PDES, SUBMANIFOLDS AND
AFFINE DIFFERENTIAL GEOMETRY
BANACH CENTER PUBLICATIONS, VOLUME 69
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2005

A CHARACTERIZATION OF n-DIMENSIONAL HYPERSURFACES IN R^{n+1} WITH COMMUTING CURVATURE OPERATORS

YULIAN T. TSANKOV

Department of Geometry, Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
Blvd. "J. Bourchier", 5, 1164 Sofia, Bulgaria
E-mail: ucankov@fmi.uni-sofia.bg

Abstract. Let M^n be a hypersurface in R^{n+1}. We prove that two classical Jacobi curvature operators J_x and J_y commute on M^n, $n > 2$, for all orthonormal pairs (x, y) and for all points $p \in M$ if and only if M^n is a space of constant sectional curvature. Also we consider all hypersurfaces with $n \geq 4$ satisfying the commutation relation $(K_{x,y} \circ K_{z,u})(u) = (K_{z,u} \circ K_{x,y})(u)$, where $K_{x,y}(u) = R(x, y, u)$, for all orthonormal tangent vectors x, y, z, w and for all points $p \in M$.

1. Introduction. Let ∇ be the Levi-Civita connection of a Riemannian manifold (M^m, g). Let x, y and z be tangent vector fields on M^m. Then the associated curvature tensor $R(x, y, z)$ is defined by

$$R(x, y, z) = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z.$$

The value of $R(x, y, z)$ at a point p of M depends only on the values of x, y and z at p.

The classical Jacobi curvature operator

$$J_x : T_pM \to T_pM,$$

induced by the unit vector $x \in T_pM$ and defined by

$$J_x(u) = R(u, x, x)$$

is a symmetric operator.

The skew-symmetric curvature operator $K_{x,y}$

$$K_{x,y} : T_pM \to T_pM,$$

2000 *Mathematics Subject Classification*: 53B20, 53B21.

Key words and phrases: commutation relation, Jacobi operator, skew-symmetric curvature operator.

Partially supported by AvH-Foundation and VolkswagenStiftung.

The paper is in final form and no version of it will be published elsewhere.

is defined by

$$K_{x,y}(u) = R(x, y, u),$$

for any orthonormal pair (x, y) of tangent vectors at any point p in M and $u \in T_pM$. The curvature operator $K_{x,y}$ does not depend on the oriented orthonormal basis chosen for the oriented 2-plane $span\{x, y\}$ [3].

Here, using the eigenvalues of the Weingarten map of M, we give a characterization of those hypersurfaces in R^{n+1} for which the operator J_x satisfies the following condition:

$$J_x \circ J_y = J_y \circ J_x$$

on T_pM for any two orthogonal vectors $x, y \in T_pM$.

Also we characterize the hypersurfaces for which the operator $K_{x,y}$ satisfies the following condition:

$$K_{x,y} \circ K_{z,u} = K_{z,u} \circ K_{x,y}$$

on T_pM for any four orthogonal vectors $x, y, z, u \in T_pM$.

2. A characterization of n-dimensional hypersurfaces in R^{n+1} with commuting Jacobi operators

THEOREM 1. *A hypersurface M in R^{n+1}, $n \geq 3$, satisfies the commutation relation:*

$$J_x \circ J_y = J_y \circ J_x \tag{1}$$

on T_pM for all orthonormal pairs $x, y \in T_pM$ and for all $p \in M$ if and only if exactly one of the following two conditions for the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the Weingarten operator of M holds:

$$\begin{aligned} &1) \qquad \lambda_1 = \ldots = \lambda_n; \\ &2) \qquad \lambda_1 = \ldots = \lambda_{n-1} = 0,\ \lambda_n \neq 0. \end{aligned}$$

Proof. Let $e_1, \ldots, e_n$ and $\lambda_1, \ldots, \lambda_n$ be the eigenvectors and eigenvalues of the Weingarten operator of M^n. Then we have [4]

$$R(e_i, e_j, e_k) = \begin{cases} 0, & k \neq i, j; \\ -\lambda_i\lambda_j e_j, & k = i; \\ \lambda_i\lambda_j e_i, & k = j. \end{cases}$$

