# A characterization of Poissonian domains 

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#### Abstract

We give a characterization of Poissonian domains in $\mathbf{R}^{n}$, i.e., those domains for which every bounded harmonic function is the harmonic extension of some function in $L^{\infty}$ of harmonic measure. We deduce several properties of such domains, including some results of Mountford and Port. In two dimensions we give an additional characterization in terms of the logarithmic capacity of the boundary. We also give a necessary and sufficient condition for the harmonic measures on two disjoint planar domains to be mutually singular.


## 1. Introduction

Suppose $\Omega$ is a domain in $\mathbf{R}^{n}$ for which harmonic measure $\omega$ is defined. Given a (real valued) function $f \in L^{\infty}(d \omega)$ we can define a bounded harmonic function $u$ on $\Omega$ by

$$
u(z)=\int_{\partial \Omega} f(x) d \omega_{z}(x)
$$

If $n>2$ and $\Omega$ is unbounded, $\omega$ may contain a point mass at $\infty$. We say $\Omega$ is Poissonian if every bounded harmonic function on $\Omega$ is of this form. For example, the unit ball is Poissonian, and a non-Poissonian domain in $\mathbf{R}^{2}$ can be constructed by removing the line segment $[0,1]$ from the unit disk, $\mathbf{D}$. There are bounded harmonic functions on this domain which have different boundary values "above" and "below" the slit, and these are not of the form described above. However, using results of [8] it is possible to build a Jordan arc $\Gamma$ connecting 0 and 1 in $\mathbf{D}$ such that $\mathbf{D} \backslash \Gamma$ is Poissonian. Thus more than just topology is important in determining whether a domain is Poissonian.

The purpose of this paper is to give a characterization of Poissonian domains and derive some of their properties. Our characterization is in terms of harmonic measures of subdomains of $\Omega$. Since the harmonic measures corresponding to

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two different points in a domain are mutually absolutely continuous (i.e., they have the same null sets) and our conditions only concern whether a set has zero or positive measure we will refer to "the" harmonic measure $\omega(E, \Omega)$ (or just $\omega(E)$ if the domain is clear from context).

Theorem 1.1. $\Omega \subset \mathbf{R}^{n}$ is Poissonian iff for every pair of disjoint subdomains $\Omega_{1}$ and $\Omega_{2}$ of $\Omega$ with $\partial \Omega_{1} \cap \partial \Omega_{2} \subset \partial \Omega$, the harmonic measures $\omega_{1}$ and $\omega_{2}$ of $\Omega_{1}$ and $\Omega_{2}$ are mutually singular.

By the measures being singular we mean that there exists $E \subset \partial \Omega_{1} \cap \partial \Omega_{2}$ with $\omega_{1}(E)=0$ and $\omega_{2}\left(\left(\partial \Omega_{1} \cap \partial \Omega_{2}\right) \backslash E\right)=0$ and we write this as $\omega_{1} \perp \omega_{2}$. It is not clear how easy this condition is to check in practice, but it is sufficient to prove the following results (the necessary definitions will be given in Section 3):

Corollary 1.2, [23]. Each component of the intersection of two Poissonian domains is Poissonian.

Corollary 1.3, [23]. If $\Omega_{1}$ and $\Omega_{2}$ are Poissonian and $\omega\left(\partial \Omega_{1} \cap \partial \Omega_{2}, \Omega_{1} \cup \Omega_{2}\right)=0$ then $\Omega=\Omega_{1} \cup \Omega_{2}$ is Poissonian.

Corollary 1.4. If $E \subset \mathbf{R}^{n}$ is closed and has zero $n-1$ dimensional measure, then $\Omega=\mathbf{R}^{\boldsymbol{n}} \backslash E$ is Poissonian.

Corollary 1.5. If $E \subset \mathbf{R}^{\boldsymbol{n}}$ is a closed subset of a Lipschitz graph, then $\Omega=\mathbf{R}^{\boldsymbol{M}} \backslash E$ is Poissonian iff $E$ has zero $n-1$ dimensional measure.

Some hypothesis is needed in Corollary 1.3 since it is easy to see that the union of Poissonian domains need not be Poissonian, e.g. $\Omega_{1}=\{0<|z|<1,0<\arg (z)<3 \pi / 2\}$ and $\Omega_{2}=\{0<|z|<1, \pi / 2<\arg (z)<2 \pi\}$ whose union is $\mathbf{D} \backslash[0,1]$. Corollary 1.5 was proven in [23] when $E$ is a subset of a $n-1$ hyperplane and this case also follows from results in [4]. Also see [2]. Mountford and Port [23] also gave a characterization of Poissonian domains in terms of the Martin boundary $\Delta$ of $\Omega$. It says that a domain is Poissonian iff there is a measurable mapping $\varphi: \partial \Omega \rightarrow \Delta$ which takes harmonic measure on $\partial \Omega$ to the harmonic measure $\mu$ on $\Delta$, i.e., iff $(\partial \Omega, \omega$ ) and $(\Delta, \mu)$ are equivalent as measure spaces.

For domains in $\mathbf{R}^{2}$, the characterization in Theorem 1.1 can be restated in terms of a Wiener type condition involving the logarithmic capacity of $\partial \Omega$, as follows. For $x \in \mathbf{R}^{2}, \delta>0, \varepsilon>0$ and $\theta \in[0,2 \pi)$ we define the cone and wedge

$$
\begin{aligned}
& C(x, \delta, \varepsilon, \theta)=\left\{x+r e^{i \psi}: 0<r<\delta,|\psi-\theta|<\varepsilon\right\} \\
& W(x, \delta, \varepsilon, \theta)=C(x, \delta, \varepsilon, \theta) \cap\{z: \delta / 2 \leqq|z-x|\}
\end{aligned}
$$

We also let cap $(E)$ denote the logarithmic capacity of $E$ (to be defined in Section 4).

For a fixed $x, \varepsilon$ and $\theta$ let

$$
\gamma_{i}(k)=\operatorname{cap}\left(2^{k-2}\left(W\left(x, 2^{-k}, \varepsilon,(-1)^{i+1} \theta\right) \backslash \Omega_{i}\right)\right),
$$

i.e., $\gamma_{i}(k)$ is capacity of $\Omega_{i}^{c} \cap W\left(x, 2^{-k}, \varepsilon, \theta\right)$ after we have dilated it to have diameter about $1 / 2$. We say a point $x \in \partial \Omega_{1} \cap \partial \Omega_{2}$ satisfies a weak double cone condition (WDCC) with respect to the pair $\Omega_{1}, \Omega_{2}$ if there exist $\varepsilon$ and $\theta$ such that

$$
\begin{equation*}
\sum_{n=1}^{\infty}\left(\gamma_{1}(n)+\gamma_{2}(n)\right)<\infty . \tag{1.1}
\end{equation*}
$$

If $\Omega_{1}=\Omega_{2}=\Omega$ we simply say $x$ satisfies a WDCC with respect to $\Omega$. We refer to this as a "weak" condition because it generalizes the double cone condition stated in [6] and [8] which requires that

$$
C\left(x, \delta, \varepsilon,(-1)^{i+1} \theta\right) \subset \Omega_{i} .
$$

It is clear that this condition implies the WDCC since all but finitely many of the terms in (1.1) will be zero.

Theorem 1.6. A domain $\Omega \subset \mathbf{R}^{2}$ is Poissonian iff the set of points $x \in \partial \Omega$ which satisfy a weak double cone condition with respect to $\Omega$ has zero 1 dimensional measure.

