, 0) + \sum_{j=2}^{k-L(\mathbf{x})} \left(\sum_{\substack{\mathbf{x}'' \in \mathbb{X}, \\ T^{j-1+m}(\mathbf{x}'') = T^{k-1}(\mathbf{x})}} \tau^{m+j} Q(\mathbf{x}'', 0) \right) \\
&\qquad + \tau^{m+1} Q(j, 0) + \sum_{j=2}^{k-L(\mathbf{x})} \left(\sum_{\substack{\mathbf{y}'' \in \mathbb{X}, \\ T^{j-1+m}(\mathbf{y}'') = T^{m}(\mathbf{y})}} \tau^{m+j} Q(\mathbf{y}'', 0) \right) \\
&= \tau^k Q(\mathbf{x}, 0) + \sum_{j=1}^{k-L(\mathbf{x})} \left(\sum_{\substack{\mathbf{x}' \in \mathbb{X}, \\ T^{j+m}(\mathbf{x}') = T^{k}(\mathbf{x})}} \tau^{m+j} Q(\mathbf{x}', 0) \right).
\end{aligned}$$

The first equality is the recursion equation. The second equality comes from the induction assumption, and the change of variable $j + 1 \to j$. The last equality is verified by comparing the terms in each sum. $\square$

Lemma 2.4 verifies (2.12). The proof of (2.13) is now presented.

LEMMA 2.5. *For each* $\mathbf{x} \in \mathbb{X}$ *with* $L(\mathbf{x}) < m$, *the measures* $\{Q(\mathbf{x}, k)\}_{L(\mathbf{x})-m \le k \le -1}$ *satisfy the recursion equation*

$$Q(\mathbf{x}, k) = \tau^{-1} Q(\mathbf{x}, k+1) + \tau^{-1} Q(\mathbf{x}', k+1) \qquad \text{for } L(\mathbf{x}) - m \le k \le -1, \tag{2.27}$$

where $\mathbf{x}'$ *is such that* $T^k(\mathbf{x}') = T^k(\mathbf{x})$ *and* $L(\mathbf{x}') = k + m + 1$. *Furthermore*, (2.13) *solves this recursion equation.*

PROOF. Lemma 2.5 is also proven inductively. First, we show (2.27) for $\mathbf{x}$ such that $L(\mathbf{x}) = m - 1$. When $Q|_{\mathbb{R}^{\{0,1,\dots,2m\}}}$ is projected onto $\mathbb{R}^{\{0,\dots,m\}}$, (2.15) gives that only the projection of $Q(\mathbf{x}, -1)$ is supported on $B_0(\mathbf{x}, -1)$. When $Q|_{\mathbb{R}^{\{0,1,\dots,2m\}}}$ is projected onto $\mathbb{R}^{\{1,\dots,1+m\}}$, (2.15) gives that the projections of both $Q(\mathbf{x}, 0)$ and $Q(\mathbf{x}', 0)$ are supported on $B_1(\mathbf{x}, -1)$ where $\mathbf{x}'$ is such that $T^{-1}(\mathbf{x}') = T^{-1}(\mathbf{x})$ and $L(\mathbf{x}') = m$. Since the sequence is stationary,

$$\begin{aligned} Q(\mathbf{x}, -1)|_{\mathbb{R}^{\{0,\dots,m\}}} &= [Q(\mathbf{x}, 0) + Q(\mathbf{x}', 0)]|_{\mathbb{R}^{\{1,\dots,1+m\}}} \\ &= [\tau^{-1} Q(\mathbf{x}, 0) + \tau^{-1} Q(\mathbf{x}', 0)]|_{\mathbb{R}^{\{0,\dots,m\}}}. \end{aligned}$$

The second inequality follows from the definition of τ. Since $Q(\mathbf{x}, -1)$, $\tau^{-1} Q(\mathbf{x}, 0)$ and $\tau^{-1} Q(\mathbf{x}', 0)$ are supported in $\mathbb{R}^{\{0,\dots,m\}} \times \{0\}^{\{m+1,\dots,2m\}}$,

$$Q(\mathbf{x}, -1) = \tau^{-1} Q(\mathbf{x}, 0) + \tau^{-1} Q(\mathbf{x}', 0). \tag{2.28}$$

Next, fix $\mathbf{x}$ arbitrary with $L(\mathbf{x}) < m - 1$. We show that if (2.27) is true for all $\mathbf{x}'$ such that $L(\mathbf{x}) < L(\mathbf{x}') < m$, then (2.27) is true for $\mathbf{x}$. The assumption that (2.27) is true for all $\mathbf{x}'$ such that $L(\mathbf{x}) < L(\mathbf{x}') < m$ will be referred to as the induction hypothesis.

For $L(\mathbf{x}) - m \le k \le 0$, (2.15) gives that

$$\sum_{\substack{\mathbf{x}' \in \mathbb{X}, \\ T^{L(\mathbf{x})-m}(\mathbf{x}') = T^{L(\mathbf{x})-m}(\mathbf{x}), \\ L(\mathbf{x}) \le L(\mathbf{x}') \le k+m}} Q(\mathbf{x}', k)|_{\mathbb{R}^{\{k-L(\mathbf{x})+m,\dots,k-L(\mathbf{x})+2m\}}}$$

is supported on $B_{k-L(\mathbf{x})+m}(\mathbf{x}, L(\mathbf{x}) - m)$. By stationarity, for $L(\mathbf{x}) - m \le k \le -1$,

$$\begin{aligned} &\sum_{\substack{\mathbf{x}' \in \mathbb{X}, \\ T^{L(\mathbf{x})-m}(\mathbf{x}') = T^{L(\mathbf{x})-m}(\mathbf{x}), \\ L(\mathbf{x}) \le L(\mathbf{x}') \le k+m}} Q(\mathbf{x}', k)|_{\mathbb{R}^{\{k-L(\mathbf{x})+m,\dots,k-L(\mathbf{x})+2m\}}} \\ &\qquad = \sum_{\substack{\mathbf{x}' \in \mathbb{X}, \\ T^{L(\mathbf{x})-m}(\mathbf{x}') = T^{L(\mathbf{x})-m}(\mathbf{x}), \\ L(\mathbf{x}) \le L(\mathbf{x}') \le k+1+m}} Q(\mathbf{x}', k+1)|_{\mathbb{R}^{\{k+1-L(\mathbf{x})+m,\dots,k+1-L(\mathbf{x})+2m\}}}. \end{aligned} \tag{2.29}$$