The matrix of the Jacobi operator J_a, where a is a unit tangent vector at point $p \in M^n$ and $a = a^1e_1 + \ldots + a^ne_n$, is:

$$\begin{pmatrix} \sum_{i=1\ i\neq 1}^n (a^i)^2\lambda_i\lambda_1 & -a^1a^2\lambda_1\lambda_2 & \cdots & -a^1a^n\lambda_1\lambda_n \\ -a^1a^2\lambda_1\lambda_2 & \sum_{i=1,\ i\neq 2}^n (a^i)^2\lambda_i\lambda_2 & \cdots & -a^2a^n\lambda_2\lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ -a^1a^n\lambda_1\lambda_n & -a^2a^n\lambda_2\lambda_n & \cdots & \sum_{i=1,\ i\neq n}^n (a^i)^2\lambda_i\lambda_n \end{pmatrix} \tag{2}$$

Let b be a unit tangent vector such that $g(a, b) = 0$. Then, we find for the elements of the matrix of the operator $J_a \circ J_b$ with respect to $e_1, \ldots, e_n$:

$$\begin{aligned} \|J_a \circ J_b\|_{s\ s\ \ s=1}^n = &\sum_{k=1,\ k\neq s}^n (-b^sb^k\lambda_s\lambda_k)(-a^sa^k\lambda_s\lambda_k) \\ &+ \sum_{i=1,\ i\neq s}^n (b^i)^2\lambda_i\lambda_s \sum_{i=1,\ i\neq s}^n (b^i)^2\lambda_i\lambda_s \end{aligned} \tag{3}$$

and

$$\|J_a \circ J_b\|_{p\ q} = \sum_{k=1,\ k\neq p,q}^{n} (-b^p b^k \lambda_p \lambda_k)(-a^q a^k \lambda_q \lambda_k) + \sum_{i=1,\ i\neq p}^{n} (b^i)^2 \lambda_i \lambda_p (-a^p a^q \lambda_p \lambda_q) + (-b^p b^q \lambda_p \lambda_q) \sum_{i=1,\ i\neq p}^{n} (a^i)^2 \lambda_i \lambda_q \tag{4}$$

where $p \neq q$ and $p, q = 1, \ldots, n$. We have similar expressions for the matrix of the operator $J_b \circ J_a$ with respect to $e_1, \ldots, e_n$. From (1) we have the following equation:

$$\|J_a \circ J_b\|_{p\ q} = \|J_b \circ J_a\|_{p\ q},$$

for $p < q$, $p, q = 1, \ldots, n$ and for an arbitrary orthonormal pair a, b. After some algebra we have:

$$\lambda_{i_1}\lambda_{i_2}\ \Big(((-a^{i_1}a^{i_2}(b^{i_1})^2 + (a^{i_1})^2 b^{i_1} b^{i_2} - (a^{i_2})^2 b^{i_1} b^{i_2} + a^{i_1}a^{i_2}(b^{i_2})^2)\lambda_{i_1}\lambda_{i_2} + \sum_{i_k=1,\ i_k\neq i_1,i_2}^{n} ((a^{i_k})^2 b^{i_1} b^{i_2} - a^{i_1}a^{i_2}(b^{i_k})^2)(\lambda_{i_1} - \lambda_{i_2})\lambda_{i_k} + \sum_{i_s=1,\ i_s\neq i_1,i_2}^{n} (-a^{i_2}a^{i_s}b^{i_1}b^{i_s} + a^{i_1}a^{i_s}b^{i_2}b^{i_s})\lambda_{i_s}^2)\Big) = 0. \tag{5}$$