The proof will also show:
Theorem 1.7. Suppose $\Omega_{1}$ and $\Omega_{2}$ are disjoint subdomains in $\mathbf{R}^{2}$ and let $\omega_{1}$ and $\omega_{2}$ be their harmonic measures. Then $\omega_{1} \perp \omega_{2}$ iff the set of points in $\partial \Omega_{1} \cap \partial \Omega_{2}$ satisfying a weak double cone condition with respect to $\Omega_{1}, \Omega_{2}$ has zero 1 dimensional measure, $\Lambda_{1}$. Moreover, if $\omega_{1}$ and $\omega_{2}$ are mutually absolutely continuous on a set $E$ then there is Besicovitch regular $F \subset E$ with $\omega_{i}(F)=\omega_{i}(E)$ and $\omega_{i}$ mutually absolutely continuous with $\Lambda_{1}$ on $F$ for $i=1,2$.

In the case when the domains are simply connected, these results follow easily from the results of $[8]$ and the fact that the capacity of a connected set can be estimated in terms of its diameter. We can also characterize the disjoint planar domains for which the two harmonic measures are mutually absolutely continuous. This happens iff $\Omega_{i}=\tilde{\Omega}_{i} \backslash E_{i}$ for $i=1,2$ where $\operatorname{cap}\left(E_{i}\right)=0$ and $\tilde{\Omega}_{1}$ and $\tilde{\Omega}_{2}$ are disjoint simply connected domains with mutually absolutely continuous harmonic measures $\tilde{\omega}_{1}$ and $\tilde{\omega}_{2}$. Such domains are characterized in [8] (also see [6]): $\tilde{\omega}_{1} \ll \tilde{\omega}_{2} \ll \tilde{\omega}_{1}$ iff for every $\varepsilon>0$ there are subdomains $D_{i} \subset \widetilde{\Omega}_{i}$ with rectifiable boundaries $\Gamma_{i}$ such that $\tilde{\omega}_{i}\left(\Gamma_{1} \cap \Gamma_{2}\right) \geqq 1-\varepsilon$ for $i=1,2$.

Theorem 1.7 implies that if $\Omega$ is not Poissonian then $\partial \Omega$ contains a Besicovitch regular set of positive length. Thus we obtain

Corollary 1.8. If $E \subset \mathbf{R}^{2}$ is a closed, Besicovitch irregular set, then $\Omega=\mathbf{R}^{2} \backslash E$ is Poissonian.

In the next section we shall prove Theorem 1.1. In Section 3 we will prove its corollaries and in Section 4 we prove Theorems 1.6 and 1.7 and Corollary 1.8. In Section 5 we prove a lemma used in Section 4 and we conclude in Section 6 with some remarks concerning the Martin boundary and possible generalizations of Theorem 1.7 to higher dimensions.

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## 2. Proof of Theorem 1.1

Before starting the proof we recall a few basic facts of potential theory (see [10], [12]). By the Newtonian kernel on $\mathbf{R}^{n}$ we mean

$$
K(|x|)= \begin{cases}\log \frac{1}{|x|}, & n=2 \\ |x|^{2-n}, & n>2\end{cases}
$$

and given a positive measure $\mu$ on $\mathbf{R}^{\boldsymbol{n}}$ we define its potential by
and its energy by

$$
u_{\mu}(x)=\int K(|x-y|) d \mu(y)
$$

$$
I(\mu)=\iint K(|x-y|) d \mu(x) d \mu(y)
$$

For a set $E \subset \mathbf{R}^{n}$, we let $\operatorname{Pr}(E)$ denote the set of probability measures on $E$. We define the capacity of $E$ as

$$
\operatorname{cap}(E)=\left(\inf _{\mu \in \operatorname{Pr}(E)} I(\mu)\right)^{-1}
$$

There exists a unique probability measure $\bar{\mu}$, called the equilibrium measure, for which the inf is attained. Moreover, the potential of this measure satisfies

$$
u_{\bar{\mu}}(x)=I(\bar{\mu})
$$

for every $x \in E$ except possibly a subset of capacity zero. In 2 dimensions the kernel is not positive, so a large $E$ may have infinite or negative capacity, and for this reason the capacity is sometimes defined as $\exp \left(-\operatorname{cap}(E)^{-1}\right)$ to make it positive and monotonic. However, we will only consider sets of diameter less that 1 , so our capacities will always be positive.

If $\Omega$ is a domain with $\operatorname{cap}(\partial \Omega)>0$ then $\Omega$ has a Green's function and the harmonic measure for $\Omega$ exists. A set $E$ with $\operatorname{cap}(E)=0$ is called polar. A property which holds everywhere of $\partial \Omega$ except possibly on a polar set is said to hold p.p.
("presque partout") on $\partial \Omega$. For example, the Green's function of $\Omega$ always tends to 0 p.p. on $\partial \Omega$. The points of $\partial \Omega$ where it does so are called regular for the Dirichlet problem on $\Omega$. If $E \subset \partial \Omega$ is polar then $E$ has zero harmonic measure in $\Omega$. If $f$ is continuous on $\partial \Omega$ and we extend it to a harmonic function $u$ on $\Omega$ via the Perron process such that $u$ extends to be continuous and agree with $f$ p.p. on $\partial \Omega$. The mapping $f \rightarrow u(z)$ turns out to be a continuous linear functional on $C(\partial \Omega)$ so by the Riesz representation theorem there is a probability measure $\omega_{z}$ on $\partial \Omega$ such that $u(z)=\int f d \omega_{z}$. This is the harmonic measure for $\Omega$ with respect to $z$. For a fixed $E \subset \partial \Omega, \omega_{z}(E)$ is a nonnegative, harmonic function in $z$, so is either 0 for all $z$ or for none if $\Omega$ is connected. Thus the harmonic measures for different points of $\Omega$ are mutually absolutely continuous, as mentioned before.

If $F \subset \partial \Omega$ is closed then the harmonic function $u(z)=\omega_{z}(F)$ tends to zero p.p. on $\partial \Omega \backslash F$. Furthermore, the maximum principle states that if $u$ is a bounded subharmonic function on $\Omega$ such that

$$
\limsup _{z \rightarrow x, z \in \Omega} u(z) \leqq M
$$

for p.p. $x \in \partial \Omega$ then $u \leqq M$ on $\Omega$. I shall also use the phrase "maximum principle" to refer to the following fact: if $\tilde{\Omega} \subset \Omega$ and $E \subset \partial \widetilde{\Omega} \cap \partial \Omega$ then $\omega(E, \widetilde{\Omega}) \leqq \omega(E, \Omega)$.

Now we begin the proof of Theorem 1.1. First we will show the stated condition implies $\Omega$ is Poissonian. Let $\mathrm{HB}(\Omega)$ denote the Banach space of bounded harmonic functions on $\Omega$ with the "sup" norm. Then there is a continuous linear mapping $P: L^{\infty}(\omega) \rightarrow \mathrm{HB}(\Omega)$ given by

$$
P(f)(z)=\int f(x) d \omega_{z}
$$

In order to show $\Omega$ is Poissonian we want to show this mapping is onto. It suffices to show that for any $u \in \mathrm{HB}(\Omega)$ with $\|u\|_{\infty} \leqq 1$ there exists an $f \in L^{\infty}(\omega)$ with $\|f\|_{\infty} \leqq 1$ and such that $\|u-P(f)\|_{\infty} \leqq 3 / 4$. Since $P$ is bounded and linear a standard successive approximation argument gives a $g$ with $P(g)=u$.

