A Characterization of Plurality-Like Rules Based on Non-Manipulability, Restricted Efficiency, and Anonymity

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Abstract

A society needs to decide which issues (laws, public projects, public facilities, etc.) in an agenda to accept. The decision can be any subset of the agenda but must reflect the preferences of its members, which are assumed to be "separable weak orderings". We characterize a family of plurality-like rules based on *strategy-proofness*, *restricted efficiency*, *anonymity*, and two weak axioms pertaining to the variable agenda feature of our model, called *dummy independence* and *division indifference*. We also characterize a wide spectrum of rules dropping *anonymity* or *restricted efficiency*.

Keywords: Plurality; Strategy-proofness; Efficiency; Restricted efficiency; Anonymity; Division indifference; Separable preferences
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1 Introduction

We consider social choice problems of the following form. There is a set of issues such as passing a law, launching a public project, building a public facility, etc. This set is called an *agenda*. A society consisting of more than two agents has to decide which issues in the agenda to choose. Any subset is a social alternative. A problem is identified by the agenda and the list of agents' preferences over alternatives. A social choice rule, or simply a *rule*, associates a single alternative with each problem.

In a variable agenda model with a restricted domain of preferences known as "separable" preferences, we study rules satisfying the following two axioms. *Strategy-proofness* (Gibbard 1973; Satterthwaite 1975) requires that no one should ever benefit by misrepresenting his preferences, independently of the others' announcements. *Efficiency* requires that it should not be possible to make an agent better off without making someone else worse off. In their work on *strategy-proofness*, Barberà, Sonnenschein, and Zhou (1991, BSZ below) established an impossibility result saying that if there are at least three issues in the agenda, no rule satisfies *strategy-proofness*, *efficiency*, and the minimal equity criterion of "non-dictatorship".[1] Thus, to avoid dictatorship, one needs to sacrifice *strategy-proofness* or *efficiency*. Here we sacrifice *efficiency*. Interestingly, we come across an axiomatic justification for plurality-like rules based on a notion of "restricted efficiency", *strategy-proofness*, *anonymity* (symmetric treatment of agents), and two additional axioms pertaining to possible variations in agendas, called "dummy independence" and "division indifference" (to be explained below). This result is established without the unnatural assumption of strict preferences (there is no indifference between alternatives), which is crucial in BSZ.

The BSZ impossibility applies to any agenda with at least three issues. Thus, one natural way of weakening *efficiency* in our variable agenda model is to restrict efficiency to agendas with at most two issues. This is our notion of *restricted efficiency*.

In our model, preferences are defined over the set of potential alternatives, which is typically larger than the set of feasible alternatives in the agenda under actual consideration. Thus, preferences describe not only how agents rank feasible alternatives but also how they rank infeasible alternatives. One could argue that the latter information is irrelevant, and require rules not to depend on it. We call the requirement *independence*. A *dummy* (player) is an agent who is indifferent between any pair of feasible alternatives (he may not be indifferent between infeasible alternatives). *Dummy independence*, a much weaker requirement than *independence*, limits its application to the preferences of dummies.

Division indifference is a robustness property of rules with respect to agenda division. It says that for any pair of agendas X and Y, making decisions on X and Y independently should result in an outcome that is indifferent, for everyone, to the

[1] See also Le Breton and Sen (1995, 1999), Le Breton and Weymark (1999), and Ju (2003).

decision on the union $X \cup Y$. It is similar, in spirit, to the path independence axiom introduced by Plott (1973), stating robustness to agenda division in the Arrovian social choice framework (see also Plott and Levine 1978; Kalai and Megiddo 1980). *Division indifference* requires rules to behave consistently across different agendas. It excludes, for example, rules based on the "plurality principle" for one agenda and on the "unanimity principle" for another. When agents have separable preferences, the impact of an issue on the welfare of each agent can be separated from the impact of the other issues. Then, *division indifference* is easily met: it suffices to make decisions issue by issue. Indeed, we identify a large family of rules satisfying this axiom. They include, among others, all the rules known as "voting by committees" (BSZ). In this sense, there is little cost of imposing this axiom, especially from the perspective of the main goal of our research. On the contrary, the axiom greatly facilitates our investigation. This is because, when combined with our other axioms, it delivers a useful representation of rules in terms of "power relations" among groups of agents.

Justifications for plurality rule have been derived in various models (May 1952, Murakami 1966, Inada 1969, Guha 1972, Ching 1996, etc.). However, these justifications are based on "neutrality" or "duality" type conditions. Our characterization involves no such conditions.

Ju (2005) considers two simple domains of separable preferences, called "dichotomous" and "trichotomous" preferences. Our results do not apply to these domains. This is because they do not have the richness that is crucial here. However, relying on the full power of *efficiency* in combination with *anonymity* and other axioms, Ju (2005) characterizes a family of rules, called "semi-plurality rules". On our domain, none of these rules satisfies *restricted efficiency*.

Shimomura (1996) studies partial efficiency (efficiency on the subdomain of problems where voters' preferences bear some degrees of resemblance or agreement) in the fixed agenda model with strict separable preferences considered by BSZ.[2] He offers a sharp characterization of partially efficient schemes of voting by committees.

The rest of this paper is organized as follows. In Section 2, we define our model, basic concepts, and some important families of rules. In Section 3, we define our axioms. In Section 4, we state our results. All the proofs are collected in Section 5. In Section 6, we obtain a corollary involving the "issue-wise voting property" and *efficiency*.

2 The Model and Basic Concepts

There is a society of n agents, $n \geq 2$, denoted by $N \equiv \{1, \cdots, n\}$. The society faces the problem of choosing a subset from a set of issues. Let A be the set of potential

[2] I thank William Thomson for bringing to my attention this work by Shimomura (1996).

issues, with $|A| \geq 2$. An *agenda* is a nonempty subset of A. Given an agenda $X \subseteq A$, the society can choose any subset of X. Each subset of X is called an *alternative*.

Let $\mathcal{A}$ be the family of all admissible agendas, referred to as the *agenda domain*. We assume *finiteness*, saying that each agenda $X \in \mathcal{A}$ contains a finite number of issues, and *divisibility*, saying that for each $X \in \mathcal{A}$ and each $X' \subseteq X$, $X' \in \mathcal{A}$. By divisibility, $\mathcal{A}$ also represents the set of alternatives. For each pair $X, X' \in \mathcal{A}$, X and X' are *linked* if there are $k \in \mathbb{N}$ agendas $X^1, \cdots, X^k \in \mathcal{A}$ such that $X \cap X^1 \neq \emptyset$, $X^1 \cap X^2 \neq \emptyset$, $\cdots$, $X^{k-1} \cap X^k \neq \emptyset$, and $X^k \cap X' \neq \emptyset$. Finally, we assume *connectedness*, saying that every two agendas in $\mathcal{A}$ are linked. Our results can be extended to the case without connectedness.[3]

A *preference* of agent $i \in N$, denoted generically by R_i, is a complete and transitive binary relation over $\mathcal{A}$. As in BSZ, we focus on the following special type of preferences. A preference R_i is *separable* if for each $X \in \mathcal{A}$ and each $x \in X$, (i) $X \; P_i \; X \backslash \{x\}$ if and only if $\{x\} \; P_i \; \emptyset$; (ii) $X \; I_i \; X \backslash \{x\}$ if and only if $\{x\} \; I_i \; \emptyset$, where P_i and I_i denote the strict and indifferent parts of R_i, respectively. Let $\mathcal{S}$ be the collection of separable preferences. A preference R_i is *strict* if for each pair of distinct alternatives $X, Y \in \mathcal{A}$, either $X \; P_i \; Y$ or $Y \; P_i \; X$. Clearly, for strict preferences, part (ii) does not apply, and so our separability property coincides with that separability property considered by BSZ. Let $\mathcal{S}_{\text{Strict}}$ be the collection of strict separable preferences. Examples of separable preferences are *additive* preferences. Such a relation for agent i is represented by a function $u_i \colon A \to \mathbb{R}$ in the following way: for each pair $X, X' \in \mathcal{A}$, $X \; R_i \; X'$ if and only if $\sum_{x \in X} u_i(x) \geq \sum_{x \in X'} u_i(x)$, where $\sum_{x \in X} u_i(x) = 0$ when $X = \emptyset$.[4] Let $\mathcal{S}_{\text{Add}}$ be the collection of additive preferences.

A collective choice problem, or briefly a *problem*, is characterized by a profile of preferences in $\mathcal{S}^N$ and an agenda in $\mathcal{A}$. Thus, we may denote the collection of problems by the Cartesian product $\mathcal{V} \equiv \mathcal{S}^N \times \mathcal{A}$. A *rule* is a function $\varphi \colon \mathcal{V} \to \mathcal{A}$ associating with each problem $(R, X) \in \mathcal{V}$ a *single* alternative that is a subset of the agenda X. We call $\mathcal{V}$ the *separable domain*. Similarly, we call $\mathcal{V}_{\text{Add}} \equiv \mathcal{S}^N_{\text{Add}} \times \mathcal{A}$ the *additive domain*, and $\mathcal{V}_{\text{Strict}} \equiv \mathcal{S}_{\text{Strict}} \times \mathcal{A}$ the *strict separable domain*.

Families of Voting Rules

We next define families of rules that are important in this paper. They are based on the following simple information about preferences. For each $R \in \mathcal{S}^N$ and each $a \in A$,

[3]The earlier version of this article, Chapter 7 of Ju (2001), considers an agenda domain without connectedness.