We want to find all solutions of (5) for all orthonormal pairs a, b. We will find an arbitrary orthonormal pair of vectors for which a non-trivial solution of (5) exists, i.e. a solution which is not zero. Let a and b have the coordinates:

$$a^i = \frac{1}{2}, \qquad a^j = \frac{\sqrt{3}}{2}, \qquad a^s = 0,$$

where $i < j,\ s \neq i, j,\ i, j, s = 1, \ldots, n$,

$$b^k = 1, \qquad b^l = 0,$$

where $k \neq l, i, j,\ k, l = 1, \ldots, n$. We have

$$\lambda_i \lambda_j \lambda_k (\lambda_i - \lambda_j) = 0. \tag{6}$$

Let a and b have the coordinates:

$$a^i = -\frac{\sqrt{3}}{2}, \qquad a^j = \frac{\sqrt{3}}{4}, \qquad a^k = -\frac{1}{4}, \qquad a^s = 0;$$

$$b^i = \frac{1}{2}, \qquad b^j = \frac{3}{4}, \qquad b^k = -\frac{\sqrt{3}}{4}, \qquad b^s = 0,$$

where $i < j,\ k \neq i, j,\ s \neq i, j, k,\ i, j, s, k = 1, \ldots, n$. We have

$$\lambda_i \lambda_j (\lambda_i - \lambda_k)(\lambda_j + \lambda_k) = 0. \tag{7}$$

All solutions of (6) and (7) are:

$$\lambda_1 = \ldots = \lambda_n \tag{8}$$

and

$$\lambda_1 = \ldots = \lambda_{n-1} = 0, \qquad \lambda_n \neq 0. \tag{9}$$

The solutions (8) and (9) are all solutions of (5). ∎

Remark 1. In the first case we obtain spaces of constant sectional curvature, in the second we have ordinary parabolic forms. They are also spaces with zero constant sectional curvature.

3. A characterization of n-dimensional hypersurfaces in R^{n+1} with commuting skew-symmetric curvature operators

Theorem 2. *A hypersurface M in R^{n+1}, $n \geq 4$, satisfies the commutation relation:*

$$K_{x,y} \circ K_{z,u} = K_{z,u} \circ K_{x,y} \tag{10}$$

on T_pM for all orthonormal vectors $x, y, z, u \in T_pM$ and for all $p \in M$ if and only if the eigenvalues $\lambda_1, \dots, \lambda_n$ of the Weingarten operator of M satisfy one of the following conditions:

$$\begin{aligned} &1) \quad |\lambda_1| = \ldots = |\lambda_n|; \\ &2) \quad \lambda_1 = \ldots = \lambda_{n-1} = 0, \quad \lambda_n \neq 0; \\ &3) \quad \lambda_1 = \ldots = \lambda_{n-2} = 0, \quad \lambda_{n-1} \neq 0, \ \lambda_n \neq 0. \end{aligned}$$

Proof. Similar to the proof of Theorem 1, let $e_1, \dots, e_n$ and $\lambda_1, \dots, \lambda_n$ be the eigenvectors and eigenvalues of the Weingarten operator of M^n. Let $a = a^1e_1 + \ldots + a^ne_n$, $b = b^1e_1 + \ldots + b^ne_n$, $c = c^1e_1 + \ldots + c^ne_n$, $d = d^1e_1 + \ldots + d^ne_n$ be four orthonormal tangent vectors. The condition (10) is true if and only if $\lambda_1, \dots, \lambda_n$ satisfy the system:

$$\lambda_i\lambda_j \sum_{k=1, k\neq i,j}^{n} ((a^kb^j - a^jb^k)(c^kd^i - c^id^k) + (a^kb^i - a^ib^k)(-c^kd^j - c^jd^k)\lambda_k^2) = 0 \tag{11}$$

where $i < j$ and $i, j = 1, \dots, n$.