The induction hypothesis implies that the LHS of (2.29) is

$$Q(\mathbf{x}, k) + \sum_{\substack{(\mathbf{x}', \mathbf{x}'') \in \mathbb{X} \times \mathbb{X}, \\ T^{L(\mathbf{x})-m}(\mathbf{x}') = T^{L(\mathbf{x})-m}(\mathbf{x}), \\ L(\mathbf{x}) < L(\mathbf{x}') \le k+m T^k(\mathbf{x}'') = T^k(\mathbf{x}'), \\ L(\mathbf{x}'') = k+m+1,}} \tau^{-1} Q(\mathbf{x}', k+1) + \tau^{-1} Q(\mathbf{x}'', k+1), \tag{2.30}$$

projected onto $\mathbb{R}^{\{k-L(\mathbf{x})+m,\dots,k-L(\mathbf{x})+2m\}}$. The RHS of (2.29) is

$$\sum_{\substack{\mathbf{x}'\in\mathbb{X},\\ T^{L(\mathbf{x})-m}(\mathbf{x}')=T^{L(\mathbf{x})-m}(\mathbf{x}),\\ L(\mathbf{x})\le L(\mathbf{x}')\le k+1+m}} \big[\tau^{-1}Q(\mathbf{x}',k+1)\big]\big|_{\mathbb{R}^{\{k-L(\mathbf{x})+m,\dots,k-L(\mathbf{x})+2m\}}}. \tag{2.31}$$

By canceling out the common terms in (2.30) and (2.31), the equation reduces to

$$\begin{aligned} Q(\mathbf{x},k)&+\sum_{\substack{(\mathbf{x}',\mathbf{x}'')\in\mathbb{X}\times\mathbb{X},\\ T^{L(\mathbf{x})-m}(\mathbf{x}')=T^{L(\mathbf{x})-m}(\mathbf{x}),\\ L(\mathbf{x})<L(\mathbf{x}')\le k+m,\\ T^k(\mathbf{x}'')=T^k(\mathbf{x}'),\\ L(\mathbf{x}'')=k+m+1}} \tau^{-1}Q(\mathbf{x}'',k+1)\\ &=\tau^{-1}Q(\mathbf{x},k+1)+\sum_{\substack{\mathbf{x}'\in\mathbb{X},\\ T^{L(\mathbf{x})-m}(\mathbf{x}')=T^{L(\mathbf{x})-m}(\mathbf{x}),\\ L(\mathbf{x}')=k+m+1}} \tau^{-1}Q(\mathbf{x}',k+1), \end{aligned} \tag{2.32}$$

where each side is projected onto $\mathbb{R}^{\{k-L(\mathbf{x})+m,\dots,k-L(\mathbf{x})+2m\}}$. The sum is further simplified by carefully comparing the terms on each side. Recall that $\mathbf{x}$ is a vector of length $m+1$ of the form

$$\left|\begin{array}{c|c|c} 0\dots0 & 1*\cdots*1 & 0\dots0\\ m-L(\mathbf{x}) & L(\mathbf{x})+1 & m \end{array}\right|.$$

For the terms on the LHS of (2.32), $\mathbf{x}'$ is a vector of the form

$$\left|\begin{array}{c|c|c|c} 0\dots0 & \text{at least one }1 & 1*\cdots*1 & 0\dots0\\ -k & k+m-L(\mathbf{x}) & L(\mathbf{x})+1 & m \end{array}\right|$$

and $\mathbf{x}''$ is a vector of the form

$$\left|\begin{array}{c|c|c|c|c} 0\dots0 & 1 & \text{at least one }1 & 1*\cdots*1 & 0\dots0\\ -k-1 & 1 & k+m-L(\mathbf{x}) & L(\mathbf{x})+1 & m \end{array}\right|.$$

For the terms on the RHS of (2.32), $\mathbf{x}'$ is a vector of the form

$$\left|\begin{array}{c|c|c|c|c} 0\dots0 & 1 & \text{0's or 1's} & 1*\cdots*1 & 0\dots0\\ -k-1 & 1 & k+m-L(\mathbf{x}) & L(\mathbf{x})+1 & m \end{array}\right|.$$

Therefore, the terms in the sums on each side are the same except for the additional term on the RHS where $\mathbf{x}'$ is

$$\left|\begin{array}{c|c|c|c|c} 0\dots0 & 1 & 0\dots0 & 1*\cdots*1 & 0\dots0\\ -k-1 & 1 & k+m-L(\mathbf{x}) & L(\mathbf{x})+1 & m \end{array}\right|.$$

Hence (2.32) further simplifies to

$$\begin{aligned} &Q(\mathbf{x},k)|_{\mathbb{R}^{\{k-L(\mathbf{x})+m,\dots,k-L(\mathbf{x})+2m\}}}\\ &\quad=\big[\tau^{-1}Q(\mathbf{x},k+1)+\tau^{-1}Q(\mathbf{x}',k+1)\big]\big|_{\mathbb{R}^{\{k-L(\mathbf{x})+m,\dots,k-L(\mathbf{x})+2m\}}}, \end{aligned}$$

where $\mathbf{x}'$ is such that $T^k(\mathbf{x}') = T^k(\mathbf{x})$ and $L(\mathbf{x}') = k + m + 1$. Since $Q(\mathbf{x}, k)$, $\tau^{-1}Q(\mathbf{x}, k+1)$ and $\tau^{-1}Q(\mathbf{x}', k+1)$ are each supported on

$$\{0\}^{\{0,\dots,k-L(\mathbf{x})+m-1\}} \times \mathbb{R}^{\{k-L(\mathbf{x})+m,\dots,k-L(\mathbf{x})+2m\}} \times \{0\}^{\{k-L(\mathbf{x})+2m+1\dots,2m\}},$$

it follows that

$$Q(\mathbf{x}, k) = \tau^{-1}Q(\mathbf{x}, k+1) + \tau^{-1}Q(\mathbf{x}', k+1). \tag{2.33}$$

Equations (2.28) and (2.33) complete the proof of (2.27). It remains to show that since $\{Q(\mathbf{x}, k)\}_{L(\mathbf{x})-m\le k\le -1}$ satisfy (2.27), $\{Q(\mathbf{x}, k)\}_{L(\mathbf{x})-m\le k\le -1}$ also satisfy (2.13). Once more, this is done by induction.

For pairs $Q(\mathbf{x}, k)$ with $k = -1$, the implication is clear. Fix $L(\mathbf{x}) - m \le k < -1$ and assume that $Q(\mathbf{x}, k+1)$ and $Q(\mathbf{y}, k+1)$ satisfy (2.13) for $\mathbf{y} \in \mathbb{X}$ such that $T^k(\mathbf{y}) = T^k(\mathbf{x})$. Note that $L(\mathbf{y}) = k + m + 1 > L(\mathbf{x})$.