[4]Let $X \in \mathcal{A}$ and $x \in X$. If $X \; P_i \; X \backslash \{x\}$, then $\sum_{y \in X} u_i(y) > \sum_{y \in X \backslash \{x\}} u_i(y)$, that is, $u_i(x) > 0$. This means $\{x\} \; P_i \; \emptyset$. On the other hand, if $\{x\} \; P_i \; \emptyset$, then $u_i(x) > 0$, which implies $\sum_{y \in X} u_i(y) > \sum_{y \in X \backslash \{x\}} u_i(y)$. Thus $X \; P_i \; X \backslash \{x\}$. Hence $X \; P_i \; X \backslash \{x\}$ if and only if $\{x\} \; P_i \; \varnothing$. Similarly, we can show that $X \; I_i \; X \backslash \{x\}$ if and only if $\{x\} \; I_i \; \varnothing$. Therefore, additive preference R_i is separable.

let

$$N_a^+(R) \equiv \{i \in N : \{a\}\ P_i\ \emptyset\} \text{ and } N_a^-(R) \equiv \{i \in N : \emptyset\ P_i\ \{a\}\}.$$

Issue $a \in A$ is a *good for* R_i if $i \in N_a^+(R)$, a *bad for* R_i if $i \in N_a^-(R)$, and a *null for* R_i, if $i \notin N_a^+(R) \cup N_a^-(R)$. When a is a null for R_i, agent i is indifferent between having a and not having a, that is, for each $X \in \mathcal{A}$ with $a \in X$, $X\ I_i\ X \backslash \{a\}$. Let

$$Null(R) \equiv \{a \in A : \text{ for each } i \in N,\ a \text{ is a null for } R_i\}$$

be the set of nulls for all agents at R.

A rule $\varphi \colon \mathcal{V} \to \mathcal{A}$ satisfies *votes-only* if for each pair $R, R' \in \mathcal{S}^N$ and each $X \in \mathcal{A}$ such that for each $x \in X$, $N_x^+(R) = N_x^+(R')$ and $N_x^-(R) = N_x^-(R')$, we have $\varphi(R, X) = \varphi(R', X)$. A weaker condition is *votes-only^o*, saying that for each pair $R, R' \in \mathcal{S}^N$ and each $X \in \mathcal{A}$ such that for each $x \in X$, $N_x^+(R) = N_x^+(R')$ and $N_x^-(R) = N_x^-(R')$, if $y \in X \backslash Null(R)$, $y \in \varphi(R, X)$ if and only if $y \in \varphi(R', X)$.[5] Note that when a rule satisfies *votes-only*$^{\text{o}}$, the decision on each non-null issue x can be implemented through a voting procedure in which agents may vote on x positively or negatively, or they may abstain. Thus, we call it a *voting rule.*

We next define families of voting rules that are similar to the family known as "voting by committees" (BSZ). For these rules, the decision on each issue is made according to a predetermined set of ordered pairs of disjoint groups, called a "power structure". Each issue x is accepted if and only if the group of "supporters", $N_x^+(R)$, and the group of "objectors", $N_x^-(R)$, constitute a pair in the power structure. Let $\bar{\mathfrak{C}} \equiv \{(C_1, C_2) : C_1, C_2 \subseteq N \text{ and } C_1 \cap C_2 = \emptyset\}$ be the set of all pairs of disjoint groups. A *power structure* is a subset of $\bar{\mathfrak{C}}$. Given a power structure $\mathfrak{C} \subseteq \bar{\mathfrak{C}}$, we say that C_1 *overpowers* C_2 if $(C_1, C_2) \in \mathfrak{C}$. A *profile of power structures* is a list $(\mathfrak{C}_x)_{x \in A}$ of power structures indexed by issues.[6]

Definition (Families Φ and Φ^{o}). A rule φ belongs to the *Family* Φ if it can be represented by a profile of power structures $(\mathfrak{C}_x)_{x \in A}$ as follows: for each $(R, X) \in \mathcal{V}$ and each $x \in X$,

$$x \in \varphi(R, X) \text{ if and only if } \left(N_x^+(R), N_x^-(R)\right) \in \mathfrak{C}_x.$$

A rule φ belongs to the *Family* Φ^{o} if there is $\varphi' \in \Phi$ such that for each $(R, X) \in \mathcal{V}$ and each $x \in X \backslash Null(R)$,

$$x \in \varphi(R, X) \text{ if and only if } x \in \varphi'(R, X).$$

[5] For *strict* separable preferences, *votes-only* and *votes-only*$^{\text{o}}$ are identical and coincide with "tops-only" by BSZ.

[6] The definition is due to Ju (2003).

In this case, we say that φ *coincides, except for nulls, with* φ'.[7]

A profile $(\mathfrak{C}_x)_{x\in A}$ satisfies *monotonicity*, if for each $x \in A$, whenever group C_1 overpowers group C_2, each supergroup of C_1 also overpowers each disjoint subgroup of C_2: that is, for each $x \in A$ and each $(C_1, C_2) \in \mathfrak{C}_x$, if $(C_1', C_2') \in \bar{\mathfrak{C}}$ is such that $C_1' \supseteq C_1$ and $C_2' \subseteq C_2$, then $(C_1', C_2') \in \mathfrak{C}_x$.

Definition (Families Φ_m and Φ_m^o). *Family* Φ_m is composed of rules in Φ represented by a *monotonic* profile $(\mathfrak{C}_x)_{x\in A}$. *Family* Φ_m^o is composed of rules that coincide, except for nulls, with rules in Φ_m.

For example, *plurality rule* is the rule in Φ_m, represented by the following profile $(\mathfrak{C}_x)_{x\in A}$: for each $x \in A$, $(C_1, C_2) \in \mathfrak{C}_x$ if and only if $|C_1| > |C_2|$.[8]

We next define rules that embody the most extreme form of discrimination among agents. For each $\mathcal{X} \subseteq \mathcal{A}$ and each $R_i \in \mathcal{S}$, let $Max[R_i\colon \mathcal{X}] \equiv \{Y \in \mathcal{X} : \text{for each } Y' \in \mathcal{X},\ Y\ R_i\ Y'\}$ be the set of best alternatives for R_i in $\mathcal{X}$. A rule φ is *dictatorial* if there is $i \in N$ such that for each $(R, X) \in \mathcal{V}$, $\varphi(R, X) \in Max[R_i\colon 2^X]$. Let $\pi\colon N \to N$ be a permutation on N. Let $R \in \mathcal{S}^N$. Let $M^1(R, X, \pi) \equiv Max[R_{\pi(1)}\colon 2^X]$. For each $k \in \{2, \cdots, n\}$, let $M^k(R, X, \pi) \equiv Max[R_{\pi(k)} : M^{k-1}(R, X, \pi)]$. A rule φ is *serially dictatorial* if there is a permutation π on N such that for each $(R, X) \in \mathcal{V}$ and each $k \in N$, $\varphi(R) \in M^k(R, X, \pi)$. Note that if φ is a serially dictatorial rule in Φ and is associated with permutation $\pi\colon N \to N$, then φ is represented by the profile $(\mathfrak{C}_x)_{x\in A}$ such that for each $x \in X$ and each pair of disjoint groups $C_1, C_2 \subseteq N$ with $C_1, C_2 \neq \emptyset$, $(C_1, C_2) \in \mathfrak{C}_x$ if and only if for some $k \in \{1, \cdots, n\}$, $\pi(k) \in C_1$ and for each $k' < k$, $\pi(k') \notin C_1 \cup C_2$. It is easily shown that each serially dictatorial rule coincides, except for nulls, with a serially dictatorial rule in Φ.

3 Axioms

We are interested in rules satisfying the following axioms. First, it should not be possible to make an agent better off without making someone else worse off.

Efficiency. For each $(R, X) \in \mathcal{V}$, there is no $Y \subseteq X$ such that for each $i \in N$, $Y\ R_i\ \varphi(R, X)$ and for some $j \in N$, $Y\ P_j\ \varphi(R, X)$.

The next axiom says that no one should ever benefit by misrepresenting his preferences, independently of the others' announcements. This axiom is studied by BSZ in

[7] A larger family of rules can be defined by using extended profiles of power structures $\left((\mathfrak{C}_{x,X})_{x\in X}\right)_{X\in\mathcal{A}}$ that may vary across agendas.

[8] Samet and Schmeidler (2003) call this rule "majority rule" on a simple domain without any null issue (when there is no null for anyone, plurality means majority). Likewise, on the domain of *strict* separable preferences, we will use the two terms, majority and plurality, interchangeably. This rule belongs to the family of rules which BSZ call voting by quota.

the fixed agenda model with *strict* separable preferences.

Strategy-Proofness. For each $(R, X) \in \mathcal{V}$ and each $i \in N$, there is no $R'_i \in \mathcal{S}$ such that $\varphi((R'_i, R_{-i}), X)\ P_i\ \varphi(R, X)$.

For each $(R, X) \in \mathcal{V}$, agent $i \in N$ is a *dummy* (player) *at* (R, X) if i is indifferent between any two subsets of X, that is, for each pair $Y, Y' \subseteq X$, $Y\ I_i\ Y'$. Note that a dummy may rank infeasible alternatives, namely alternatives outside 2^X, in various ways. Now, consider two problems that only differ in the preference of a dummy. Then, there is no reason for making different (in terms of welfare) choices for the two problems.

Dummy Independence. For each pair $(R, X), (R', X') \in \mathcal{V}$ and each $i \in N$ with $R_{-i} = R'_{-i}$ and $X = X'$, if agent i is a dummy at both problems, then for each $j \in N$, $\varphi(R, X)\ I_j\ \varphi(R', X')$.