So fix a $u \in \mathrm{HB}(\Omega)$ with $\|u\|_{\infty} \leqq 1$. We may assume that $u(\Omega)=(-1,1)$. Let

$$
\begin{aligned}
& \Omega_{1}=\left\{z \in \Omega: u(z)>\frac{1}{3}\right\} \\
& \Omega_{2}=\left\{z \in \Omega: u(z)<-\frac{1}{3}\right\} .
\end{aligned}
$$

Note that $\partial \Omega_{1} \cap \partial \Omega_{2}$ must lie in $\partial \Omega$ and that $\Omega_{1}$ and $\Omega_{2}$ may have countable many components. By hypothesis the harmonic measure for any component of $\Omega_{1}$ is singular to harmonic measure of any component of $\Omega_{2}$. Therefore, given compact subsets $K_{i}$ of $\Omega_{i}(i=1,2)$ and $\varepsilon>0$ we can find disjoint closed sets $E_{i} \subset \partial \Omega_{1} \cap$
$\partial \Omega_{2} \subset \partial \Omega$ such that

$$
\omega\left(z, E_{i}, \Omega_{i}\right) \geqq \omega\left(z, \partial \Omega \cap \partial \Omega_{i}, \Omega_{i}\right)-\varepsilon
$$

for every $z \in K_{i}$ and $i=1,2$.
Now define a function $g$ by setting $g(z)=1 / 3$ for $z \in E_{1}, g(z)=-1 / 3$ on $E_{2}$ and extending $g$ to be continuous on all of $\mathbf{R}^{n}$ and satisfying $-1 / 3 \leqq g \leqq 1 / 3$. Now restrict $g$ to $\partial \Omega$ and let $v=P(g)$. By the maximum principle $-1 / 3 \leqq v \leqq 1 / 3$ so for $z \in \Omega \backslash\left(\Omega_{1} \cup \Omega_{2}\right),|u(z)-v(z)| \leqq 2 / 3$. For $z \in \partial \Omega_{1} \cap \Omega$ we have $u(z)=1 / 3$, so $|u(z)-v(z)| \leqq 2 / 3$. Also, for p.p. $x \in E_{1} v(z) \rightarrow 1 / 3$ as $z \rightarrow x$ in $\Omega_{1}$. Thus for p.p. $x \in E_{1}$

$$
\limsup _{z \rightarrow x, z \in \Omega_{1}}(u(z)-v(z)) \leqq 1-1 / 3=2 / 3
$$

and for every $x \in\left(\partial \Omega_{1} \cap \partial \Omega\right) \backslash E_{1}$,

$$
\limsup _{z \rightarrow x, z \in \Omega_{1}} u(z)-v(z) \leqq 2
$$

From this and the maximum principle we obtain

$$
|u(z)-v(z)| \leqq 2 / 3+2 \omega\left(z, \partial \Omega \backslash E_{1}, \Omega_{1}\right) \leqq 2 / 3+2 \varepsilon, \quad z \in K_{1}
$$

Similarly for $\Omega_{2}$. Thus if we take an exhaustion of $\Omega_{1}$ and $\Omega_{2}$ by compact sets and a sequence of $\varepsilon$ 's tending to zero, we obtain a sequence $\left\{g_{n}\right\} \in L^{\infty}(\omega)$ with $\left|P\left(g_{n}\right)(z)-u(z)\right| \leqq 3 / 4$ for every $z \in \Omega$ and $n=n(z)$ large enough. By passing to a subsequence we may assume the $\left\{g_{n}\right\}$ converge weakly in $L^{\infty}(\omega)$ and the limit $g$ clearly satisfies $\|u-P(g)\|_{\infty} \leqq 3 / 4$. Thus $\Omega$ is Poissonian.

Next we prove the converse. We will show that if the condition in Theorem 1.1 fails then $\Omega$ is not Poissonian. So suppose $\Omega_{1}$ and $\Omega_{2}$ are subdomains of $\Omega$, that $\partial \Omega_{1} \cap \partial \Omega_{2} \subset \partial \Omega$ and that $\omega_{1}$ and $\omega_{2}$ are not singular. Then there is a set $E \subset \partial \Omega_{1} \cap \partial \Omega_{2}$ such that $\omega_{1}(E)>0, \omega_{2}(E)>0$ and $\omega_{1}$ and $\omega_{2}$ are mutually absolutely continuous on $E$.

Let $\mathscr{F}$ be the family of subharmonic functions $v$ on $\Omega$ which satisfy

$$
v(z) \leqq w(z)= \begin{cases}1-2 \omega\left(z, E, \Omega_{2}\right), & z \in \Omega_{2} \\ 1, & z \in \Omega \backslash \Omega_{2}\end{cases}
$$

Note $w$ is a superharmonic function on $\Omega$, and so $\mathscr{F}$ is a Perron family (see [1, page 248]). Thus

$$
-1 \leqq u \equiv \sup _{\mathscr{F}} v \leqq w
$$

exists and is harmonic. We claim that $u$ cannot be uniformly approximated on $\Omega$ by functions of the form $P(f), f \in L^{\infty}(\omega)$.

First note that if we define $v$ on $\Omega$ by

$$
v(z)= \begin{cases}2 \omega\left(z, E, \Omega_{1}\right)-1, & z \in \Omega_{1} \\ -1, & z \in \Omega \backslash \Omega_{1}\end{cases}
$$

then $v$ is subharmonic on $\Omega$ and is in $\mathscr{F}$. Thus $v \leqq u$. Let $D_{i} \subset \Omega_{i}$ be defined by

$$
D_{i}=\left\{z \in \Omega_{i}: \omega\left(z, E, \Omega_{i}\right)>3 / 4\right\}
$$

for $i=1,2$. Note that $u \geqq 1 / 2$ on $D_{1}$ and $u \leqq-1 / 2$ on $D_{2}$.
Now suppose there exists $f \in L^{\infty}(\omega)$ such that $\|u-P(f)\|_{\infty}<1 / 4$. Then $P(f) \geqq$ $1 / 4$ on $D_{1}$. Therefore $f \geqq 1 / 4$ a.e. $\left(\omega_{1}\right)$ on $E$, for if $F=\{f<1 / 4-\varepsilon\}$ and $\omega_{1}(F)>0$ for some $\varepsilon>0$, there would be a sequence $\left\{z_{n}\right\} \subset D_{1}$ with $\omega\left(z_{n}, F, \Omega_{1}\right) \rightarrow 1$ and hence $P(f)\left(z_{n}\right)<1 / 4$ for some $n$, a contradiction.

The same argument shows that $P(f) \leqq-1 / 4$ on $D_{2}$ and therefore $f \leqq-1 / 4$ a.e. $\left(\omega_{2}\right)$ on $E$. But $f \geqq 1 / 4$ a.e. $\omega_{1}$ and $\omega_{1}$ and $\omega_{2}$ are mutually absolutely continuous on $E$ ! This is a contradiction, and so no such $f$ can exist. Hence $\Omega$ is not Poissonian.

## 3. Proof of the corollaries

Before proving the corollaries it is convenient to record the following simple results.