Then

$$\begin{aligned} Q(\mathbf{x}, k) &= \tau^{-1}Q(\mathbf{x}, k+1) + \tau^{-1}Q(\mathbf{y}, k+1) \\ &= \sum_{\substack{\mathbf{x}''\in\mathbb{X},\\ T^{k+1}(\mathbf{x}'')=T^{k+1}(\mathbf{x})}} \tau^k Q(\mathbf{x}'', 0) + \sum_{\substack{\mathbf{y}''\in\mathbb{X},\\ T^{k+1}(\mathbf{y}'')=T^{k+1}(\mathbf{y})}} \tau^k Q(\mathbf{y}'', 0) \\ &= \sum_{\substack{\mathbf{x}'\in\mathbb{X},\\ T^{k}(\mathbf{x}')=T^{k}(\mathbf{x})}} \tau^k Q(\mathbf{x}', 0). \end{aligned}$$

The first equality is the recursion equation. The second equality comes from the induction assumption. The last equality is verified by comparing the terms in each sum.

This concludes the proof of Theorem 2.1. $\square$

REMARK 2.1. In the proof of Theorem 2.1, a ξ with a particular Lévy measure $\widehat{Q}$ is shown to generate $\{X_j\}_{j\in\mathbb{Z}}$. Instead, let $\widehat{Q}_1$ be the Lévy measure of $\{X_j\}_{j=-m}^m$ restricted to

$$\{0\}^{\{-m,\dots,-1\}} \times (\mathbb{R} \setminus \{0\})^{\{0\}} \times \mathbb{R}^{\{1,\dots,m\}}$$

and let ξ_1 be the infinitely divisible vector with characterizing triplet (0, 0, $\widehat{Q}_1|_{\mathbb{R}^{\{0,\dots,m\}}}$). Then ξ_1 also generates $\{X_j\}_{j\in\mathbb{Z}}$.

We now come to the proof of Corollary 2.1.

PROOF OF COROLLARY 2.1. The classes of α-stable and compound Poisson vectors are closed under linear combinations. The class of infinitely divisible vectors such that all the one-dimensional marginals are Poisson is also closed under linear combinations. Therefore, if ξ is in one of these classes, all of the finite-dimensional marginals of the sequence generated by ξ (and thus the sequence itself) are in the same class of distributions.

To prove the converse, we assume that the stationary m-dependent sequence $\{X_j\}_{j\in\mathbb{Z}}$ is α-stable, compound Poisson, or infinitely divisible with Poisson one-dimensional marginals, and we show that there exists a ξ from the same class of distributions such that ξ generates $\{X_j\}_{j\in\mathbb{Z}}$.

If $\{X_j\}_{j\in\mathbb{Z}}$ is α-stable, then $\{X_j\}_{j=0}^{2m}$ has characterizing triplet $(\mu\mathbf{1}, 0, Q)$ where μ is the location parameter of X_0, $\mathbf{1} = (1, 1, \dots, 1)$, and Q is a Borel measure on $\mathbb{R}^{2m+1}$ given by

$$Q(d\mathbf{x}) = \frac{1}{r^{1+\alpha}}\Gamma(d\mathbf{s}),$$

where Γ is a measure on the unit sphere in S_{2m+1} [see, e.g., Samorodnitsky and Taqqu (1994)]. Let

$$\widehat{Q}(d\mathbf{x}) = \frac{1}{r^{1+\alpha}}\widehat{\Gamma}(d\mathbf{s}),$$

where

$$\widehat{\Gamma}(d\mathbf{s}) = \begin{cases} \Gamma(d\mathbf{s}), & \text{for } \mathbf{s} \in S_{2m+1} \cap \bigl(\mathbb{R}^m \times (\mathbb{R}\setminus\{0\}) \times \{0\}^m\bigr), \\ 0, & \text{otherwise.} \end{cases}$$

Then $(0, 0, \widehat{Q})$ is the characterizing triplet of $(\xi^0, \dots, \xi^{2m})$ where $(\xi^0, \dots, \xi^m)$ is an α-stable random vector in $\mathbb{R}^{m+1}$ and $(\xi^{m+1}, \dots, \xi^{2m}) \equiv \mathbf{0}$. Lemma 2.3 shows that $(\xi^0, \dots, \xi^m)$ generates an m-dependent stationary infinitely divisible sequence with characterizing triplet $(0, 0, Q)$. It thus follows that the α-stable vector $(\xi^0 + \mu, \xi^1, \dots, \xi^m)$ generates $\{X_j\}_{j\in\mathbb{Z}}$.

If $\{X_j\}_{j\in\mathbb{Z}}$ is compound Poisson, then $\{X_j\}_{j=0}^{2m}$ has characteristic function

$$\exp\bigl(\lambda(\varphi(\mathbf{t}) - 1)\bigr),$$

where $\lambda > 0$ and $\varphi(\mathbf{t})$ is the characteristic function of a vector $\mathbf{Y}$ with $P(\{\mathbf{Y} = 0\}) = 0$. Moreover, $\{X_j\}_{j=0}^{2m}$ has characterizing triplet $(b\mathbf{1}, 0, Q)$ where Q is given by

$$Q(A) = \lambda P(\mathbf{Y} \in A),$$
$$b = \int_{\mathbb{R}^{2m+1}\setminus\{\mathbf{0}\}} x_1 \mathbf{1}_{\|\mathbf{x}\|\le 1} Q(d\mathbf{x})$$

and $\mathbf{1} = (1, 1, \dots, 1)$. Let $\widehat{\mathbf{Y}}$ be $(Y_0, Y_1, \dots, Y_m)$ conditioned on

$$\begin{cases} Y_m \neq 0, \\ Y_k = 0, & \text{for } m+1 \le k \le 2m, \end{cases}$$

and let

$$\widehat{Q} = \lambda\, P\bigl(\mathbf{Y} \in \mathbb{R}^m \times (\mathbb{R}\setminus\{0\}) \times \{0\}^m\bigr)\, P(\widehat{\mathbf{Y}} \in A).$$

Then $(0, 0, \widehat{Q})$ is the characterizing triplet of a vector $(\xi^0, \dots, \xi^m)$ in $\mathbb{R}^{m+1}$. Lemma 2.3 shows that $(\xi^0, \dots, \xi^m)$ generates an m-dependent stationary infinitely divisible sequence with characterizing triplet $(0, 0, Q)$. Thus, the compound Poisson vector with Lévy measure $\widehat{Q}$ generates a stationary m-dependent compound Poisson sequence. Hence, this sequence is $\{X_j\}_{j\in\mathbb{Z}}$, since a compound Poisson sequence is uniquely determined by its Lévy measure.