Preferences in each problem (R, X) contain information on how agents rank infeasible alternatives. Some rules may depend on this information. *Dummy independence* prevents rules from depending on the irrelevant information of a dummy's preference. Of course, one may wish to prevent rules to depend on the irrelevant information of all agents (both dummies and non-dummies). To formally define such a requirement, we use the following notation. For each pair $R_i, R'_i \in \mathcal{S}$ and each $X \in \mathcal{A}$, we write $R_i|_X \equiv R'_i|_X$ if R_i and R'_i rank alternatives in 2^X identically, that is, for each pair $Y, Y' \subseteq X$, $Y\ R_i\ Y'$ if and only if $Y\ R'_i\ Y'$. A rule φ satisfies *independence* (of irrelevant issues) if for each pair $R, R' \in \mathcal{S}^N$ and each $X \in \mathcal{A}$, if $R_i|_X = R_i|_X$ for each $i \in N$, then $\varphi(R, X)\ I_i\ \varphi(R', X)$ for each $i \in N$. Both *dummy independence* and *independence* are stated in welfare terms. They could be stated in terms of set equality, replacing "for each $i \in N$, $\varphi(R, X)\ I_i\ \varphi(R', X)$" with "$\varphi(R, X) = \varphi(R', X)$". Obviously, this replacement strengthens the axioms.[9]

The next axiom says that the names of agents should not matter. Let $\pi \colon N \to N$ be a permutation on N. For each $R \in \mathcal{S}^N$, let R^π be such that for each $i \in N$, $R^\pi_i \equiv R_{\pi(i)}$.

Anonymity.[10] For each $(R, X) \in \mathcal{V}$, each permutation π on N, and each $i \in N$,

$$\varphi(R, X)\ I_i\ \varphi(R^\pi, X).$$

When all rules in a family violate *anonymity*, we may look for those members of the

[9] To explain why the axiom is stronger after the replacement, consider the following example. For each $i \in N$, let $\{a\}\ I_i\ \{b\}$ and $R'_i|_{\{a,b\}} = R_i|_{\{a,b\}}$. Let $\varphi(R, \{a, b\}) \equiv \{a\}$ and $\varphi(R', \{a, b\}) \equiv \{b\}$. Then for each $i \in N$, $\varphi(R, \{a, b\})\ I_i\ \varphi(R', \{a, b\})$, but $\varphi(R, \{a, b\}) \neq \varphi(R', \{a, b\})$.

[10] A stronger formulation is *strong anonymity* saying that for each $(R, X) \in \mathcal{V}$ and each permutation π on N, $\varphi(R, X) = \varphi(R^\pi, X)$. Our results involving *anonymity* do not hold with *strong anonymity* because of nulls.

family that minimally violate the property. We say that a rule φ *violates anonymity at* $(R, X) \in \mathcal{V}$ if there are a permutation $\pi \colon N \to N$ and an agent $i \in N$ such that $\varphi(R, X) \; P_i \; \varphi(R^\pi, X)$ or $\varphi(R^\pi, X) \; P_i \; \varphi(R, X)$.

Definition (Minimal Violation of Anonymity within Family Ψ). Given a family of rules Ψ, a rule $\varphi \in \Psi$ *minimally violates anonymity within* Ψ if for each $(R, X) \in \mathcal{V}$, whenever φ violates *anonymity* at (R, X), all other rules in Ψ also violate *anonymity* at (R, X).

In concrete environments, some unforeseen features associated with institutional specifications, informational imperfection, etc. may have an undesired impact on the outcome of the rule adopted. This concern calls for robustness requirements. Several axioms of this kind have been introduced in various models. Examples are "path independence" in the Arrovian social choice model (Plott 1973), its extension for social choice functions (Kalai and Megiddo 1980), "composition up" and "composition down" axioms in the model of claims adjudication (Thomson 2003), etc. The problems we consider here often occur in organizations where provisions regulate how proposals or issues are raised and agendas formed. These provisions enable some agents inside or outside the organization to exert influence on agenda formation, in particular, separation of some issues in an agenda from other issues. Our next axiom is a robustness requirement associated with agenda division.

To motivate the axiom from a different angle, suppose that the members of a society have chosen a decision from an agenda X. After the decision, however, they realize that some issues have been missed or that some new issues have arisen. Denote the set of these missed or newly raised issues by X'. Some members may argue that the original decision on X should be respected and X' should be considered independently as a new agenda. Others may argue that a new decision should be made reconsidering the whole set $X \cup X'$. The next axiom frees society from this potential conflict. It requires rules not to depend on which perspective is taken.

Division Indifference. For each $R \in \mathcal{S}^N$, each disjoint pair $X, X' \in \mathcal{A}$, and each $i \in N$,

$$\varphi(R, X \cup X') \;\; I_i \;\; [\varphi(R, X') \cup \varphi(R, X)] \, .$$

Among the rules that depend only on preference information over feasible alternatives, those satisfying *division indifference* have the advantage of "informational efficiency". This is because dividing a large agenda into subagendas enables society to solve its problem at a smaller informational cost (after an agenda division, preference information associated with any set of two divided issues becomes irrelevant). Of course, if the solution obtained after the agenda division differs from the solution for the original problem, then this idea of dividing the agenda will be problematic. But, this is not something society needs to worry about as long as it adopts a rule satisfying *division indifference*.

Finally, note that since preferences are separable, *division indifference* is *very easy* to satisfy. Indeed, any rule in Φ^o satisfies this axiom.

4 Results

We state first our preliminary results (Theorems 1-3) and the main result (Theorem 4) on the separable domain. Then we state corollaries (Corollaries 2-4) on the strict separable domain. All proofs are collected in Section 5. All our results on the separable domain hold on the additive domain.

First is a characterization of the family of rules satisfying *strategy-proofness*, *division indifference*, and *dummy independence*. This family is Φ^o_m.

Theorem 1. *A rule on the separable domain satisfies strategy-proofness, division indifference, and dummy independence if and only if it coincides, except for nulls, with a rule that is represented by a monotonic profile of power structures.*

This theorem extends the possibility result by BSZ established on the *strict* separable domain (their Theorem 1). However, to deal with the difficulty caused by the admissibility of non-strict preferences (Le Breton and Sen 1995), we have to rely on the variable agenda feature of our model and *division indifference*. On the other hand, we do not need the full-range condition, known also as "voter sovereignty", which plays a critical role in BSZ.

Adding *efficiency*, we obtain a dictatorship result parallel to one in BSZ (Theorem 4).

Theorem 2. *Assume that the agenda domain $\mathcal{A}$ has an agenda with at least three issues. Then a rule on the separable domain satisfies strategy-proofness, division indifference, dummy independence, and efficiency if and only if it is serially dictatorial.*

Remark 1. The two theorems characterize the same families of rules as in Theorems 1 and 2 by Ju (2003). This paper imposes *strategy-proofness*, *votes-only*, and "null-independence" (a weaker version of the *issue-wise voting property* to be defined in Section 6) in a fixed agenda model. The three axioms together with *division indifference* will lead to similar results in our variable agenda model. In fact, it can be shown that given *division indifference*, the three axioms in Ju (2003) are logically "almost" equivalent to *strategy-proofness* and *dummy independence* (this can be proven using Lemma 1).[11] Note that our *dummy independence* is much weaker and has clearer

[11] The three axioms in Ju (2003) (defined in our variable agenda model) together imply *dummy independence* but not *division indifference*. *Strategy-proofness* and *dummy independence* together imply neither *votes-only* nor *null-independence* in Ju (2003). But given *division indifference*, this implication *almost* holds, as we can show using Lemma 1: "almost" because we get *votes-only*^{o}

motivation than *votes-only* and *null-independence*.

By Theorem 2, no *efficient* rule satisfies *strategy-proofness*, *division indifference*, *dummy independence*, and *non-dictatorship*. As stated in the assumption of the agenda domain, this impossibility occurs because we require *efficiency* for problems whose agendas contain more than two issues.

The next axiom restricts *efficiency* to agendas with at most two issues.

Restricted Efficiency. For each $(R, X) \in \mathcal{V}$ with $|X| \leq 2$, there is no $Y \subseteq X$ such that for each $i \in N$, $Y \; R_i \; \varphi(R, X)$ and for some $j \in N$, $Y \; P_j \; \varphi(R, X)$.

We show that within rules in Φ^{o}, *restricted efficiency* implies *monotonicity* as well as the following additional properties of power structures. A profile $(\mathfrak{C}_x)_{x \in A}$ satisfies *unanimity* if for each $x \in A$ and each non-empty $S \subseteq N$, $(S, \emptyset) \in \mathfrak{C}_x$ and $(\emptyset, S) \notin \mathfrak{C}_x$. It satisfies *neutrality* if for each pair $x, y \in A$, $\mathfrak{C}_x = \mathfrak{C}_y$. It satisfies *duality* if for each $x \in A$ and each $(C_1, C_2) \in \bar{\mathfrak{C}}$ with $(C_1, C_2) \neq (\emptyset, \emptyset)$, $(C_1, C_2) \in \mathfrak{C}_x$ if and only if $(C_2, C_1) \notin \mathfrak{C}_x$.