Lemma 3.1. Suppose $\Omega \subset \mathbf{R}^{n}$ and $E \subset \partial \Omega$ has positive harmonic measure. Fix $0<a<1$ and set $\Omega=\{z \in \Omega: \omega(z, E, \Omega)>a\}$. Then $F \subset E$ has positive harmonic measure in $\Omega$ iff it has positive harmonic measure in some component of $\widetilde{\Omega}$.

Lemma 3.2. Suppose $\Omega_{1}$ and $\Omega_{2}$ are disjoint domains with mutually continuous harmonic measures $\omega_{1}$ and $\omega_{2}$ on a compact subset $E \subset \partial \Omega_{1} \cap \partial \Omega_{2}$ of positive measure. Then there exist $\widetilde{\Omega}_{i} \subset \Omega_{i}$ for $i=1,2$ with mutually continuous harmonic measures $\tilde{\omega}_{i}$ on a subset $F \subset E$ of positive $\tilde{\omega}_{i}$ measure and $\partial \widetilde{\Omega}_{1} \cap \partial \widetilde{\Omega}_{2} \subset E$. Moreover, given any open neighborhood $U$ of $E$, we may take $\widetilde{\Omega}_{i} \subset U$ for $i=1,2$.

To prove the first lemma, let $F \subset E$ be compact and note by the maximum principle that for each component $\Omega_{j}$ of $\tilde{\Omega}, \omega\left(F, \Omega_{j}\right) \leqq \omega(F, \Omega)$, so $\omega(F)=0$ implies $\omega_{j}(F)=0$ for every $j$. On the other hand if $\omega(F)>0$ then there exists $z_{0} \in \widetilde{\Omega}$ with $\omega\left(z_{0}, F, \Omega\right)>(1+a) / 2$. Suppose $z_{0}$ is in a component $\Omega_{j}$ of $\tilde{\Omega}$. First note that $\omega(z, F, \Omega)$ is harmonic on $\Omega_{j}$ but less than $a$ on $\partial \Omega_{j} \cap \Omega$. Also observe that cap $\left.\left(\partial \Omega_{j} \cap \partial \Omega\right) \backslash F\right)=0$ (this holds since $\omega(z, F, \Omega) \rightarrow 0$ p.p. on $\left.\partial \Omega \backslash F\right)$. Thus $\omega\left(z_{0}, F, \Omega_{j}\right)>0$. This proves Lemma 3.1.

To prove Lemma 3.2 let $\widetilde{\Omega}_{i}$ be components of $\left\{z \in \Omega_{i}: \omega\left(z, E, \Omega_{i}\right)>1 / 2\right\}$ for $i=1,2$. By Lemma 3.1 they can be chosen so they have mutually absolutely continuous harmonic measures on some subset $F \subset E$ of positive measure. Now suppose $x \in \partial \widetilde{\Omega}_{1} \cap \partial \widetilde{\Omega}_{2}$ but not in $E$. Then $x \in \partial \Omega_{1} \cap \partial \Omega_{2}$. After the proof of Lemma 3.3 we shall observe that a point of the common boundary to two disjoint domains must be regular for the Dirichlet problem for at least one of the domains. So as-
sume $x$ is regular for $\Omega_{1}$. Then since $x \notin E, \omega\left(z, E, \Omega_{1}\right) \rightarrow 0$ as $z \rightarrow x$ in $\Omega_{1}$. Since $\omega\left(z, E, \Omega_{1}\right)>1 / 2$ for $z \in \tilde{\Omega}_{1}, x \notin \tilde{\Omega}_{1}$, a contradiction. Thus $x \in E$, as required.

To prove the last claim we take $\tilde{\Omega}_{i}$ to be components of

$$
\left\{z \in \Omega_{i}: \omega\left(z, E, \Omega_{1}\right)>1-\varepsilon\right\}
$$

for small $\varepsilon$. To show $\widetilde{\Omega}_{i} \subset U$ if $\varepsilon$ is small enough observe

$$
\omega\left(z, E, \Omega_{1}\right) \leqq \frac{u_{\bar{\mu}}(z)}{I(\bar{\mu})}
$$

where $\bar{\mu}$ is the equilibrium measure for $E$. This inequality holds by the maximum principle since both sides are 1 p.p. on $E$ and the the right-hand side is positive on $\partial \Omega_{1} \backslash E$ while the left-hand side is 0 p.p. on $\partial \Omega_{1} \backslash E$. The right-hand side is strictly less than 1 off $E$ and so is $\leqq 1-\varepsilon$ on the complement of $U$. This argument needs to be slightly modified in 2 dimensions because the potential is not positive. However in this case we may assume that $\Omega_{1}$ is bounded (use a Möbius transformation to put $\left.\infty \in \Omega_{2}\right)$ and then compare $\omega\left(z, E, \Omega_{1}\right)$ to $1-\eta\left(1-u_{\bar{\mu}}(z) / I(\bar{\mu})\right)$ which is positive on $\Omega_{1}$ if $\eta$ is small enough. This proves the second lemma.

To prove Corollary 1.2 suppose $\Omega$ is a component of $\Omega_{1} \cap \Omega_{2}$ and that $\Omega$ is not Poissonian. Then there exist disjoint domains $D_{1}, D_{2} \subset \Omega$ whose harmonic measures $\omega_{1}$ and $\omega_{2}$ are not singular. Thus there is a subset $E \subset \partial D_{1} \cap \partial D_{2}$ on which $\omega_{1}$ and $\omega_{2}$ are mutually absolutely continuous and which has positive measure. Since $\partial \Omega \subset \partial \Omega_{1} \cup \partial \Omega_{2}$ either $E_{1}=E \cap \partial \Omega_{1}$ or $E_{2}=E \cap \partial \Omega_{2}$ must have positive measure with respect to $\omega_{1}$. Without loss of generality suppose it is $E_{1}$. Then $D_{1}$, $D_{2}$ are disjoint subdomains of $\Omega_{1}$ whose harmonic measures (restricted to $E_{1}$ ) are not singular. We may not have $\partial D_{1} \cap \partial D_{2} \subset \partial \Omega_{1}$, but we can fix this by using Lemma 3.2. Thus $\Omega_{1}$ is not Poissonian.

To prove Corollary 1.3 suppose $\Omega_{1}$ and $\Omega_{2}$ are domains, and define $\Omega=\Omega_{1} \cup \Omega_{2}$. Suppose also that $\omega\left(\partial \Omega_{1} \cap \partial \Omega_{2}, \Omega\right)=0$, but that $\Omega$ is not Poissonian. Then there exist disjoint subdomains $D_{1}$ and $D_{2}$ and a set $E \subset \partial D_{1} \cap \partial D_{2}$ as above. Since $\omega_{1}(E)>0, \omega(E, \Omega)>0$ by the maximum principle. Since $\omega\left(\partial \Omega_{1} \cap \partial \Omega_{2}, \Omega\right)=0$ either $E_{1}=E \cap \partial \Omega_{1} \backslash \partial \Omega_{2}$ or $E_{2}=E \cap \partial \Omega_{2} \backslash \partial \Omega_{1}$ must have positive harmonic measure in $\Omega$. Assume it is $E_{1}$. We may also assume $E_{1}$ is compact and so does not meet $\bar{\Omega}_{2}$. Therefore there is an open neighborhood $U$ of $E_{1}$ which also does not meet $\Omega_{2}$. By the second part of Lemma 3.2 we can find subdomains of $U \cap \Omega_{1}$ which have mutually continuous harmonic measures on a set of positive measure. This proves Corollary 1.3.