We aim to find all solutions of (11) for all orthonormal tangent vectors a, b, c, d and b. First, we will find an arbitrary 4-tuple of orthonormal tangent vectors for which a non-trivial solution of (11) exists, i.e. a solution which is not generated by a space of constant sectional curvature. We put:

$$a^s = -1, \qquad a^i = 0,$$

where s is fixed and $i = 1, \dots, n$, $i \neq s$;

$$b^p = \frac{\sqrt{3}}{2}, \qquad b^q = \frac{1}{2}, \qquad b^i = 0,$$

where $p \neq q$, $i = 1, \dots, n$ and $i \neq p, q$;

$$c^p = \frac{1}{2}, \qquad c^q = -\frac{\sqrt{3}}{2}, \qquad c^i = 0,$$

where $p \neq q$, $i = 1, \dots, n$ and $i \neq p, q$;

$$d^k = 1, \qquad a^i = 0$$

where k is fixed and $i = 1, \dots, n$, $i \neq k$. We have the system:

$$\lambda_s\lambda_k(\lambda_p^2 - \lambda_q^2) = 0 \tag{12}$$

where $s, k, p, q = 1, \dots, n$ are pairwise different. All solutions of (12) are:

$$|\lambda_1| = \ldots = |\lambda_n|; \tag{13}$$

$$\lambda_1 = \ldots = \lambda_{n-1} = 0, \quad \lambda_n \neq 0; \tag{14}$$

$$\lambda_1 = \ldots = \lambda_{n-2} = 0, \quad \lambda_{n-1} \neq 0, \ \lambda_n \neq 0. \tag{15}$$

It is easy to see that (14) and (15) are solutions of (11) for all orthogonal tangent vectors. Using

$$\sum_{k=1,k\neq i,j}^{n} ((a^k b^j - a^j b^k)(c^k d^i - c^i d^k) + (a^k b^i - a^i b^k)(-c^k d^j + c^j d^k) = 0$$

we see that (13) is a solution of (11) for all orthogonal tangent vectors. ■

Remark 2. Examples of hypersurfaces that fulfill the condition 1) of Theorem 2 are IP-hypersurfaces. An IP-hypersurface in R^{n+1} is a hypersurface in R^{n+1} such that its induced metric is an IP-metric [2], [1]. The IP-metric is a warped product metric of the form $ds^2 = dt^2 + f(t)ds_K^2$ where ds_K^2 is a metric of constant sectional curvature K and $f(t)$ is a suitably chosen warping function $f(t) = Kt^2 + Ct + D$. An example of a hypersurface in R^{n+1} with induced IP-metric of the standard metric of R^{n+1} is a rotated hypersurface

$$\begin{cases} x^1 &= f(u^1)\sin(u^2)\sin(u^3)\ldots\sin(u^n) \\ x^2 &= f(u^1)\sin(u^2)\sin(u^3)\ldots\cos(u^n) \\ \vdots & \\ x^{n-1} &= f(u^1)\sin(u^2)\cos(u^3) \\ x^n &= f(u^1)\cos(u^2) \\ x^{n+1} &= h(u^1) \end{cases}$$

$$u^i \in V_i, \qquad V_i \subset R^1, \qquad i = 1, \ldots, n,$$

where

$$f(u^1) = \sqrt{(u^1)^2 + Cu^1 + D}, \qquad 4D - C^2 > 0,$$
$$h(u^1) = \frac{1}{2}\sqrt{4D - C^2}\ln(C + 2(u^1 + \sqrt{(u^1)^2 + Cu^1 + D})).$$

By a direct check it is seen that relation 1) of Theorem 2 holds for the eigenvalues of the Weingarten operator.

In the second case we have a flat hypersurface.

References

[1] P. Gilkey, J. V. Leahy and H. Sadofsky, *Riemannian manifolds whose skew-symmetric curvature operator has constant eigenvalues*, Indiana Univ. Math. J. 48 (1999), 615–634.

[2] S. Ivanov and I. Petrova, *Riemannian manifold in which the skew-symmetric curvature operator has constant eigenvalues*, Geometriae Dedicata 70 (1998), 269–282.

[3] R. Ivanova and G. Stanilov, *A skew-symmetric curvature operator in Riemannian geometry*, in: Symposia Gaussiana, Behara *et al.* (eds.), de Gruyter, Berlin, 1995, 392–395.

[4] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. 2, Interscience Publishers, New York, 1969.