If $\{X_j\}_{j\in\mathbb{Z}}$ is infinitely divisible and all the one-dimensional marginals are Poisson, then $\{X_j\}_{j\in\mathbb{Z}}$ is compound Poisson where the Lévy measure of $\{X_j\}_{j=0}^{2m}$ is supported on the points $\{0, 1\}^{2m+1}$. In this case $\widehat{Q}$ is the restriction of Q to $\{0, 1\}^m \times \{1\} \times \{0\}^m$, and the unique infinitely divisible vector with Lévy measure $\widehat{Q}$ generates $\{X_j\}_{j\in\mathbb{Z}}$. All of the one-dimensional marginals of this vector are Poisson. $\square$

3. Weak convergence of partial sums for stationary infinitely divisible sequences. Partial sum convergence for m-dependent stationary infinitely divisible sequences is studied next. These results are extended to stationary infinitely divisible sequences having a form of asymptotic independence in Section 3.2.

3.1. *m-dependent sequences.* Let $\{X_j\}_{j\in\mathbb{Z}}$ be an m-dependent stationary sequence with partial sums $S_n = \sum_{j=0}^{n-1} X_j$. While Theorem 1.1 gives sufficient conditions for the distributional convergence of the partial sums properly centered and normalized, the following theorem provides necessary and sufficient conditions.

THEOREM 3.1. *Let $\{X_j\}_{j\in\mathbb{Z}}$ be an m-dependent stationary infinitely divisible sequence with generating vector ξ and partial sums S_n. Then the characteristic function of $\sum_{k=0}^{m} \xi_0^k$ is equal to*

$$\frac{Ee^{itS_{m+1}}}{Ee^{itS_m}}.$$

Moreover, there exists a centering sequence $\{A_n\}$ and a normalizing sequence $\{B_n\}$ with $B_n \to \infty$ such that $\frac{S_n - A_n}{B_n}$ converges in distribution to a nondegenerate $S_\alpha(\sigma, \beta, \mu)$ random variable if and only if $\sum_{k=0}^{m} \xi_0^k$ is in the domain of attraction of the same $S_\alpha(\sigma, \beta, \mu)$ distribution.

PROOF. Let ξ be the generating vector of $\{X_j\}_{j\in\mathbb{Z}}$. Then

$$\begin{aligned} S_{m+1} &\stackrel{\mathcal{L}}{=} \sum_{j=0}^{m} \sum_{k=0}^{m} \xi_{j-k}^k \\ &= \sum_{i=-m}^{-1} \sum_{k=-i}^{m} \xi_i^k + \sum_{k=0}^{m} \xi_0^k + \sum_{i=1}^{m} \sum_{k=0}^{m-i} \xi_i^k. \end{aligned}$$

The sum of the first and third term is equal in distribution to

$$S_m \stackrel{\mathcal{L}}{=} \sum_{j=0}^{m-1}\sum_{k=0}^{m}\xi_{j-k}^k = \sum_{i=-m}^{-1}\sum_{k=-i}^{m}\xi_i^k + \sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\xi_i^k$$

and is independent of the second term. Thus,

$$\frac{Ee^{itS_{m+1}}}{Ee^{itS_m}}$$

is the characteristic function of $\sum_{k=0}^{m}\xi_0^k$.

Let $\{\widetilde{X}_j\}_{j\in\mathbb{Z}}$ be an i.i.d. sequence with $\widetilde{X}_0 \stackrel{\mathcal{L}}{=} \sum_{k=0}^{m}\xi_0^k$ and partial sums $\widetilde{S}_n$. Then, for $n\in\mathbb{N}$,

$$\begin{aligned}
S_n &\stackrel{\mathcal{L}}{=} \sum_{j=0}^{n-1}\sum_{k=0}^{m}\xi_{j-k}^k \\
&= \sum_{i=-m}^{-1}\sum_{k=-i}^{m}\xi_i^k + \sum_{i=0}^{n-1}\sum_{k=0}^{m}\xi_i^k - \sum_{i=n-m}^{n-1}\sum_{k=n-i}^{m}\xi_i^k \\
&\stackrel{\mathcal{L}}{=} \sum_{i=-m}^{-1}\sum_{k=-i}^{m}\xi_i^k + \widetilde{S}_n - \sum_{i=n-m}^{n-1}\sum_{k=n-i}^{m}\xi_i^k
\end{aligned}$$

and

$$\frac{S_n - A_n}{B_n} \stackrel{\mathcal{L}}{=} \frac{1}{B_n}\sum_{i=-m}^{-1}\sum_{k=-i}^{m}\xi_i^k + \frac{\widetilde{S}_n - A_n}{B_n} - \frac{1}{B_n}\sum_{i=n-m}^{n-1}\sum_{k=n-i}^{m}\xi_i^k.$$

If $B_n\to\infty$, then

$$P\left(\sum_{i=-m}^{-1}\sum_{k=-i}^{m}\xi_i^k > \varepsilon B_n^{-1}\right)\to 0,$$

$$\begin{aligned}
&P\left(\sum_{i=1}^{m}\sum_{k=m-i+1}^{m}\xi_{i+n-1-m}^k > \varepsilon B_n^{-1}\right) \\
&\qquad = P\left(\sum_{i=1}^{m}\sum_{k=m-i+1}^{m}\xi_{i-1-m}^k > \varepsilon B_n^{-1}\right)\to 0
\end{aligned}$$

and therefore

$$\frac{S_n - A_n}{B_n} \Rightarrow S_\alpha(\sigma,\beta,\mu)$$

if and only if

$$\frac{\widetilde{S}_n - A_n}{B_n} \Rightarrow S_\alpha(\sigma,\beta,\mu),$$

which is equivalent to $\sum_{k=0}^{m}\xi_0^k$ being in the domain of attraction of the $S_\alpha(\sigma,\beta,\mu)$ distribution. Since the limiting distribution is nondegenerate, the classical theory implies that $B_n\to\infty$ as $n\to\infty$. $\square$

REMARK 3.1. It follows from Theorem 2.1 that the ratio $\frac{Ee^{itS_{m+1}}}{Ee^{itS_m}}$ is a characteristic function. This is not necessarily true for noninfinitely divisible sequences of stationary m-dependent random variables.

REMARK 3.2. It is easy to check that the assumptions of Theorem 1.1 imply that the random variable with characteristic function $\frac{Ee^{itS_{m+1}}}{Ee^{itS_m}}$ is in the domain of attraction of a $S_\alpha(\sigma, \beta, \mu)$ distribution. An example of an m-dependent stationary infinitely divisible sequence that satisfies the assumptions of Theorem 3.1 but does not satisfy the assumptions of Theorem 1.1 has not been constructed.