Before proving Corollary 1.4 we we recall the definition of Hausdorff measure. For a set $E \subset \mathbf{R}^{n}$ we let

$$
\Lambda_{s}(E)=\lim _{\delta \rightarrow 0}\left(\inf \left\{\sum\left(r_{j}\right)^{s}: E \subset \cup B\left(x_{j}, r_{j}\right), r_{j} \leqq \delta\right\}\right)
$$

Here $B(x, r)$ denotes a solid ball of radius $r$ and center $x$. We call $\Lambda_{s}$ the $s$ dimensional Hausdorff measure. See [10] or [13] for further details. We also need the following estimate on harmonic measure.

Lemma 3.3. Suppose $\Omega_{1}$ and $\Omega_{2}$ are disjoint domains in $\mathbf{R}^{n}$. Fix points $z_{i} \in \Omega_{i}$ for $i=1,2$. There is a $C>0$ so that for $x \in \partial \Omega_{1} \cap \partial \Omega_{2}$ and

$$
\begin{gathered}
r<\min \left(\operatorname{dist}\left(z_{1}, \partial \Omega_{1}\right), \operatorname{dist}\left(z_{2}, \partial \Omega_{2}\right)\right), \\
\omega\left(z_{1}, B(x, r) \cap \partial \Omega_{1}, \Omega_{1}\right) \omega\left(z_{2}, B(x, r) \cap \partial \Omega_{2}, \Omega_{2}\right) \leqq C r^{2(n-1)} .
\end{gathered}
$$

The constant $C$ depends only on dist $\left(z_{1}, \partial \Omega_{1}\right)$ and $\operatorname{dist}\left(z_{2}, \partial \Omega_{2}\right)$.
First normalize so that $\operatorname{dist}\left(z_{i}, \partial \Omega_{i}\right) \geqq 1$ for $i=1,2$. For domains in $\mathbf{R}^{2}$ this result is proven in [8]. For higher dimensions it follows from estimates of Huber [17] and Friedland and Hayman [14]. Suppose $u$ is positive and subharmonic on $\mathbf{R}^{n}$ and vanishes on $\partial \Omega$, and for $r>0$ let $S(x, r)=\partial B(x, r)$ and define

$$
m_{r}(u)=\left(\int_{S(x, r)} u^{2} d \sigma\right)^{1 / 2}
$$

where $\sigma$ is surface measure on the sphere normalized to have mass 1 . Then we have

$$
\begin{equation*}
m_{r}(u) \leqq C m_{1}(u) \exp \left(-\int_{r}^{1 / 2} \alpha(t) \frac{d t}{t}\right) \tag{3.1}
\end{equation*}
$$

where $\alpha(t)$ is the characteristic constant of the $n-1$ dimensional set $\Omega(t)$ which is the radial projection of $\Omega \cap S(x, t)$ onto the unit sphere. This can be defined by $\alpha(\alpha+n-2)=\lambda$ where

$$
\lambda(\Omega(t))=\inf \frac{\int\left|\nabla_{S} f\right|^{2} d \sigma}{\int|f|^{2} d \sigma}
$$

where the "inf" is over all Lipschitz, nonnegative functions vanishing off $\Omega(t)$ and $\nabla_{S} f$ denotes the spherical gradient of $f$. The constant $-\lambda$ is also the first eigenvalue for the Dirichlet problem with vanishing boundary conditions, at least if $\Omega(t)$ is smooth enough. If $f$ is the eigenfunction corresponding to $\lambda$ then $u(x)=|x|^{\alpha} f(x| | x \mid)$ is harmonic in the cone defined by $\Omega(t)$ iff $\alpha(\alpha+n-2)=\lambda$. This is because for a homogeneous function $u$ the spherical Laplacian is given by $\Delta_{S}=-u_{r r}-(n-1) u_{r}$ on the unit sphere. Thus $\Delta_{S} f=(-\alpha(\alpha-1)-(n-1) \alpha) f=-\alpha(\alpha+n-1) f$. See [14] for details.

To deduce our estimate from this result, let $\tilde{\Omega}=(\Omega \backslash \overline{B(x, r)}) \cup\{1 / 2<|z-x|<2\}$ and let $v(z)=\omega(z, S(x, r), \widetilde{\Omega})$ for $z \in \widetilde{\Omega}$ and $v(z)=0$ elsewhere. Then $v$ is subharmonic on $\mathbf{R}^{r} \backslash B(x, r)$ and an easy application of the maximum principle shows $\omega(z, B(x, r), \Omega) \leqq v(z)$ for $z \in \Omega$. Without loss of generality, suppose $x=0$ and define $u(z)=v\left(z /|z|^{2}\right)|z|^{2-d}$. Since $u$ is obtained from $v$ by a Kelvin transforma-
tion (reflection across a sphere) $u$ is harmonic at $z$ iff $v$ is harmonic at $z^{*}=z /|z|^{2}$. Therefore $u$ is positive and subharmonic on $B(0, R)(R=1 / r)$ and is zero on $\partial \Omega^{*}$ and equals $R^{2-d}$ on $S(0, R) \cap \Omega^{*}$. Applying the result from [14] gives

$$
m_{1}(u) \leqq C m_{R}(u) \exp \left(-\int_{1}^{R / 2} \alpha(1 / t) \frac{d t}{t}\right)
$$

Note that $m_{1}(u)=m_{1}(v), m_{R}(u) \leqq R^{2-d}$ and by Harnack's inequality

$$
m_{1}(v) \sim \min _{S(0,1)} v \sim \max _{S(0,1)} v
$$

Thus for $|z|>1$ the maximum principle gives

$$
\begin{equation*}
\omega(z, B(0, r) \cap \partial \Omega, \Omega) \leqq \max _{S(0,1)} v \leqq C r^{d-2} \exp \left(-\int_{2 r}^{1} \alpha(t) \frac{d t}{t}\right) \tag{3.2}
\end{equation*}
$$

To obtain Lemma 3.3, we use Theorem 3 of [14] which states that $\alpha_{i}(t) \geqq$ $2\left(1-S_{i}(t)\right)$ where $S_{i}(t)$ is the $(n-1)$ dimensional surface area of $\Omega_{i}(t)$ (normalized so the whole sphere has area 1). This is proven using a result of Sperner [25] that among all domains on the sphere with equal area, the spherical cap is the one with smallest characteristic constant and then estimating $\alpha$ for a spherical cap. Since $\Omega_{1}(t)$ and $\Omega_{2}(t)$ are disjoint we have $S_{1}(t)+S_{2}(t) \leqq 1$, and hence $\alpha_{1}(t)+\alpha_{2}(t) \geqq 2$. Multiplying the two estimates in (3.2) and using $\exp \left(-\int_{r}^{1} d t / t\right)=r$ proves Lemma 3.3.

If we apply the arguments of the above paragraphs to the Green's functions $G_{1}$ and $G_{2}$ of two disjoint domains we see that

$$
\left(\max _{B(x, r)} G_{1}(z)\right)\left(\max _{B(x, r)} G_{2}(z)\right) \leqq C r^{2(n-1)}
$$

This implies that at least one of the two functions tends to 0 at $x$ and hence $x$ is regular for the Dirichlet problem on at least one of the domains. This is a fact which we used earlier. A similar argument shows that if $\Omega_{1}$ and $\Omega_{2}$ are disjoint, unbounded domains, their harmonic measures cannot both have point masses at infinity. Therefore the point at infinity is not important in deciding whether a domain is Poissonian or not.