Theorem 3. *A rule on the separable domain satisfies strategy-proofness, division indifference, dummy independence, and restricted efficiency if and only if it coincides, except for nulls, with a rule that is represented by a profile of power structures satisfying monotonicity, unanimity, duality, and neutrality.*

Note that *duality* embodies a form of discrimination among agents; for example, when two disjoint subsets $C_1, C_2 \subseteq N$ have the same number of agents, different decisions have to be made on x for (C_1, C_2) and for (C_2, C_1). Therefore, we obtain:

Corollary 1. *No anonymous rule on the separable domain satisfies strategy-proofness, division indifference, dummy independence, and restricted efficiency.*

However, we show that there are rules that violate *anonymity* in a minimal way. Our main result, Theorem 4, is a characterization of these rules. To define them, let $\mathfrak{C}^{\mathrm{Tie}} \equiv \{(C_1, C_2) \in \bar{\mathfrak{C}} : |C_1| = |C_2| \neq 0\}$ be the set of pairs of disjoint non-empty groups of the same size. A *tie-breaking function* is a function $\tau \colon \mathfrak{C}^{\mathrm{Tie}} \to \{0, 1\}$ such that for each $(C_1, C_2) \in \mathfrak{C}^{\mathrm{Tie}}$, either $\tau(C_1, C_2) = 1$ or $\tau(C_2, C_1) = 1$ but not both (or equivalently, for each $(C_1, C_2) \in \mathfrak{C}^{\mathrm{Tie}}$, $\tau(C_1, C_2) = 1$ if and only if $\tau(C_2, C_1) = 0$). We call this property the *tie-breaking condition.* We now define a family of rules that make the same decision as plurality rule on each issue unless the sets of supporters and objectors have the same size. In that case, the decision on the issue relies on a tie-breaking function. For example, there is a chair, say agent 1, who decides which one of the tied groups wins, that is, for each $(C_1, C_2) \in \mathfrak{C}^{\mathrm{Tie}}$ with $1 \in C_1 \cup C_2$, $\tau\,(C_1, C_2) = 1$

instead of *votes-only*. *Division indifference* in our variable agenda model plays a similar role to the combination of *votes-only* and *null-independence* in the fixed agenda model of Ju (2003).

if and only if $1 \in C_1$.[12]

Definition (TBD-Plurality Rules). A rule is a "tie-breaking-discrimination-plurality rule", briefly, a *TBD-plurality rule* associated with a tie-breaking function τ, denoted by TBD^{τ}, if for each $(R, X) \in \mathcal{V}$ and each $x \in X$,
(i) when $|N_x^+(R)| \neq |N_x^-(R)|$,

$$x \in TBD^{\tau}(R, X) \text{ if and only if } |N_x^+(R)| > |N_x^-(R)|;$$

(ii) when $|N_x^+(R)| = |N_x^-(R)|$,

$$x \in TBD^{\tau}(R, X) \text{ if and only if } \tau\left(N_x^+(R), N_x^-(R)\right) = 1.$$

Although plurality rule is very close to all TBD-plurality rules, it is not one of these rules. Clearly, all profiles of power structures representing TBD-plurality rules satisfy *monotonicity*, *unanimity*, and *neutrality*. By the condition imposed in the definition of tie-breaking functions, they also satisfy *duality*. And because of *duality*, the TBD-plurality rules violate *anonymity*. However, violations occur only when the sets of supporters and objectors for an issue have the same size, which is exactly when all rules represented by a profile of power structures with *duality* violate *anonymity*. This is why we speak of minimal violations of anonymity.

Theorem 4. *A rule on the separable domain minimally violates anonymity within rules satisfying strategy-proofness, division indifference, dummy independence, and restricted efficiency if and only if it coincides, except for nulls, with a TBD-plurality rule.*

We next establish the independence of our axioms in this theorem.

Example 1 (Dropping *strategy-proofness*). We define a rule that satisfies *division indifference*, *dummy independence*, and *restricted efficiency* and that violates *anonymity* at least as rarely as any TBD-plurality rule. This will show that Theorem 4 fails without *strategy-proofness*. Assume that A is finite and $\mathcal{A} \equiv 2^A$. Let τ be a tie-breaking function. Let φ be defined as follows. For each $R \in \mathcal{S}^N$, if $\{Y \subseteq A : \text{for each } i \in N, Y \; I_i \; TBD^{\tau}(R, A)\} \backslash \{Y \subseteq A : Y \backslash Null\,(R) = TBD^{\tau}\,(R, A) \backslash Null\,(R)\} \neq \emptyset$, pick any element from this set and call it $Y(R, A)$. If this set is empty, let $Y\,(R, A) \equiv TBD^{\tau}\,(R, A)$. Thus, when $Y\,(R, A) \neq TBD^{\tau}\,(R, A)$, there are non-null issues in $Y\,(R, A) \backslash TBD^{\tau}\,(R, A)$ or $TBD^{\tau}\,(R, A) \backslash Y\,(R, A)$. For each $R \in \mathcal{S}^N$, let $\varphi\,(R, A) \equiv Y\,(R, A)$. For each $(R, X) \in \mathcal{V}$ with $X \neq A$, let $\varphi\,(R, X) \equiv TBD^{\tau}\,(R, X)$. Then, since

[12] To complete the definition of such a tie-breaking function, we need to specify a priorty ordering π of agents, say, $\pi\,(1) = 1$, $\cdots$, $\pi\,(n) = n$. Then for each $(C_1, C_2) \in \mathfrak{C}^{\text{Tie}}$, whenever $1 \notin C_1 \cup C_2$ and $2 \in C_1 \cup C_2$, $\tau\,(C_1, C_2) = 1$ if and only if $2 \in C_1$. Whenever $1, 2 \notin C_1 \cup C_2$ and $3 \in C_1 \cup C_2$, $\tau\,(C_1, C_2) = 1$ if and only if $3 \in C_1$. We proceed in this way, according to π.

for each $(R,X) \in \mathcal{V}$, $\varphi(R,X)\ I_i\ TBD^{\tau}(R,X)$, φ violates *anonymity* at least as rarely as any TBD-plurality rule. It is easy to show that φ satisfies *division indifference*, *dummy independence*, and *restricted efficiency*. But φ violates *strategy-proofness*. To see this, suppose that $R \in \mathcal{S}$ is such that $Y(R,A) = \emptyset$ and $TBD^{\tau}(R,A) = A$. Let $i \in N$ and R'_i be such that $Y(R'_i, R_{-i}, A) = A$ and $\emptyset\ P'_i\ A$. Then $\varphi(R,A) = \emptyset\ P'_i\ \varphi(R'_i, R_{-i}, A) = A$.

Example 2 (Dropping *division indifference*). Let φ be a rule that coincides with a TBD-plurality rule if the agenda has at most two issues and with an *anonymous* rule (such as plurality rule) in Φ_m otherwise. Then φ violates *division indifference*. Since φ satisfies *anonymity* for any problem with more than two issues, φ violates *anonymity* less often than any TBD-plurality rule.

Example 3 (Dropping *dummy independence*). For simplicity, let us assume $N \equiv \{1,2,3\}$. Define a tie-breaking function τ^1 as follows. For each $i = 2,3$, let $\tau^1(\{1\},\{i\}) \equiv 1$ and $\tau^1(\{i\},\{1\}) \equiv 0$. Let $\tau^1(\{2\},\{3\}) \equiv 1$ and $\tau^1(\{3\},\{2\}) \equiv 0$. Define another tie-breaking function τ^2 as follows. For each $i = 2,3$, let $\tau^2(\{1\},\{i\}) \equiv 1$ and $\tau^2(\{i\},\{1\}) \equiv 0$. Let $\tau^2(\{3\},\{2\}) \equiv 1$ and $\tau^2(\{2\},\{3\}) \equiv 0$. Let $a,b \in A$. For each $(R,X) \in \mathcal{V}$, let φ be defined as follows:

$$\varphi(R,X) \equiv \begin{cases} TBD^{\tau^1}(R,X), \text{ if } \{a\}\ P_1\ \{b\}, \\ TBD^{\tau^2}(R,X), \text{ otherwise.} \end{cases}$$

Thus φ depends on the ordering between $\{a\}$ and $\{b\}$, even if both a and b are not in the agenda. So φ violates *dummy independence*. It is easy to show that φ satisfies *division indifference* and *restricted efficiency*, and that φ violates *anonymity* at least as rarely as any TBD-plurality rule. To show *strategy-proofness*, it is enough to consider agent 1. Let (R,X) be such that $\{a\}\ P_1\ \{b\}$ (the same argument applies for the other case). If there is $x \in X$ such that $(N_x^+(R), N_x^-(R)) = (\{1\},\{i\})$ or $(\{i\},\{1\})$ for some $i = 2,3$, then the decision on x is made according to whether it is a good or a bad for agent 1. Thus agent 1 has no incentive to misrepresent his preference. If there is no such issue, then for each $x \in X$, either x is a null for R_1 or $|N_x^+(R)| \neq |N_x^-(R)|$. In the former case, agent 1 is indifferent to the decision on x. In the latter case, if agent 1 is in the larger group between $N_x^+(R)$ and $N_x^-(R)$, he has no incentive to change the decision on x; if agent 1 is in the smaller group, he has no power to change the decision on x.

Example 4 (Dropping *restricted efficiency*). Let $(\mathfrak{C}_x)_{x \in A}$ be a profile of power structures satisfying *monotonicity* and for each $x \in A$, if (C_1, C_2) and (C'_1, C'_2) are pairs of disjoint subsets of N with $|C_1| = |C'_1|$ and $|C_2| = |C'_2|$, then

$$(C_1, C_2) \in \mathfrak{C}_x \text{ if and only if } (C'_1, C'_2) \in \mathfrak{C}_x.$$

Let φ be the rule represented by $(\mathfrak{C}_x)_{x\in A}$, called a *counting rule* (Ju 2003). It is easy to show that φ satisfies the remaining four axioms.

Example 5 (Dropping *anonymity*). Any rule φ in the family characterized in Theorem 3 satisfies the remaining four axioms.

Strict Separable Preferences

Next are three corollaries of Theorems 1 and 4 on the strict separable domain $\mathcal{V}_{\text{Strict}}$. If preferences are strict, there is no dummy. Thus *dummy independence* is satisfied vacuously by any rule, and our corollaries are stated without this axiom.

We first define a family of rules introduced by BSZ in a fixed agenda model. We extend their definition to our variable agenda model.