3.2. *Approximation of infinitely divisible sequences by m-dependent ones.* We extend the weak convergence results of the previous section to a larger class of stationary infinitely divisible sequences. In particular, we study the class of such sequences that can be weakly approximated by m-dependent ones in a particular way.

Note that all stationary infinitely divisible sequences can be approximated weakly by m-dependent stationary infinitely divisible sequences as shown in the following proposition.

PROPOSITION 3.1. *Let $\{X_j\}_{j\in\mathbb{Z}}$ be a stationary infinitely divisible sequence. Let*

$$\varphi_n : \mathbb{R}^{n+1} \to \mathbb{R}$$

denote the characteristic function of $(X_0, X_1, \dots, X_n)$. Let

$$\{X_j(m)\}_{j\in\mathbb{Z}}$$

be the m-dependent stationary infinitely divisible sequence generated by the $(m+1)$-dimensional vector with characteristic function $(\varphi_m)^{1/m}$. Then

$$\{X_j(m)\}_{j\in\mathbb{Z}} \Rightarrow \{X_j\}_{j\in\mathbb{Z}},$$

as $m \to \infty$.

PROOF. Fix $n \in \mathbb{N}$ and $(t_0, t_1, \dots, t_n) \in \mathbb{R}^n$ arbitrary. Then

$$Ee^{i(t_0X_0(m)+t_1X_1(m)+\cdots+t_nX_n(m))} = \prod_{j=-m}^{n} \varphi_m^{1/m}(t_j, t_{j+1}, \dots, t_{j+m}), \tag{3.1}$$

where $t_j = 0$ for $j \notin \{0, 1, \dots, n\}$. Since $\{X_j\}_{j\in\mathbb{Z}}$ is stationary, when $m > n$ (3.1) reduces to

$$\prod_{j=-m}^{n-m-1} \varphi_{m+j}^{1/m}(t_0, \dots, t_{j+m}) \prod_{j=n-m}^{0} \varphi_n^{1/m}(t_0, \dots, t_n) \prod_{j=1}^{n} \varphi_{n-j}^{1/m}(t_j, \dots, t_n). \tag{3.2}$$

Note that

$$\left(\prod_{j=-m}^{n-m-1} \varphi_{m+j}(t_0, \dots, t_{j+m})\right)^{1/m} \to 1,$$

since the product does not depend on m, does not vanish and its absolute value is bounded by 1. Hence the first product in (3.2) goes to 1 as $m \to \infty$. The third product in (3.2) goes to 1 as $m \to \infty$ by a similar argument. The second product in (3.2) simplifies to $(\varphi_n(t_0, \dots, t_n))^{(m-n+1)/m}$ which converges to $\varphi_n(t_0, \dots, t_n)$, as $m \to \infty$. Thus

$$\lim_{m\to\infty} Ee^{i(t_0 X_0(m)+t_1 X_1(m)+\cdots t_n X_n(m))} = \varphi_n(t_0, t_1, \dots, t_n),$$

which completes the proof. $\square$

This proposition does not really help in the study of partial sum convergence for the original sequence $\{X_j\}_{j\in\mathbb{Z}}$. Thus, we turn our attention to stationary infinitely divisible sequences that can be approximated by m-dependent sequences in such a way that the results of Section 3.1 can be extended.

We assume that

$$\{X_j\}_{j\in\mathbb{Z}} \stackrel{\mathcal{L}}{=} \sum_{m=0}^{\infty} \{Y_j(m)\}_{j\in\mathbb{Z}},$$

where for each $m \in \mathbb{N}, \{Y_j(m)\}_{j\in\mathbb{Z}}$ is an m-dependent stationary infinitely divisible sequence and the collection is independent in m.

For each m, let $\xi(m) = (\xi^0(m), \dots, \xi^m(m))$ be the infinitely divisible vector that generates $\{Y_j(m)\}_{j\in\mathbb{Z}}$ according to Theorem 2.1. Suppose that $\xi(m)$ has characterizing triplet $(\widehat{\mathbf{b}}_m, \widehat{\Sigma}_m, \widehat{Q}_m)$, where

$$\widehat{\mathbf{b}}_m = \big(\widehat{b}_m(0), \widehat{b}_m(1), \dots, \widehat{b}_m(m)\big),$$
$$\widehat{\Sigma}_m = \{\widehat{\sigma}_m(k,l)\}_{k,l\in\{0,1,\dots,m\}},$$

and $\widehat{Q}_m$ is a Lévy measure on $\mathbb{R}^{\{0,1,\dots,m\}}$. The characterizing triplet of $\{Y_j(m)\}_{j\in\mathbb{Z}}$ is given by

$$b_m = \sum_{k=0}^{m} \widehat{b}_m(k),$$

$$\Sigma_m(j) = \begin{cases} \displaystyle\sum_{k=0}^{m-j} \widehat{\sigma}_m(k, j+k), & \text{for } 0 \le j \le m, \\ 0, & \text{for } j > m, \end{cases}$$

and the Lévy measure Q_m on $\mathbb{R}^{\mathbb{Z}}$ is defined by

$$Q_m = \sum_{k=-\infty}^{\infty} \tau^k \widehat{Q}_{m,\infty},$$

where $\widehat{Q}_{m,\infty}$ is a Lévy measure on $\mathbb{R}^{\mathbb{Z}}$ given by

$$\cdots \otimes \delta_{\{0\}} \otimes \widehat{Q}_m \otimes \delta_{\{0\}} \otimes \cdots.$$

Note that

$$\sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}^{\mathbb{Z}}\setminus\{\mathbf{0}\}} \min(1, x_0^2)\tau^k \widehat{Q}_{m,\infty}(d\mathbf{x}) \right) = \sum_{k=0}^{m} \left(\int_{\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}} \min(1, x_k^2) \widehat{Q}_m(d\mathbf{x}) \right).$$

Thus the sequence $\{X_j\}_{j\in\mathbb{Z}}$ is well defined if and only if

$$\sum_{m=0}^{\infty} \sum_{k=0}^{m} \widehat{b}_m(k) < \infty, \tag{3.3a}$$

$$\sum_{m=0}^{\infty} \sum_{k=0}^{m} \widehat{\sigma}_m(k,k) < \infty \tag{3.3b}$$

and

$$\sum_{m=0}^{\infty} \sum_{k=0}^{m} \left(\int_{\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}} \min(1, x_k^2) \widehat{Q}_m(d\mathbf{x}) \right) < \infty. \tag{3.3c}$$

Although all stationary infinitely divisible sequences can be approximated weakly by m-dependent ones, not all stationary infinitely divisible sequences can be approximated by m-dependent ones in the cumulative manner just described. In fact, sequences that can be approximated this way are strongly mixing in the sense of ergodic theory.