To prove Corollary 1.4 suppose $E$ has zero ( $n-1$ ) dimensional measure and let $\Omega=\mathbf{R}^{\boldsymbol{n}} \backslash E$. Suppose $\Omega_{1}$ and $\Omega_{2}$ are disjoint subdomains with $\partial \Omega_{1} \cap \partial \Omega_{2} \subset E$. Fix $\varepsilon$ small and let $\mathscr{D}=\left\{B_{j}\right\}=\left\{B\left(x_{j}, r_{j}\right)\right\}$ be a covering of $E$ with $\sum r_{j}^{n-1}<\varepsilon$. If $\omega_{1}(E)=0$ we are done so suppose $\omega_{1}(E)>0$. Let $\mathscr{C} \subset \mathscr{D}$ be the subcollection of balls such that $\omega_{1}\left(B_{j}\right) \geqq r_{j}^{n-1}$. By Lemma $3.3 \omega_{2}\left(B_{j}\right) \leqq C r^{n-1}$ for $B_{j} \in \mathscr{C}$. Thus $\mathscr{C}$ covers a subset $F$ of $E$ with $\omega_{1}(F) \geqq \omega_{1}(E)-\varepsilon$ and

$$
\omega_{2}(F) \leqq \sum_{c} \omega_{2}\left(B_{j}\right) \leqq C \sum_{c} r_{j}^{n-1} \leqq C \varepsilon
$$

Taking $\varepsilon \rightarrow 0$ we see that $\omega_{1} \perp \omega_{2}$. Thus $\Omega$ is Poissonian.

Recall that a real valued function $A$ on $\mathbf{R}^{\boldsymbol{n}}$ is called Lipschitz if there is a constant $C>0$ such that $|A(x)-A(y)| \leqq C|x-y|$ for all $x, y \in \mathbf{R}^{n}$. A Lipschitz graph in $\mathbf{R}^{n}$ is a set of the form $\left\{(x, A(x)): x \in \mathbf{R}^{n-1}\right\}$ where $A$ is a Lipschitz function on $\mathbf{R}^{n-1}$.

If $\Lambda_{n-1}(E)=0$ then $\Omega$ is Poissonian by Corollary 1.4. To prove the other direction of Corollary 1.5, suppose $E=\{(x, A(x)): x \in F\}$ and let $\Omega_{1}$ be the region above the graph of the Lipschitz function $A(x)$ and $\Omega_{2}$ the region below it. Then the harmonic measures are mutually absolutely continuous because of Dahlberg's theorem that harmonic measure on a Lipschitz domain and ( $n-1$ ) dimensional measure are mutually absolutely continuous ([11], [18]). Using Lemma 3.2 we may assume $\partial \Omega_{1} \cap \partial \Omega_{2} \subset E$ and so $\Omega$ is not Poissonian.

## 4. Poissonian domains in $\mathbf{R}^{\mathbf{2}}$

We start by reviewing some related material from [6] and [8]. Suppose $\Omega_{1}$ and $\Omega_{2}$ are domains and that $x \in \partial \Omega_{1} \cap \partial \Omega_{2}$. We say $x$ satisfies a double cone condition with respect to the pair $\Omega_{1}, \Omega_{2}$ if there exists $\delta, \varepsilon>0$ and $\theta \in[0,2 \pi)$ such that

$$
C(x, \delta, \varepsilon, \theta)=\left\{x+r e^{i \psi}: 0<r<\delta,|\theta-\psi|<\varepsilon\right\} \subset \Omega_{1}
$$

and $C(x, \delta, \varepsilon,-\theta) \subset \Omega_{2}$. The point $x$ is called a tangent point if there is a fixed $\theta$ for which can take $\varepsilon$ as close to $\pi$ as we wish (if $\delta$ is small enough). Up to a set of $\Lambda_{1}$ measure zero, the set of tangent points and the set of points satisfying the DCC are the same. From [8] we have

Lemma 4.1. If $\Omega_{1}$ and $\Omega_{2}$ are simply connected domains with harmonic measures $\omega_{1}$ and $\omega_{2}$ then $\omega_{1} \perp \omega_{2}$ iff the set of points satisfying a DCC with respect to $\Omega_{1}, \Omega_{2}$ has zero $\Lambda_{1}$ measure.

Simply connected Poissonian domains in $\mathbf{R}^{2}$ were considered by Glicksberg in [15] in connection with certain function algebras. He called them "nicely connected" and defined them as those domains for which the Riemann mapping from the unit disk to $\Omega$ is $1-1$ on a full measure subset of $\mathbf{T}$. For further details see [7] and its references.

It follows from the lemma that if $\omega_{1}$ and $\omega_{2}$ are not mutually singular then $\partial \Omega_{1} \cap \partial \Omega_{2}$ "looks like" a Lipschitz graph. More precisely, if $\omega_{1}$ and $\omega_{2}$ are not singular then for any $\varepsilon>0$ we can find real valued functions $f_{1}$ and $f_{2}$ on $[-1,1]$ such that
(1) $f_{i}$ are Lipschitz with constant $\varepsilon, i=1,2$.
(2) $f_{1}=f_{2}$ except on a set of length $\leqq \varepsilon$.

Moreover, if

$$
D_{1}=\left\{z:|z|<1, \operatorname{Im}(z)>f_{1}(\operatorname{Re}(z))\right\}
$$

and

$$
D_{2}=\left\{z:|z|<1, \operatorname{Im}(z)<f_{2}(\operatorname{Re}(z))\right\}
$$

then after translating, rotating and dilating $\Omega_{1} \cup \Omega_{2}$ we have $D_{i} \subset \Omega_{i}$ for $i=1,2$. The subarc of $\partial D_{i}$ corresponding to the graph of $f_{i}$ will be denoted $\Gamma_{i}$. See [6] for details.

Note in particular that if $\omega_{1}$ and $\omega_{2}$ are not mutually singular then $\partial \Omega_{1} \cap \partial \Omega_{2}$ contains a positive length subset of a Lipschitz graph.

Given a set $E \subset \mathbf{R}^{2}$ with $0<\Lambda_{1}(E)<\infty$ we call a point $x \in E$ Besicovitch regular if

$$
\lim _{r \rightarrow 0} \frac{\Lambda_{1}(E \cap B(x, r))}{r}=1
$$

and irregular otherwise. The set $E$ is called Besicovitch regular if a.e. $\left(\Lambda_{1}\right)$ point of $E$ is regular. $E$ is called irregular if a.e. point is irregular. One can show that a set is regular iff it consists of a subset of zero $\Lambda_{1}$ measure plus a subset of a countable union of rectifiable curves. Conversely, the intersection of an irregular set with any rectifiable curve has $\Lambda_{1}$ measure zero. Furthermore, any set $E$ with $\Lambda_{1}(E)<\infty$ can be divided into two sets, one of which is regular and the other irregular. For the proofs of these facts and further details see [13].