Definition (Voting by Committees). A rule φ is a *scheme of voting by committees* if for each $x \in A$, there is a *nonempty* collection $\mathcal{C}_x$ of *groups* of agents such that (i) for each $C_0 \in \mathcal{C}_x$ and each $C_0' \supseteq C_0$, $C_0' \in \mathcal{C}_x$; (ii) for each $(R, X) \in \mathcal{V}$, $x \in \varphi(R, X)$ if and only if $N_x^+(R) \in \mathcal{C}_x$.[13]

We call $\mathcal{C}_x$ the *committee structure for* x. Any scheme of voting by committees φ represented by a profile $(\mathcal{C}_x)_{x\in A}$ can also be represented by the profile of power structures $(\mathfrak{C}_x)_{x\in A}$ defined as follows: for each $x \in A$, let $\mathfrak{C}_x \equiv \{(C_1, C_2) \in \bar{\mathfrak{C}} : C_1 \in \mathcal{C}_x, C_2 \subseteq N \backslash C_1\}$. By condition (i), $(\mathfrak{C}_x)_{x\in A}$ satisfies *monotonicity*. Hence schemes of voting by committees are in Φ_m. It is easy to show that the converse inclusion also holds on the strict separable domain $\mathcal{V}_{\text{Strict}}$, because any pair of disjoint groups (C_1, C_2) with $C_1 \cup C_2 \neq N$ is useless.

Applying the same argument as that used for proving Theorem 1, we obtain:

Corollary 2. *A rule on the strict separable domain satisfies strategy-proofness and division indifference if and only if it is a scheme of voting by committees.*

Majority rule (Samet and Schmeidler 2003) is the scheme of voting by committees represented by $(\mathcal{C}_x)_{x\in A}$ such that for each $x \in A$ and each $C_0 \subseteq N$, $C_0 \in \mathcal{C}_x$ if and only if $|C_0| \geq (n+1)/2$. On the strict separable domain $\mathcal{V}_{\text{Strict}}$, majority rule coincides with plurality rule. So, we can use the two terms interchangeably. For the same reason, we call each TBD-plurality rule as a *TBD-majority rule.*

Adding *restricted efficiency* and *anonymity*, we obtain:

Corollary 3. *A rule on the strict separable domain minimally violates anonymity within rules satisfying strategy-proofness, division indifference, and restricted efficiency if and only if it is a TBD-majority rule.*

[13] Our definition of voting by committees is slightly weaker than the definition by BSZ. To get their definition, we need to add the following: for each $x \in A$, $\emptyset \notin \mathcal{C}_x$.

When the number of agents is odd, there is no need for tie-breaking. Thus, majority rule is the only TBD-majority rule. Therefore, we obtain:

Corollary 4. *When the number of agents is odd, a rule on the strict separable domain satisfies strategy-proofness, division indifference, restricted efficiency, and anonymity if and only if it is majority rule.*

5 Proofs

We use the following additional properties of rules. A rule φ satisfies *division invariance* if for each $(R, X) \in \mathcal{V}$ and each $X' \subseteq X$, $\varphi(R, X') \cup \varphi(R, X \backslash X') = \varphi(R, X)$.[14] It satisfies *division invariance*^{o} if for each $(R, X) \in \mathcal{V}$, each $i \in N$, each $x \in X \backslash Null(R)$, and each $X' \subseteq X$, $x \in \varphi(R, X') \cup \varphi(R, X \backslash X')$ if and only if $x \in \varphi(R, X)$. For each $(R, X) \in \mathcal{V}$ and each $i \in N$, let

$$G(R_i, X) \equiv \{x \in X : \{x\} \ P_i \ \emptyset\} \text{ and } B(R_i, X) \equiv \{x \in X : \emptyset \ P_i \ \{x\}\}$$

be the set of goods and the set of bads in X for R_i, respectively.

We first establish two useful lemmas.

Lemma 1. *Strategy-proofness, division indifference, and dummy independence together imply division invariance^{o}, votes-only^{o}, and independence.*

Proof. Let φ be a rule satisfying strategy-proofness, division indifference, and dummy independence. We first show that φ satisfies division invariance^{o} and then, that φ satisfies votes-only^{o}. Finally, independence follows directly from votes-only^{o}. Let $(R, X) \in \mathcal{V}$ be such that $|X| \geq 2$. By division indifference, for each $i \in N$,

$$\varphi(R, X) \ I_i \ \bigcup_{x \in X} \varphi(R, \{x\}).$$

Suppose by contradiction that there is $a \in X \backslash Null(R)$ such that $a \in \varphi(R, X) \backslash \varphi(R, \{a\})$ or $a \in \varphi(R, \{a\}) \backslash \varphi(R, X)$. Consider the case $a \in \varphi(R, X) \backslash \varphi(R, \{a\})$ (the same argument applies for the other case). Since $a \notin Null(R)$, there is $j \in N$ for whom a is either a good or a bad for R_j. Assume $\{a\} \ P_j \ \emptyset$ (the same argument applies when $\emptyset \ P_j \ \{a\}$). Let R'_j be such that $G(R'_j, X) = G(R_j, X)$, $B(R'_j, X) = B(R_j, X)$, and for each pair $Y, Y' \subseteq X \backslash \{a\}$,

$$Y \cup \{a\} \ P'_j \ Y' \text{ or } Y' \ P'_j \ Y \cup \{a\}, \qquad (\dagger)$$

[14] This is similar to Plott's (1973) *path independence*, which can be formulated as follows: for each $R \in \mathcal{S}^N$ and each $X, Y \in \mathcal{A}$, $\varphi(R, X \cup Y) = \varphi(R, \varphi(R, X) \cup \varphi(R, Y))$. *Division invariance* implies *path independence* but the converse is not true.

and

$$\text{if } Y \cup \{a\} \; R_j \; Y', \; Y \cup \{a\} \; P'_j \; Y'.^{15} \tag{††}$$

Let $x \in X$. If $x \in G(R_j, X) \cup B(R_j, X)$, then since $G(R'_j, X) = G(R_j, X)$ and $B(R'_j, X) = B(R_j, X)$, by *strategy-proofness*, $\varphi(R'_j, R_{-j}, \{x\}) = \varphi(R, \{x\})$. If $x \notin G(R_j, X) \cup B(R_j, X)$, then since j is a dummy for both $(R'_j, R_{-j}, \{x\})$ and $(R, \{x\})$, by *dummy independence*, $\varphi(R'_j, R_{-j}, \{x\}) \; I_i \; \varphi(R, \{x\})$ for each $i \in N$. Thus, for each $x \in X \backslash Null\,(R)$, $\varphi(R'_j, R_{-j}, \{x\}) = \varphi(R, \{x\})$. Since $a \notin \bigcup_{x \in X} \varphi\,(R, \{x\})$ and $a \notin Null\,(R)$, $a \notin \bigcup_{x \in X} \varphi\left(R'_j, R_{-j}, \{x\}\right)$. On the other hand, by *division indifference*,

$$\varphi\left(R'_j, R_{-j}, X\right) \; I'_j \; \bigcup_{x \in X} \varphi\left(R'_j, R_{-j}, \{x\}\right). \tag{†††}$$

Thus by (†) and (†††), $a \notin \varphi\left(R'_j, R_{-j}, X\right)$. Then since $a \in \varphi\,(R, X)$, again by (†),

$$\varphi\,(R, X) \; P'_j \; \varphi\left(R'_j, R_{-j}, X\right) \text{ or } \varphi\left(R'_j, R_{-j}, X\right) \; P'_j \; \varphi\,(R, X).$$

In the latter case, by (††), $\varphi\left(R'_j, R_{-j}, X\right) P_j \, \varphi\,(R, X)$. Thus, $\varphi\,(R, X) \; P'_j \; \varphi\left(R'_j, R_{-j}, X\right)$ or $\varphi\left(R'_j, R_{-j}, X\right) \; P_j \; \varphi\,(R, X)$, contradicting *strategy-proofness*.

Therefore for each $a \in X \backslash Null\,(R)$, $a \in \varphi\,(R, X)$ if and only if $a \in \varphi\,(R, \{a\})$. This shows that φ satisfies *division invariance*$^{\text{o}}$.

To show *votes-only*$^{\text{o}}$, let $R, R' \in \mathcal{S}^N$ and $X \in \mathcal{A}$ be such that for each $i \in N$, $G\,(R_i, X) = G\,(R'_i, X)$ and $B\,(R_i, X) = B\,(R'_i, X)$. Assume $x \in X \backslash Null\,(R)$ (and $x \in X \backslash Null\,(R')$). By *division invariance*$^{\text{o}}$, $x \in \varphi\,(R, X)$ if and only if $x \in \varphi\,(R, \{x\})$. Similarly, $x \in \varphi\,(R', X)$ if and only if $x \in \varphi\,(R', \{x\})$. Applying *strategy-proofness* and *dummy independence*, and changing R_i into R'_i for each $i \in N$, we deduce that $x \in \varphi\,(R, \{x\})$ if and only if $x \in \varphi\,(R', \{x\})$. Thus, $x \in \varphi\,(R, X)$ if and only if $x \in \varphi(R', X)$. □

Lemma 2. *A rule satisfies division invariance^{o} and votes-only^{o} if and only if it is in Φ^{o}.*

Proof. It is easy to show that each rule in Φ^{o} satisfies *division invariance*$^{\text{o}}$ and *votes-only*$^{\text{o}}$. Conversely, let φ be a rule satisfying the two axioms. For each $x \in A$, let $\mathfrak{C}_x \equiv \{(C_1, C_2) \in \bar{\mathfrak{C}} : (C_1, C_2) \equiv (N^+_x\,(R)\,, N^-_x\,(R))$ and $x \in \varphi\,(R, \{x\})$, for some $R \in \mathcal{S}^N\}$. We show that φ coincides, except for nulls, with the rule $\hat{\varphi}$ that is represented by $(\mathfrak{C}_x)_{x \in A}$. Let $(R, X) \in \mathcal{V}$. Let $x \in X \backslash Null\,(R)$. Suppose $x \in \varphi\,(R, X)$. Then by *division invariance*$^{\text{o}}$, $x \in \varphi\,(R, \{x\})$ and so $(N^+_x\,(R)\,, N^-_x\,(R)) \in \mathfrak{C}_x$. Hence $x \in \hat{\varphi}\,(R, X)$. Conversely, let $x \in \hat{\varphi}\,(R, X)$. Then $(N^+_x\,(R)\,, N^-_x\,(R)) \in \mathfrak{C}_x$. Hence for