THEOREM 3.2. *For each $m \in \mathbb{N}$, let $\xi(m)$ be an $(m+1)$-dimensional infinitely divisible vector with characterizing triplet $(\widehat{\mathbf{b}}_m, \widehat{\Sigma}_m, \widehat{Q}_m)$. Let the sequence of characterizing triplets satisfies the conditions* (3.3). *Also let*

$$\{X_j\}_{j\in\mathbb{Z}} \stackrel{\mathcal{L}}{=} \sum_{m=0}^{\infty} \{Y_j(m)\}_{j\in\mathbb{Z}},$$

where $\{Y_j(m)\}_{j\in\mathbb{Z}}$ is the m-dependent stationary infinitely divisible sequence generated by $\xi(m)$, and the sequences $\{Y_j(m)\}_{j\in\mathbb{Z}}$ are taken to be independent in m. Then $\{X_j\}_{j\in\mathbb{Z}}$ is strongly mixing in the sense of ergodic theory.

PROOF. Assume that the Lévy measure of X_0 has no atoms in $2\pi\mathbb{Z}$. This assumption is without loss of generality since there exists $a \in \mathbb{R}$ such that the Lévy measure of aX_0 has no atoms in $2\pi\mathbb{Z}$, and $\{X_j\}_{j\in\mathbb{Z}}$ is mixing if and only if $\{aX_j\}_{j\in\mathbb{Z}}$ is mixing. Rosiński and Żak (1996) showed that a stationary infinitely divisible sequence is strongly mixing if and only if $\operatorname{Cod}(X_0, X_j) \to 0$ as $n \to \infty$, where Cod denotes the codifference function, that is,

$$\operatorname{Cod}(X_0, X_j) = \ln E e^{i(X_0 - X_j)} - \ln E e^{iX_0} - \ln E e^{-iX_j},$$

To show that this last condition is verified, note that

$$
\begin{aligned}
\operatorname{Cod}(X_0, X_j) &= \operatorname{Cod}\left(\sum_{m=0}^{\infty} Y_0(m), \sum_{m=0}^{\infty} Y_j(m)\right) \\
&= \sum_{m=j}^{\infty} \operatorname{Cod}\big(Y_0(m), Y_j(m)\big) \\
&= \sum_{m=j}^{\infty} \operatorname{Cod}\left(\sum_{k=0}^{m} \xi_{-k}^{k}(m), \sum_{k=0}^{m} \xi_{j-k}^{k}(m)\right) \qquad (3.4)\\
&= \sum_{m=j}^{\infty} \sum_{k=0}^{m-j} \operatorname{Cod}\big(\xi^{k}(m), \xi^{j+k}(m)\big) \\
&= \sum_{m=j}^{\infty} \sum_{k=0}^{m-j} \widehat{\sigma}_m(k, j+k) + \int_{\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}} (e^{ix_k} - 1)(e^{ix_{j+k}} - 1)\widehat{Q}_m(d\mathbf{x}).
\end{aligned}
$$

This is shown to go to zero in two steps. First,

$$
\begin{aligned}
\sum_{k=0}^{m-j} |\widehat{\sigma}_m(k, j+k)| &\leq \sum_{k=0}^{m-j} \left(\big(\widehat{\sigma}_m(k,k)\big)^{1/2}\big(\widehat{\sigma}_m(j+k, j+k)\big)^{1/2}\right) \\
&\leq \left(\sum_{k=0}^{m-j} \widehat{\sigma}_m(k,k)\right)^{1/2} \left(\sum_{k=0}^{m-j} \widehat{\sigma}_m(j+k, j+k)\right)^{1/2} \\
&\leq \sum_{k=0}^{m} \widehat{\sigma}_m(k,k),
\end{aligned}
$$

where the first inequality holds since for each m, $\widehat{\Sigma}_m$ is a positive definite matrix, while the second inequality is Cauchy–Schwarz. Thus,

$$
\left|\sum_{m=j}^{\infty} \sum_{k=0}^{m-j} \widehat{\sigma}_m(k, j+k)\right| \leq \sum_{m=j}^{\infty} \sum_{k=0}^{m} \widehat{\sigma}_m(k,k) \to 0, \tag{3.5}
$$

as $j \to \infty$ since $\sum_{m=j}^{\infty} \sum_{k=0}^{m} \widehat{\sigma}_m(k,k)$ is the tail end of the convergent sum in (3.3b). Also, the Cauchy–Schwarz inequality implies that

$$
\begin{aligned}
&\sum_{k=0}^{m-j} \left(\int_{\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}} \big|(e^{ix_k} - 1)(e^{ix_{j+k}} - 1)\big| \widehat{Q}_m(d\mathbf{x})\right) \\
&\quad \leq \left(\sum_{k=0}^{m-j} \left(\int_{\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}} |(e^{ix_k} - 1)|^2 \widehat{Q}_m(d\mathbf{x})\right)\right. \\
&\qquad \left.\times \sum_{k=0}^{m-j} \left(\int_{\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}} |(e^{ix_{j+k}} - 1)|^2 \widehat{Q}_m(d\mathbf{x})\right)\right)^{1/2}.
\end{aligned}
$$

This last expression is no larger than

$$\sum_{k=0}^{m}\left(\int_{\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}}|(e^{ix_k}-1)|^2\ \widehat{Q}_m(d\mathbf{x})\right)$$

which is no larger than

$$4\sum_{k=0}^{m}\left(\int_{\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}}\min(1,x_k^2)\ \widehat{Q}_m(d\mathbf{x})\right).$$

Thus,

$$\begin{aligned}&\left|\sum_{m=n}^{\infty}\sum_{k=0}^{m}\left(\int_{\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}}(e^{ix_k}-1)(e^{ix_{j+k}}-1)\ \widehat{Q}_m(d\mathbf{x})\right)\right|\\ &\qquad\le 4\sum_{m=j}^{\infty}\sum_{k=0}^{m}\left(\int_{\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}}\min(1,x_k^2)\widehat{Q}_m(d\mathbf{x})\right)\to 0,\end{aligned}\tag{3.6}$$

as $j\to\infty$ since

$$\sum_{m=j}^{\infty}\sum_{k=0}^{m}\left(\int_{\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}}\min(1,x_k^2)\widehat{Q}_m(d\mathbf{x})\right),$$

is the tail end of the convergent sum in (3.3c). Now, (3.4) follows from (3.5) and (3.6). □