We now prove Corollary 1.8. Suppose $\Omega_{1}$ and $\Omega_{2}$ are two general domains whose harmonic measures $\omega_{1}$ and $\omega_{2}$ are not singular. Since $\Omega_{2}$ is connected it is contained in exactly one of $\Omega_{1}$ 's complementary components. Let $\tilde{\Omega}_{1}$ be the simply connected domain containing $\Omega_{1}$ obtained by removing all the complementary components of $\Omega_{1}$ except the one containing $\Omega_{2}$. Similarly we define $\widetilde{\Omega}_{2}$ by removing all of $\Omega_{2}$ 's complementary components except the one containing $\Omega_{1}$. (More precisely, let $F_{1}$ be the component of $\partial \Omega_{1}$ separating $\Omega_{1}$ from $\Omega_{2}$. Define $F_{2}$ similarly. Then $F_{1} \cap F_{2}=\partial \Omega_{1} \cap \partial \Omega_{2} \neq \emptyset$, so $F_{1} \cup F_{2}$ is connected. Thus the components $\widetilde{\Omega}_{1}$ and $\widetilde{\Omega}_{2}$ of $\left(F_{1} \cup F_{2}\right)^{c}$ containing $\Omega_{1}$ and $\Omega_{2}$ are disjoint, simply connected domains and $\partial \tilde{\Omega}_{1} \cap$ $\partial \widetilde{\Omega}_{2}=F_{1} \cap F_{2}$.) The harmonic measures $\tilde{\omega}_{1}$ and $\tilde{\omega}_{2}$ for the new domains cannot be mutually singular (if they were, then so would $\omega_{1}$ and $\omega_{2}$ by the maximum principle). Therefore $\partial \widetilde{\Omega}_{1} \cap \partial \widetilde{\Omega}_{2}$ hits a Lipschitz graph in positive length. Thus $\partial \Omega_{2} \cap \partial \Omega_{2}$ contains a Besicovitch regular set of positive length. This proves Corollary 1.8.

Let $K(x)=\log \frac{1}{|x|}$. Recall from Section 2 that

$$
\begin{gathered}
I(\mu)=\int K(|x-y|) d \mu(y) d \mu(x) \\
\operatorname{cap}(E)=\left(\inf _{\operatorname{Pr}(E)} I(\mu)\right)^{-1}
\end{gathered}
$$

and that there exists a unique probability measure $\bar{\mu}$ which minimizes the energy
integral. It also has the property that for p.p. $z \in E$

$$
u_{\bar{\mu}}(z)=\int \log |z-w|^{-1} d \bar{\mu}(w)=I(\bar{\mu})=\operatorname{cap}(E)^{-1}
$$

If $\Omega$ is a domain a theorem of Wiener [27] says that $x \in \partial \Omega$ is regular for the Dirichlet problem on $\Omega$ iff

$$
\sum_{n=1}^{\infty} n \cdot \operatorname{cap}\left(\partial \Omega \cap\left\{2^{-n} \leqq|z-x| \leqq 2^{-n+1}\right\}\right)<\infty .
$$

We will not need this result, but the computations we will do are quite similar to those in the proof of Wiener's theorem. (Note however that the series in Wiener's theorem is not quite the same as in Theorem 1.6.)

Recall from the introduction that given $\delta, \varepsilon>0$ and $\theta \in[0,2 \pi)$ we define a cone and wedge

$$
\begin{gathered}
C(x, \delta, \varepsilon, \theta)=\left\{x+r e^{i \psi}: 0<r<\delta,|\theta-\psi|<\varepsilon\right\} \\
W(x, \delta, \varepsilon, \theta)=C(x, \delta, \varepsilon, \theta) \backslash C(x, \delta / 2, \varepsilon, \theta)
\end{gathered}
$$

Given an open sets $\Omega_{1}, \Omega_{2}$ we define

$$
\gamma_{i}(k)=\operatorname{cap}\left(2^{k-1}\left(W\left(x, 2^{-k}, \varepsilon,(-1)^{i+1} \theta\right) \backslash \Omega_{i}\right)\right)
$$

and say $x \in \partial \Omega_{1} \cap \partial \Omega_{2}$ satisfies a weak double cone condition if there exists $\varepsilon$ and $\theta$ such that

$$
\sum_{k=1}^{\infty}\left(\gamma_{1}(k)+\gamma_{2}(k)\right)<\infty
$$

Now we start the proof of Theorem 1.6. We will start by proving that if $\Omega$ is not Poissonian then the set of points satisfying the WDCC must have positive $\Lambda_{1}$ measure. So let $\Omega_{1}$ and $\Omega_{2}$ be, as usual, two disjoint subdomains with non-singular harmonic measures $\omega_{1}$ and $\omega_{2}$. As in the paragraph above we construct simply connected domains $\widetilde{\Omega}_{1}, \widetilde{\Omega}_{2}$ containing $\Omega_{1}, \Omega_{2}$ and then the corresponding Lipschitz subdomains $D_{1}$ and $D_{2}$. Let $\tilde{D}_{i}=\Omega_{i} \cap D_{i}$ for $i=1,2$. Let $\Phi_{1}$ denote a Riemann mapping of $D_{1}$ onto the upper half plane, say with $\Gamma_{1}$ going to the interval $[-1,1]$. Since $D_{1}$ is a Lipschitz domain the mapping $\Phi$ is conformal almost everywhere on $\partial D_{1}$, i.e., if $\theta$ is the inward normal angle at a boundary point $x$ and $r$ is small enough then the image of a cone $C(x, r, \varepsilon, \theta)$ contains and is contained in a cone at $\Phi(x) \in \mathbf{R}$, centered on a vertical line segment (see e.g., [24]). Moreover the series for $C$ diverges if the corresponding series for a cone inside $\Phi(C)$ diverges, and it converges if the series for a cone containing $\Phi(C)$ converges (we are using the fact that the capacity is changed by at most a constant factor under a smooth mapping. Since the mapping in question is conformal, the Koebe $1 / 4$ theorem provides the necessary uniform bound on the derivative).

Now let $E \subset \Gamma_{1} \cap \Gamma_{2}$ be a set where the harmonic measures for $\Omega_{1}$ and $\Omega_{2}$ are mutually continuous. We have seen that on this set these measures are also mutually
absolutely continuous with $\Lambda_{1}$. Therefore $E$ has positive length and $\Phi(E) \subset \mathbf{R}$ has positive length and positive harmonic measure in $U_{1}=\Phi\left(\tilde{D}_{1}\right)$. If we can show that a.e. $x \in \Phi_{1}(E)$ is the vertex of a "vertical cone" with convergent Wiener series, we deduce the same for a.e. $x \in E$ with cones in the direction normal to $\Gamma_{1}$ at $x$. Applying the same argument to $D_{2}$ and using the fact that $\Gamma_{1}$ and $\Gamma_{2}$ agree on a set of positive length (and hence have opposite inward normals on a set of positive length) finishes the proof of this direction.

Thus we need only the first part of:
Lemma 4.2. Suppose $\Omega$ is a subdomain of $\mathbf{H}$, the upper half-plane, and let $\omega$ denote harmonic measure on $\partial \Omega$. Suppose $E \subset \mathbf{R}$ and $\Lambda_{1}(E)>0$. Then $\omega(E)=0$ if for a.e. $\left(\Lambda_{1}\right) x \in E$ there is a cone $C(x, 1, \varepsilon, \theta)$ in the upper half-plane for which the corresponding series diverges. Conversely, if a.e. $x \in E$ is the vertex of a cone in $\mathbf{H}$ with a convergent series then $\omega(E)>0$.