[15] For example, let R'_j be such that $G(R'_j, X) = G(R_j, X)$, $B(R'_j, X) = B(R_j, X)$, and for each $Y, Y' \subseteq X \backslash \{a\}$, $Y \cup \{a\} \; P'_j \; Y'$. That is, for R'_j, having a is always better than not having a, independently of other issues.

some $R' \in \mathcal{S}^N$, $(N_x^+(R'), N_x^-(R')) = (N_x^+(R), N_x^-(R))$ and $x \in \varphi(R', \{x\})$. By *votes-only*$^{\text{o}}$, $x \in \varphi(R, \{x\})$. By *division invariance*$^{\text{o}}$, $x \in \varphi(R, X)$. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let φ be a rule in Φ_m^{o}. Then there is $\hat{\varphi} \in \Phi_m$ that coincides, except for nulls, with φ. Let $(\mathfrak{C}_x)_{x\in A}$ be the *monotonic* profile representing $\hat{\varphi}$. Since $\hat{\varphi}$ satisfies *division invariance*, φ satisfies *division indifference*. Since $\hat{\varphi}$ satisfies *independence*, φ satisfies *independence* and so *dummy independence*. To prove *strategy-proofness*, let $(R, X) \in \mathcal{V}$. Let $i \in N$. Let $R_i' \in \mathcal{S}$. If $x \in \hat{\varphi}(R_i', R_{-i}, X) \cap G(R_i, X)$, then since $x \in G(R_i, X)$, by *monotonicity*, $x \in \hat{\varphi}(R, X)$. Thus,

$$\hat{\varphi}(R_i', R_{-i}, X) \cap G(R_i, X) \subseteq \hat{\varphi}(R, X) \cap G(R_i, X). \tag{$\star$}$$

If $x \in B(R_i, X) \cap \hat{\varphi}(R, X)$, then since $x \in B(R_i, X)$, by *monotonicity*, $x \in \hat{\varphi}(R_i', R_{-i}, X)$. Thus,

$$\hat{\varphi}(R_i', R_{-i}, X) \cap B(R_i, X) \supseteq \hat{\varphi}(R, X) \cap B(R_i, X). \tag{$\star\star$}$$

Let Y be the set of nulls for R_i in $\hat{\varphi}(R, X)$. Let Y' be the set of nulls for R_i in $\hat{\varphi}(R_i', R_{-i}, X)$. By ($\star$), ($\star\star$), and separability, $\hat{\varphi}(R, X) \backslash Y \; R_i \; \hat{\varphi}(R_i', R_{-i}, X) \backslash Y'$. Since all issues in Y or Y' are nulls for R_i, then by separability, $\hat{\varphi}(R, X) \; R_i \; \hat{\varphi}(R_i', R_{-i}, X)$. Thus, $\varphi(R, X) \; R_i \; \varphi(R_i', R_{-i}, X)$.

To prove the converse, let φ be a rule satisfying *strategy-proofness*, *division indifference*, and *dummy independence*. Then by Lemma 1, φ satisfies *division invariance*$^{\text{o}}$ and *votes-only*$^{\text{o}}$. Hence by Lemma 2, there is $\hat{\varphi} \in \Phi$ such that $\hat{\varphi}$ coincides, except for nulls, with φ. Let $(\mathfrak{C}_x)_{x\in A}$ be the profile representing $\hat{\varphi}$. To show that $(\mathfrak{C}_x)_{x\in A}$ satisfies *monotonicity*, let $x \in A$ and $(C_1, C_2) \in \mathfrak{C}_x$. Let $R \in \mathcal{S}^N$ be such that $N_x^+(R) = C_1$ and $N_x^-(R) = C_2$. Then $\varphi(R, \{x\}) = \{x\}$. For each $i \notin C_1 \cup C_2$, if $(C_1 \cup \{i\}, C_2) \notin \mathfrak{C}_x$, then for each $R_i' \in \mathcal{S}$ with $\{x\} \; P_i' \; \emptyset$, $\varphi(R_i', R_{-i}, \{x\}) = \emptyset$ and $\varphi(R, \{x\}) = \{x\}$. Then $\varphi(R, \{x\}) \; P_i' \; \varphi(R_i', R_{-i}, \{x\})$, contradicting *strategy-proofness*. Thus, for each $i \notin C_1 \cup C_2$,

$$(C_1 \cup \{i\}, C_2) \in \mathfrak{C}_x. \tag{$\dagger$}$$

By the same argument, for each $i \in C_2$,

$$(C_1, C_2 \backslash \{i\}) \in \mathfrak{C}_x. \tag{$\dagger\dagger$}$$

It is easy to show that ($\dagger$) and ($\dagger\dagger$) together imply *monotonicity*. □

Lemma 3. *Given a rule $\varphi \in \Phi^o$, the following are equivalent*:
(i) *Rule φ satisfies restricted efficiency.*
(ii) *There is a profile of power structures $(\mathfrak{C}_x)_{x\in A}$ satisfying monotonicity, unanimity,*

neutrality, and duality, such that φ coincides, except for nulls, with the rule that is represented by $(\mathfrak{C}_x)_{x\in A}$.

Proof. Let φ be a rule in Φ^o. Let $\hat{\varphi}$ be the rule in Φ with representation $(\mathfrak{C}_x)_{x\in A}$. Assume that φ and $\hat{\varphi}$ coincide, except for nulls, with each other. The proof is in two steps.

Step 1: (ii) *implies* (i).

Assume that $(\mathfrak{C}_x)_{x\in A}$ satisfies *monotonicity*, *unanimity*, *neutrality*, and *duality*. By *unanimity*, no alternative in each singleton agenda problem Pareto dominates the choice made by φ. So, we only have to consider agendas with two issues. Let $X \equiv \{a, b\}$. Let $R \in \mathcal{S}^N$. We show, case by case, that no subset of X Pareto dominates $\varphi(R, X)$.

Case 1: $\varphi(R, X) = \emptyset$. Then $(N_a^+(R), N_a^-(R)) \notin \mathfrak{C}_a$ and $(N_b^+(R), N_b^-(R)) \notin \mathfrak{C}_b$. Hence by *duality*, $(N_a^-(R), N_a^+(R)) \in \mathfrak{C}_b$ and $(N_b^-(R), N_b^+(R)) \in \mathfrak{C}_a$. Thus, $N_a^-(R) \not\subseteq N_b^+(R)$ or $N_a^+(R) \not\supseteq N_b^-(R)$. Let $i \in N_a^-(R)\backslash N_b^+(R)$ or $i \in N_b^-(R)\backslash N_a^+(R)$. Then $\emptyset$ P_i $\{a, b\}$. Since there is at least one such i, then $\{a, b\}$ does not Pareto dominate $\emptyset$. Clearly, neither $\{a\}$ nor $\{b\}$ Pareto dominates $\emptyset$ (if $\{a\}$ does, then a should have been chosen because of *unanimity*).

Case 2: $\varphi(R, X) = \{a\}$ *or* $\{b\}$. Let $\varphi(R, X) = \{a\}$ (a similar argument applies to the case $\varphi(R, X) = \{b\}$). Suppose that for each $i \in N$, $\{b\}$ R_i $\{a\}$ and for some $j \in N$, $\{b\}$ P_j $\{a\}$. Then clearly, $N_b^+(R) \supseteq N_a^+(R)$ and $N_b^-(R) \subseteq N_a^-(R)$. Hence by *neutrality*, $(N_b^+(R), N_b^-(R)) \in \mathfrak{C}_b$ and so $b \in \varphi(R)$, contradicting the assumption. Thus $\{b\}$ does not Pareto dominate $\{a\}$. If $\emptyset$ Pareto dominates $\{a\}$, then a should not have been chosen because of *unanimity*. If $\{a, b\}$ Pareto dominates $\{a\}$, then b should have been chosen. Thus, no alternative Pareto dominates $\{a\}$.

Case 3: $\varphi(R) = \{a, b\}$. Then $(N_a^+(R), N_a^-(R)) \in \mathfrak{C}_a$ and $(N_b^+(R), N_b^-(R)) \in \mathfrak{C}_b$. Hence by *duality*, $(N_a^-(R), N_a^+(R)) \notin \mathfrak{C}_b$ and $(N_b^-(R), N_b^+(R)) \notin \mathfrak{C}_a$. Thus, $N_a^+(R) \not\subseteq N_b^-(R)$ or $N_a^-(R) \not\supseteq N_b^+(R)$. Let $i \in N_a^+(R)\backslash N_b^-(R)$ or $i \in N_b^+(R)\backslash N_a^-(R)$. Then $\{a, b\}$ P_i $\emptyset$. Since there is at least one such i, then $\emptyset$ does not Pareto dominate $\{a, b\}$. Clearly, neither $\{a\}$ nor $\{b\}$ Pareto dominates $\{a, b\}$ (if $\{a\}$ or $\{b\}$ does, then a or b should not have been chosen because of *unanimity*).

Step 2: (i) *implies* (ii).

Unanimity of $(\mathfrak{C}_x)_{x\in A}$ follows immediately from *restricted efficiency*. In the next three substeps, we show *neutrality*, *duality*, and *monotonicity*, successively. Let $X \in \mathcal{A}$ be an agenda with at least two issues, say x, y.