In order to extend Theorem 3.1 to stationary infinitely divisible sequences $\{X_j\}_{j\in\mathbb{Z}}$ that can be cumulatively approximated by m-dependent ones, additional restrictions on the sequence of characteristic triplets $(\widehat{\mathbf{b}}_m,\widehat{\Sigma}_m,\widehat{Q}_m)$ are required. These are as follows:

$$\begin{aligned}&\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\widehat{b}_m(k)\right.\\ &\qquad\left.+\int_{\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}}\left(\sum_{k=0}^{m}x_k\right)\left(\mathbf{1}_{(\sum_{k=0}^{m}x_k)^2\le 1}-\mathbf{1}_{\|\mathbf{x}\|\le 1}\right)\widehat{Q}_m(d\mathbf{x})\right)<\infty,\end{aligned}\tag{3.7a}$$

$$\sum_{m=0}^{\infty}\sum_{k,l=0}^{m}\widehat{\sigma}_m(k,l)<\infty,\tag{3.7b}$$

$$\sum_{m=0}^{\infty}\int_{\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}}\min\left(1,\left(\sum_{k=0}^{m}x_k\right)^2\right)\widehat{Q}_m(d\mathbf{x})<\infty\tag{3.7c}$$

and

$$\sum_{m=0}^{\infty}\sum_{i=1}^{m}\left(\sum_{k=i}^{m}\widehat{b}_m(k) + \int_{\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}}\left(\sum_{k=i}^{m}x_k\right)\left(\mathbf{1}_{(\sum_{k=i}^{m}x_k)^2\le 1}-\mathbf{1}_{\|\mathbf{x}\|\le 1}\right)\widehat{Q}_m(d\mathbf{x})\right)<\infty, \tag{3.8a}$$

$$\sum_{m=0}^{\infty}\sum_{i=1}^{m}\sum_{k,l=i}^{m}\widehat{\sigma}_m(k,l)<\infty, \tag{3.8b}$$

$$\sum_{m=0}^{\infty}\sum_{i=1}^{m}\int_{\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}}\min\left(1,\left(\sum_{k=i}^{m}x_k\right)^2\right)\widehat{Q}_m(d\mathbf{x})<\infty. \tag{3.8c}$$

While these conditions look intricate, they are exactly what is required so that all the limiting random variables in Theorem 3.3, below, have a well-defined characterizing triplet. Necessary and sufficient conditions for the partial sum convergence of such sequences are now given.

THEOREM 3.3. *For each $m\in\mathbb{N}$, let $\xi(m)$ be an $(m+1)$-dimensional infinitely divisible vector with characterizing triplet $(\widehat{\mathbf{b}}_m,\widehat{\Sigma}_m,\widehat{Q}_m)$. Let the sequence of characterizing triplets $\{(\widehat{\mathbf{b}}_m,\widehat{\Sigma}_m,\widehat{Q}_m)\}_{m=0}^{\infty}$ satisfy* (3.3), (3.7) *and* (3.8). *Let*

$$\{X_j\}_{j\in\mathbb{Z}}\stackrel{\mathcal{L}}{=}\sum_{m=0}^{\infty}\{Y_j(m)\}_{j\in\mathbb{Z}},$$

where $\{Y_j(m)\}_{j\in\mathbb{Z}}$ is an m-dependent stationary infinitely divisible sequence generated by $\xi(m)$ and where the sequences $\{Y_j(m)\}_{j\in\mathbb{Z}}$ are taken to be independent in m. Consider $S_n=\sum_{j=0}^{n-1}X_j$. There exists a centering sequence $\{A_n\}$ and a normalizing sequence $\{B_n\}$ with $B_n\to\infty$ such that $\frac{S_n-A_n}{B_n}$ converges in distribution to a nondegenerate $S_\alpha(\sigma,\beta,\mu)$ random variable if and only if $\sum_{m=0}^{\infty}\sum_{k=0}^{m}\xi^k(m)$ is in the domain of attraction of a $S_\alpha(\sigma,\beta,\mu)$ distribution.

PROOF. Let $\widetilde{S}_n$ be a random variable with the same distribution as the sum of n independent copies of $\sum_{m=0}^{\infty}\sum_{k=0}^{m}\xi^k(m)$. Note that (3.7) implies that $\sum_{m=0}^{\infty}\sum_{k=0}^{m}\xi^k(m)$ is a well defined random variable. Now consider the convergence of

$$\sum_{m=0}^{\infty}\left(\sum_{i=-m}^{-1}\sum_{k=-i}^{m}\xi_i^k(m),\sum_{i=0}^{n-1}\sum_{k=0}^{m}\xi_i^k(m),\sum_{i=n-m}^{n-1}\sum_{k=n-i}^{m}\xi_i^k(m)\right) \tag{3.9}$$

to an infinitely divisible vector in $\mathbb{R}^3$, say with characterizing triplet $(\mathbf{b}',\Sigma',Q')$. It is clear that $\mathbf{b}'$ is finite if and only if each coordinate is finite. Also $\Sigma'=\{\sigma'_{k,l}\}_{k,l=1}^{3}$

is finite (and positive definite) if and only if $\sigma'_{k,k} < \infty$ for each $k \in \{1, 2, 3\}$. Finally, Q' assigns values, possibly infinite, to each Borel set in $\mathbb{R}^3$. Thus Q' is a Lévy measure if and only if each one-dimensional marginal is a Lévy measure on $\mathbb{R}$. Therefore, (3.9) is a well defined infinitely divisible vector if and only if each one-dimensional marginal is a well defined infinitely divisible random variable.

The marginal

$$\sum_{m=0}^{\infty}\sum_{i=0}^{n-1}\sum_{k=0}^{m}\xi_i^k(m)$$

converges since it is equal to n independent copies of $\sum_{m=0}^{\infty}\sum_{k=0}^{m}\xi_0^k(m)$, which converges by (3.7). The marginals

$$\sum_{m=0}^{\infty}\sum_{i=-m}^{-1}\sum_{k=-i}^{m}\xi_i^k(m)$$

and

$$\sum_{m=0}^{\infty}\sum_{i=n-m}^{n-1}\sum_{k=n-i}^{m}\xi_i^k(m)$$

converge since each limit is equal in distribution to $\sum_{m=0}^{\infty}\sum_{i=1}^{m}\sum_{k=i}^{m}\xi_i^k(m)$, which converges by (3.8).