Substep 2.1: *Neutrality.* Without loss of generality, we may assume that $(\emptyset, \emptyset) \in \mathfrak{C}_x$ if and only if $(\emptyset, \emptyset) \in \mathfrak{C}_y$.[16] We now show $\mathfrak{C}_x\backslash\{(\emptyset, \emptyset)\} = \mathfrak{C}_y\backslash\{(\emptyset, \emptyset)\}$. Let $(C_1, C_2) \in$

[16] When $(\emptyset, \emptyset) \in \mathfrak{C}_x$ and $(\emptyset, \emptyset) \notin \mathfrak{C}_y$, we can change $\mathfrak{C}_y$ to $\mathfrak{C}_y \cup \{(\emptyset, \emptyset)\}$ or change $\mathfrak{C}_x$ to $\mathfrak{C}_x\backslash\{(\emptyset, \emptyset)\}$.

$\mathfrak{C}_x \setminus \{(\emptyset,\emptyset)\}$. Suppose $(C_1, C_2) \notin \mathfrak{C}_y$. Then there is $R \in \mathcal{S}^N$ such that $(N_x^+(R), N_x^-(R)) = (N_y^+(R), N_y^-(R)) \equiv (C_1, C_2)$ and for each $i \in C_1 \cup C_2$, $\{y\}\ P_i\ \{x\}$. Then since $(C_1, C_2) \in \mathfrak{C}_x \setminus \mathfrak{C}_y$, $\varphi(R, \{x, y\}) \equiv \{x\}$. Since everyone weakly prefers $\{y\}$ to $\{x\}$ and all agents in $C_1 \cup C_2$, which is non-empty, prefer $\{y\}$ to $\{x\}$, then we have a contradiction to *restricted efficiency*. Thus, $\mathfrak{C}_x \setminus \{(\emptyset,\emptyset)\} \subseteq \mathfrak{C}_y \setminus \{(\emptyset,\emptyset)\}$. The proof of the opposite inclusion is similar. To complete the proof, we use the connectedness assumption on the agenda domain.

Substep 2.2: *Duality.* By *neutrality*, we may suppose that $\mathfrak{C}_0 \equiv \mathfrak{C}_x$ for each $x \in A$. We show that for each $(C_1, C_2) \in \bar{\mathfrak{C}}$ with $(C_1, C_2) \neq (\emptyset,\emptyset)$, $(C_1, C_2) \in \mathfrak{C}_0$ if and only if $(C_2, C_1) \notin \mathfrak{C}_0$. Suppose to the contrary that $(C_1, C_2) \in \mathfrak{C}_0$ and $(C_2, C_1) \in \mathfrak{C}_0$. Let $R \in \mathcal{S}^N$ be such that $(N_x^+(R), N_x^-(R)) \equiv (C_1, C_2)$, $(N_y^+(R), N_y^-(R)) \equiv (C_2, C_1)$, and for each $i \in C_1 \cup C_2$, $\emptyset\ P_i\ \{x, y\}$. Then since $(C_1, C_2) \in \mathfrak{C}_0$ and $(C_2, C_1) \in \mathfrak{C}_0$, $\hat{\varphi}(R, \{x, y\}) \equiv \{x, y\}$. Since everyone weakly prefers $\emptyset$ to $\{x, y\}$ and all agents in $C_1 \cup C_2$, which is nonempty, prefer $\emptyset$ to $\{x, y\}$, then we have a contradiction to *restricted efficiency*.

Substep 2.3: *Monotonicity.* Let $(C_1, C_2) \in \mathfrak{C}_0$ and $(C_1', C_2') \in \bar{\mathfrak{C}}$ be such that $C_1' \supseteq C_1$ and $C_2' \subseteq C_2$. Suppose, by contradiction, that $(C_1', C_2') \notin \mathfrak{C}_0$. By *duality*, $(C_2', C_1') \in \mathfrak{C}_0$. Since $C_1' \supseteq C_1$ and $C_2' \subseteq C_2$, then there is $R \in \mathcal{S}^N$ such that $(N_x^+(R), N_x^-(R)) = (C_1, C_2)$, $\left(N_y^+(R), N_y^-(R)\right) = (C_2', C_1')$, for each $i \in C_1' \cup C_2$, $\emptyset\ P_i\ \{x, y\}$, and for each $i \notin C_1' \cup C_2$, $\emptyset\ I_i\ \{x, y\}$. Since $(C_1, C_2) \in \mathfrak{C}_0$ and $(C_2', C_1') \in \mathfrak{C}_0$, then $\varphi(R, \{x, y\}) = \{x, y\}$. Then by construction of R, $\emptyset$ Pareto dominates $\{x, y\}$, contradicting *restricted efficiency*. □

A rule satisfies *issue-wise monotonicity* if for each $(R, X) \in \mathcal{V}$, each $R' \in \mathcal{S}^N$, and each $x \in X$ satisfying $N_x^+(R) \subseteq N_x^+(R')$ and $N_x^-(R) \supseteq N_x^-(R')$, $x \in \varphi(R, x)$ implies $x \in \varphi(R', x)$. For each agenda $X \in \mathcal{A}$ with at least three issues, each *efficient* and *issue-wise monotonic* rule is serially dictatorial on the set of problems with agenda X (Lemma 9 in Ju 2003). Let $\mathcal{V}(X) \equiv \{(R, X) : R \in \mathcal{S}^N\}$ be the subdomain of problems with the fixed agenda X.

Lemma 4 (Ju 2003). *If a rule satisfies efficiency and issue-wise monotonicity, then for each $X \in \mathcal{A}$ with $|X| \geq 3$, it is serially dictatorial on the fixed agenda subdomain $\mathcal{V}(X)$.*

Now, we are ready to prove Theorem 2.

Proof of Theorem 2. Assume that $\mathcal{A}$ has at least one agenda with at least three issues, say X. We will prove only the non-trivial direction. Let φ be a rule satisfying *strategy-proofness*, *division indifference*, *dummy independence*, and *efficiency*. By

Then the new profile of power structures represents another rule that still coincides, except for nulls, with φ.

Lemmas 1, 2, and 3, there is a profile $(\mathfrak{C}_x)_{x\in A}$ that satisfies *monotonicity*, *unanimity*, *neutrality*, and *duality*, and φ coincides, except for nulls, with the rule $\hat{\varphi}$ that is represented by $(\mathfrak{C}_x)_{x\in A}$.

Since $\hat{\varphi}$ satisfies *efficiency* and *objectwise monotonicity*, then by Lemma 4, $\hat{\varphi}$ is serially dictatorial on the fixed agenda subdomain $\mathcal{V}(X)$. Thus, there is a permutation $\pi\colon N \to N$ such that for each $R \in \mathcal{S}^N$, each $X' \subseteq X$, and each $k \in \{2, \cdots, n\}$, $\varphi(R, X') \in M^k(R, X', \pi)$, where $M^k(R, X', \pi) \equiv Max[R_{\pi(k)} : M^{k-1}(R, X', \pi)]$ and $M^1(R, X', \pi) \equiv Max[R_{\pi(1)}\colon 2^{X'}]$. This implies that for each $x \in X$ and each disjoint pair of non-empty groups $C_1, C_2 \subseteq N$, $(C_1, C_2) \in \mathfrak{C}_x$ if and only if there is $k \in \{1, \cdots, n\}$ such that $\pi(k) \in C_1$ and for each $k' < k$, $\pi(k') \notin C_1 \cup C_2$. By *neutrality*, the same holds for $\mathfrak{C}_y$ for each $y \in A$. Thus, $\hat{\varphi}$ is serially dictatorial on the entire domain $\mathcal{V}$. □

Proof of Theorem 3. Theorem 3 follows directly from Theorem 1 and Lemma 3. □

We next prove the main result, Theorem 4.

Proof of Theorem 4. Let φ be a rule that coincides, except for nulls, with a TBD-plurality rule $\hat{\varphi}$. Note that each TBD-plurality rule is represented by a profile of power structures that satisfies *monotonicity*, *unanimity*, *neutrality*, and *duality*. Thus, by Theorem 3, $\hat{\varphi}$ satisfies *strategy-proofness*, *division indifference*, *dummy independence*, and *restricted efficiency*. Finally, we show that $\hat{\varphi}$ (and φ) *minimally violates anonymity* within the family of rules satisfying the four axioms.

Let $(R, X) \in \mathcal{V}$. If for each $x \in X$, $|N_x^+(R)| \neq |N_x^-(R)|$, then the choice made by φ coincides with the choice made by plurality rule. Since plurality rule satisfies *anonymity*, $\hat{\varphi}$ satisfies *anonymity at* (R, X). Now, suppose that there is $x \in X \backslash Null(R)$ such that $|N_x^+(R)| = |N_x^-(R)|$. Then there is a permutation π on N such that $(N_x^+(R^\pi), N_x^-(R^\pi)) = (N_x^-(R), N_x^+(R))$. By Theorem 3, each rule φ' satisfying *strategy-proofness*, *division indifference*, *dummy independence*, and *restricted efficiency* coincides, except for nulls, with the rule $\hat{\varphi}'$ that is represented by a profile of power structures with *duality*. Because of *duality*, $x \in \varphi'(R, X)$ if and only if $x \notin \varphi'(R^\pi, X)$. Since $x \notin Null(R)$, there is $i \in N$ such that x is either a good or a bad for R_i. Suppose that x is a good for R_i (the same argument applies when x is a bad). Let $\bar{R}_i \in \mathcal{S}$ be such that $G(\bar{R}_i, X) = G(R_i, X)$, $B(\bar{R}_i, X) = B(R_i, X)$, and for each pair $Y, Y' \subseteq X \backslash \{x\}$,

$$Y \cup \{x\} \ \bar{P}_i \ Y' \text{ or } Y' \ \bar{P}_i \ Y \cup \{x\}. \tag{$\star$}$$

Let $\bar{R} \equiv (\bar{R}_i, R_{-i})$. Then $x \in \varphi'(\bar{R}, X)$ if and only if $x \notin \varphi'(\bar{R}^\pi, X)$. By ($\star$), either $\varphi'(\bar{R}, X) \ \bar{P}_i \ \varphi'(\bar{R}^\pi, X)$ or $\varphi'(\bar{R}^\pi, X) \ \bar{P}_i \ \varphi'(\bar{R}, X)$. Thus, φ' violates *anonymity at* (R, X).