Therefore, (3.9) is a well-defined infinitely divisible vector. Moreover,

$$\begin{aligned}
S_n &\stackrel{\mathcal{L}}{=} \sum_{m=0}^{\infty}\sum_{j=0}^{n-1}\sum_{k=0}^{m}\xi_{j-k}^k(m)\\
&\stackrel{\mathcal{L}}{=} \sum_{m=0}^{\infty}\left[\sum_{i=-m}^{-1}\sum_{k=-i}^{m}\xi_i^k(m)+\sum_{i=0}^{n-1}\sum_{k=0}^{m}\xi_i^k(m)-\sum_{i=n-m}^{n-1}\sum_{k=n-i}^{m}\xi_i^k(m)\right]\\
&\stackrel{\mathcal{L}}{=} \sum_{m=0}^{\infty}\sum_{i=-m}^{-1}\sum_{k=-i}^{m}\xi_i^k(m)+\sum_{i=0}^{n-1}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\xi_i^k(m)-\sum_{m=0}^{\infty}\sum_{i=n-m}^{n-1}\sum_{k=n-i}^{m}\xi_i^k(m)\\
&\stackrel{\mathcal{L}}{=} \sum_{m=0}^{\infty}\sum_{i=-m}^{-1}\sum_{k=-i}^{m}\xi_i^k(m)+\widetilde{S}_n-\sum_{m=0}^{\infty}\sum_{i=n-m}^{n-1}\sum_{k=n-i}^{m}\xi_i^k(m).
\end{aligned}\tag{3.10}$$

The first and last terms have distributions that do not depend on n, thus for $B_n \to \infty$, those terms divided by B_n converge to zero in probability. Thus

$$\frac{S_n - A_n}{B_n} \Rightarrow S_\alpha(\sigma, \beta, \mu)$$

if and only if

$$\frac{\widetilde{S}_n - A_n}{B_n} \Rightarrow S_\alpha(\sigma, \beta, \mu).$$

Since the limiting distribution is nondegenerate, the classical theory implies that $B_n \to \infty$ as $n \to \infty$. $\square$

REMARK 3.3. It would be interesting to know conditions on the stationary infinitely divisible sequence $\{X_j\}_{j\in\mathbb{Z}}$ that are sufficient for the sequence to admit a decomposition

$$\{X_j\}_{j\in\mathbb{Z}} \stackrel{\mathcal{L}}{=} \sum_{m=0}^{\infty}\sum_{k=0}^{m} \xi_{j-k}^{k}(m),$$

that satisfies (3.3), (3.7) and (3.8). We point out that all three sets of conditions are guaranteed by the following:

$$\max\left\{\sum_{m=0}^{\infty}\sum_{k=0}^{m}\widehat{b}_m(k), \sum_{m=0}^{\infty}\sum_{k=0}^{m}k\widehat{b}_m(k)\right\} < \infty, \tag{3.11a}$$

$$\sum_{m=0}^{\infty} m(m+1)\sum_{k=0}^{m}\widehat{\sigma}_m(k,k) < \infty \tag{3.11b}$$

and

$$\sum_{m=0}^{\infty} m(m+1)\int_{\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}} \min\left(1, \left(\sum_{k=0}^{m} x_k\right)^2\right)\widehat{Q}_m(d\mathbf{x}) < \infty. \tag{3.11c}$$

4. Concluding remarks. A stationary sequence of the form

$$\{f(Y_j, Y_{j+1}, \ldots, Y_{j+m})\}_{j\in\mathbb{Z}},$$

where $\{Y_j\}_{j\in\mathbb{Z}}$ is i.i.d. and $f:\mathbb{R}^{m+1} \to \mathbb{R}$, is called an $(m+1)$-block factor. Sequences that are $(m+1)$-block factors are clearly m-dependent and stationary. Ibragimov and Linnik (1971) state that there exist m-dependent stationary sequences that are not $(m+1)$-block factors. Examples of one-dependent sequences that are not two-block factors are constructed by Aaronson, Gilat, Keane and de Valk (1989). Examples of one-dependent sequences that are not k-block factors for any k are constructed by Burton, Goulet and Meester (1993). It is then natural to try to characterize m-dependent stationary sequences which are $(m+1)$-block factors. Aaronson, Gilat and Keane (1992) show that every one-dependent Markov chain of no more than four states is a two-block factor. Theorem 2.1 is another result in this direction.

Many weak dependence conditions for stationary Gaussian sequences have been characterized in terms of the covariance function or the spectral measure of the sequence [see, e.g., Bradley (1986), Cornfeld, Fomin and Sinaĭ (1982), Ibragimov and Rozanov (1978)]. For Gaussian sequences, Theorem 2.1 follows from the characterization of the spectral measure for m-dependent stationary

Gaussian sequences and the Riesz factorization lemma. For infinitely divisible sequences without Gaussian component, Theorem 2.1 is proven by a different technique. However, it should be noted that Rosiński and Żak (1996, 1997) characterized ergodic properties in terms of the positive definite codifference function and its spectral measure. (For a stationary Gaussian sequence, the codifference function and the covariance function coincide.) A direct consequence of their work is that a stationary infinitely divisible sequence is m-dependent if and only if $\mathrm{Cod}(X_0, \pm X_j) = 0$ for $|j| > m$. Characterizing the spectral measure of the codifference function of m-dependent stationary infinitely divisible sequences is an open problem. Also, the characterizations for mixing stationary Gaussian sequences in terms of the spectral measure of their covariance function have not been extended to the general infinitely divisible case. Since the codifference is only a parameter and does not characterize an infinitely divisible distribution, it is unclear if these extensions are possible.

Finally, we also remark that all the results of the present paper have natural extensions to stationary sequences of d-dimensional infinitely divisible vectors. The proofs follow from the same techniques as used here. However, extending the results of this paper to stationary infinitely divisible random fields require different techniques. We say that an infinitely divisible random field is generated by ξ if

$$\{X_j\}_{j\in\mathbb{Z}^d} \stackrel{\mathcal{L}}{=} \left\{ \sum_{\mathbf{k}\in\{0,1,\dots,m\}^d} \xi^{\mathbf{k}}_{\mathbf{j}-\mathbf{k}} \right\}_{\mathbf{j}\in\mathbb{Z}^d}, \tag{4.1}$$

where $\xi = \{\xi^{\mathbf{k}}\}_{\mathbf{k}\in\{0,1,\dots,m\}^d}$ is an infinitely divisible vector and $\xi_{\mathbf{j}} = \{\xi^{\mathbf{k}}_{\mathbf{j}}\}_{\mathbf{k}\in\{0,1,\dots,m\}^d}$ are independent copies of ξ. It is clear that all random fields given by (4.1) are stationary and m-dependent. Moreover, the techniques used in the present paper can be extended to show that a stationary m-dependent infinitely divisible field without Gaussian component is necessarily generated by some infinitely divisible vector ξ without Gaussian component. However, in the absence of a multidimensional Riesz factorization lemma, the stationary m-dependent Gaussian results of Section 2.1 cannot be directly extended to fields. This problem deserves further attention.

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