In order to prove the converse, let φ be a rule that *minimally violates anonymity* within rules satisfying *strategy-proofness*, *division indifference*, *dummy independence*, and *restricted efficiency*. By Theorem 3, φ coincides, except for nulls, with the rule $\hat{\varphi}$ that is represented by a profile of power structures $(\mathfrak{C}_x)_{x\in A}$ satisfying *monotonicity*, *unanimity*, *neutrality*, and *duality*. Let $(R, X) \in \mathcal{V}$. Suppose $x \in X \backslash Null\,(R)$ and $|N_x^+(R)| \neq |N_x^-(R)|$. Then each TBD-plurality rule satisfies *anonymity at* $(R, \{x\})$. Thus, $\hat{\varphi}$ (and φ) also satisfies *anonymity at* $(R, \{x\})$. That is, for each permutation $\pi \colon N \to N$ and each $i \in N$, $\hat{\varphi}(R, \{x\})\ I_i\ \hat{\varphi}(R^\pi, \{x\})$. Since $x \notin Null\,(R)$, $\hat{\varphi}(R, \{x\}) = \hat{\varphi}(R^\pi, \{x\})$. Now, using *division invariance* of $\hat{\varphi}$, we obtain:

Condition A. For each $(R, X) \in \mathcal{V}$, each $x \in X \backslash Null\,(R)$ with $|N_x^+(R)| \neq |N_x^-(R)|$, and each permutation $\pi \colon N \to N$, $x \in \hat{\varphi}(R, X)$ if and only if $x \in \hat{\varphi}(R^\pi, X)$.

By *neutrality*, there is a power structure $\mathfrak{C}_0$ such that for each $x \in A$, $\mathfrak{C}_x = \mathfrak{C}_0$. Without loss of generality, we may assume that $(\emptyset, \emptyset) \notin \mathfrak{C}_0$.[17]

Claim 1. For each pair of disjoint groups $C_1, C_2 \subseteq N$ with $C_1 \cup C_2 \neq \emptyset$ and $|C_1| \neq |C_2|$, $(C_1, C_2) \in \mathfrak{C}_0$ if and only if $|C_1| > |C_2|$.

Proof. Let (C_1, C_2) be given as above. In order to show that $(C_1, C_2) \in \mathfrak{C}_0$ implies $|C_1| > |C_2|$, suppose by contradiction that $(C_1, C_2) \in \mathfrak{C}_0$ and $|C_1| < |C_2|$. Then by *Condition A* and *monotonicity*, for each disjoint pairs, C_1' and C_2', if $|C_1'| \geq |C_1|$ and $|C_2'| \leq |C_2|$, then $(C_1', C_2') \in \mathfrak{C}_0$. Thus $(C_2, C_1) \in \mathfrak{C}_0$, contradicting *duality*.

In order to show the converse, suppose $|C_1| > |C_2|$. Then as shown above, $(C_2, C_1) \notin \mathfrak{C}_0$. Thus, by *duality*, $(C_1, C_2) \in \mathfrak{C}_0$. □

We next consider pairs of disjoint groups of the same size. We skip the trivial proof of Claim 2.

Claim 2. For each $(C_1, C_2) \in \mathfrak{C}^{Tie}$, $(C_1, C_2) \in \mathfrak{C}_0$ if and only if there are $R \in \mathcal{S}^N$ and $x \in A$ such that $(|N_x^+(R)|, |N_x^-(R)|) = (C_1, C_2)$ and $x \in \hat{\varphi}(R, \{x\})$.

Now, let τ be the tie-breaking function defined as follows: for each $(C_1, C_2) \in \mathfrak{C}^{\text{Tie}}$, $\tau(C_1, C_2) = 1$ if and only if $(C_1, C_2) \in \mathfrak{C}_0$. Then by *duality*, τ satisfies the tie-breaking condition and it is clear by construction that $\hat{\varphi} = TBD^\tau$. □

Proof of Corollary 2. We only show the non-trivial direction. Let φ be a rule satisfying *strategy-proofness* and *division indifference*. We first show that φ satisfies *independence*. Let $X \in \mathcal{A}$ and $R, R' \in \mathcal{S}_{\text{Strict}}^N$ be such that for each $i \in N$, $R_i|_X \equiv R_i'|_{X'}$. Let $Y \equiv \varphi(R, X)$ and $Y^1 \equiv \varphi((R_1', R_{-1}), X)$. If $Y \neq Y^1$, then as preferences are strict, $Y\ P_1\ Y^1$ or $Y^1\ P_1\ Y$. In the former case, since $R_1|_X \equiv R_1'|_X$, $Y\ P_1'\ Y^1$, agent 1 with true preference R_1' is better off reporting R_1; in the latter case, agent 1 with true

[17] If $(\varnothing, \varnothing) \in \mathfrak{C}_0$, then we can define another rule that also coincides, except for nulls, with φ, by using the power structure $\mathfrak{C}_0 \backslash \{(\varnothing, \varnothing)\}$.

preference R_1 is better off reporting R_1'. This contradicts *strategy-proofness*. Thus, $Y = Y^1$. Similarly, changing preferences of all other agents $i \in N \backslash \{1\}$ from R_i to R_i' successively, we can show that $Y = Y'$.

Since preferences are strict, *division indifference* implies *division invariance*. Thus, using Lemmas 1 and 2, we show that φ is represented by a profile of power structures $(\mathfrak{C}_x)_{x \in A}$. Since φ satisfies *strategy-proofness*, the profile $(\mathfrak{C}_x)_{x \in A}$ satisfies *monotonicity*. Note that for strict preferences, only those elements (C_1, C_2) of $\mathfrak{C}_x$ with $C_1 \cup C_2 = N$ are used for defining φ. For each $x \in A$, let $\mathcal{C}_x \equiv \{C_1 \subseteq N : (C_1, N \backslash C_1) \in \mathfrak{C}_x\}$. By *monotonicity* of $(\mathfrak{C}_x)_{x \in A}$, $(\mathcal{C}_x)_{x \in A}$ satisfies condition (i) in the definition of voting by committees. Thus φ is the scheme of voting by committees represented by $(\mathcal{C}_x)_{x \in A}$. □

Proof of Corollary 3. The proof can be completed by applying Corollary 2 and the same arguments as in the proofs of Lemma 3 and Theorem 4. □

6 Issue-wise Voting versus Efficiency

Lemma 3 played a critical role in establishing our main result. In this section, we use this lemma to derive an impossibility result involving the "issue-wise voting property", *efficiency*, and *non-dictatorship*.

Issue-wise Voting Property. For each pair $(R, X), (R', X') \in \mathcal{V}$ and each $x \in X \cap X'$ with $N_x^+(R) = N_x^+(R')$ and $N_x^-(R) = N_x^-(R')$, $x \in \varphi(R, X)$ if and only if $x \in \varphi(R', X')$.[18]

It is easily shown that a rule satisfies the *issue-wise voting property* if and only if it is in Φ. Thus, this axiom implies both *division indifference* and *dummy independence*. The proof of Lemma 3 is easily modified to show that if a rule on a fixed agenda domain $\mathcal{V}(X)$ with $|X| \geq 3$, satisfies the *issue-wise voting property* and *efficiency*, then it is in Φ_m.[19] Similarly, on the domain with strict separable preferences and a fixed agenda $X \in \mathcal{A}$, denoted by $\mathcal{V}_{\text{Strict}}(X)$, if a rule in Φ satisfies *efficiency*, then it is a scheme of voting by committees. Thus, the *issue-wise voting property* and *efficiency* together imply *strategy-proofness*.

On each fixed agenda domain $\mathcal{V}(X)$ with $|X| \geq 3$, Ju (2003) showed that among rules in Φ_m, only serially dictatorial rules satisfy *efficiency*.[20] This result can be strengthened by replacing Φ_m with the larger family Φ.[21] On $\mathcal{V}_{\text{Strict}}(X)$ with $|X| \geq 3$, BSZ showed that *strategy-proofness* and the full-range condition together imply the

[18] *Issue-wise voting property* is called "independence" in Ju (2003) and "decomposability" in Le Breton and Sen (1999).

[19] We just need to consider preferences where all issues other than $x, y \in X$ are bads and follow the same argument as in Step 2 of the proof of Lemma 3.

[20] See Propositions 1 and 2 in Ju (2003).

[21] This means that "monotonicity" axiom in Proposition 2 of Ju (2003) can be dropped.

issue-wise voting property. Combining this with the logical relation established above, we can conclude that among *efficient* rules, the *issue-wise voting property* is a necessary and sufficient condition for *strategy-proofness*. Thus, their impossibility result (Theorem 4 in BSZ) still holds after replacing *strategy-proofness* with the *issue-wise voting property*. Therefore, we obtain:

Proposition 1. *Let $X \in \mathcal{A}$ be an agenda containing at least three issues.*
(i) *A rule on $\mathcal{V}(X)$ satisfies the* issue-wise voting property *and* efficiency *if and only if it is a serially dictatorial rule in Φ.*
(ii) *A rule on $\mathcal{V}_{Strict}(X)$ satisfies the* issue-wise voting property *and* efficiency *if and only if it is dictatorial.*

This result still holds for the variable agenda domains $\mathcal{V}$ and $\mathcal{V}_{\text{Strict}}$. Like our other results, it still holds for domains with additive preferences.

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