This will be proven in the next section. Note that if a.e. $x \in E$ has some "convergent cone" then every cone is convergent for a.e. $x \in E$. The other direction of Theorem 1.6 will follow from the second claim in Lemma 4.2. We want to show that if the set of points satisfying a WDCC has positive $\Lambda_{1}$ measure then $\Omega$ is not Poissonian. Our first step is to prove:

Lemma 4.3. Suppose $\Omega$ is a domain, and the WDCC is satisfied on a set $E$ with $\Lambda_{1}(E)<\infty$. Then $E$ is Besicovitch regular.

The proof of this just uses a few basic facts about regular and irregular sets from [13]. The first fact is that if $\Lambda_{1}(E)<\infty$ then for $\Lambda_{1}$ a.e. $x \in E$ ([13, Corollary 2.5])

$$
\begin{equation*}
\frac{1}{2} \leqq \limsup _{r \rightarrow 0} \frac{\Lambda_{1}(E \cap D(x, r))}{2 r} \leqq 1 \tag{4.1}
\end{equation*}
$$

The second fact is that if $E$ is an irregular set then for $\Lambda_{1}$ a.e. $x \in E$ and any $\varepsilon>0$ and $\theta \in[0,2 \pi$ ) ([13, Corollary 3.30$])$

$$
\begin{equation*}
\limsup _{r \rightarrow 0} \frac{\Lambda_{1}(E \cap C(x, r, \varepsilon, \theta))}{2 r}+\limsup _{r \rightarrow 0} \frac{\Lambda_{1}(E \cap C(x, r, \varepsilon,-\theta))}{2 r} \geqq \frac{\varepsilon}{10} . \tag{4.2}
\end{equation*}
$$

Choose a set $F$ with $0<\Lambda_{1}(F)<\infty$ where a WDCC is satisfied. Suppose $F$ is Besicovitch irregular. By passing to a positive measure subset if necessary, we may assume (4.1) and (4.2) hold on $F$ and that

$$
\begin{equation*}
\Lambda_{1}(F \cap B(x, r)) \leqq 4 r \tag{4.3}
\end{equation*}
$$

for all $x \in F$ and $r \leqq r_{0}$. Now fix $x \in F$ and let $\varepsilon$ and $\theta$ be as in the WDCC at $x$. By (4.2) (and replacing $\theta$ by $-\theta$ if necessary) we can choose a small $r \leqq r_{0}$ such that

$$
\Lambda_{1}(E \cap C(z, r, \varepsilon, \theta)) \geqq \frac{\varepsilon r}{40} .
$$

Then by (4.3)

$$
\Lambda_{1}(E \cap(C(x, r, \varepsilon, \theta) \backslash B(x, r \varepsilon / 320))) \geqq \frac{\varepsilon r}{80}
$$

if $r$ is small enough. But this implies

$$
\Lambda_{1}\left(E \cap W\left(x, 2^{-n}, \varepsilon, \theta\right)\right) \geqq\left(\frac{r \varepsilon}{160}\right) / \log _{2}(\varepsilon / 320)
$$

for some $r \varepsilon / 320 \leqq 2^{-n} \leqq r$. The capacity of $2^{n-2}\left(E \cap W\left(x, 2^{-n}, \varepsilon, \theta\right)\right)$ is easily seen to be bounded below by some absolute constant $A$ by using (4.3) to estimate the potential for $\left.\Lambda_{1}\right|_{E}$. Thus the corresponding term in the series is bounded below by $A$. Since this happens infinitely often, the series must diverge, a contradiction. Thus $F$ must be regular and the Lemma 4.3 is proven.

Now suppose $E \subset \partial \Omega$ is a set of positive $\Lambda_{1}$ measure where the WDCC holds. By dividing $E$ into a countable number of subsets and choosing one of positive measure we may also assume we have the same $\varepsilon$ and $\theta$ for every $x \in E$. By the previous lemma we may assume $E$ lies on a rectifiable curve, even on a Lipschitz graph $\Gamma$ with small constant. Since almost every point of $E$ is a point of density, the cones must be disjoint from $\Gamma$ for a.e. $x$. By taking a small neighborhood around a point of density of $E$ and recalling we may assume $\Gamma$ is the graph of a Lipschitz function on $[-1,1]$. Let $D_{1}$ be the part of $B(0,1)$ lying above $\Gamma$ and $D_{2}$ the part lying below. Let $\Phi_{1}$ be a Riemann mapping from $D_{1}$ to the upper half-plane which maps $\Gamma \cap \partial D_{1}$ to $[-1,1]$. Since each point of $E$ has a cone in $D_{1}$ for which the Wiener series converges, a.e. point of $\Phi_{1}(E)$ has a cone in the upper half-plane with a convergent series. If $\tilde{D}_{1}=D_{1} \cap \Omega$ and $U_{1}=\Phi_{1}\left(\tilde{D}_{1}\right)$ then the second part of Lemma 4.2 implies $\Phi_{1}(E)$ has positive harmonic measure in $U_{1}$, hence in $\tilde{D}_{1}$. In fact the harmonic measure for $\tilde{D}_{1}$ is mutually absolutely continuous with $\Lambda_{1}$ on $E$. The same argument applies to $\tilde{D}_{2}$ so the harmonic measures for $\tilde{D}_{1}$ and $\tilde{D}_{2}$ are not singular. Hence $\Omega$ is not Poissonian.

The proof of Theorem 1.7 is clearly similar.

## 5. Proof of Lemma 4.2

First we will show that if every point of $E$ is the vertex of a cone in $\mathbf{H}$ with a divergent series, then the harmonic measure of $E$ in $\Omega$ is zero. Suppose $i \in \Omega$ and fix an $x \in E$. We will show that

$$
\frac{\omega(i, B(x, r) \cap \partial \Omega, \Omega)}{r} \rightarrow 0
$$

as $r \rightarrow 0$. Since $E$ has finite length and $\Omega \subset \mathbf{H}$, this implies $\omega(E)=0$. In this section, if $E \subset \mathbf{R}$ we will let $|E|=\Lambda_{1}(E)$ denote its Lebesgue measure. The notation $a \sim b$ will mean the ratio $a / b$ is bounded and bounded away from zero by some absolute constant.

Fix a $x \in E$ with a cone $C(x)=C(x, 1, \varepsilon, \theta)$ for which the associated series diverges. For each integer $k>0$ let $A(k)=\partial \Omega \cap C(x) \cap\left\{2^{-k-1} \leqq|z-x| \leqq 2^{-k}\right\}$ and let $B(k)=2^{k-2} A(k)$ (since $x$ is fixed we will omit it from our notation to simplify matters). Let $\gamma(k)=\operatorname{cap}\left(B_{k}\right)$. Then our hypothesis is that $\sum_{k} \gamma(k)=\infty$.

Let $\mu_{k}$ be the equilibrium measure for $B(k)$. This is the unique probability measure supported on $B(k)$ which minimizes the energy integral discussed in the last section. It also has the property that its potential

$$
u_{\mu_{k}}(z)=\int \log |z-w|^{-1} d \mu_{k}(w)
$$

is equal to $\operatorname{cap}(B(k))^{-1}$ p.p. on $B(k)$. Now set

$$
\begin{aligned}
\tilde{F}_{k}(z) & =\int \log \left|\frac{z-\bar{w}}{z-w}\right| d \mu_{k}(w) \\
& =\int \log |z-w|^{-1} d \mu\left|k(w)+\int \log \right| z-\bar{w} \mid d \mu_{k}(w) \\
& =u_{\mu_{k}}(z)+\int \log |z-\bar{w}| d \mu_{k}(w) .
\end{aligned}
$$

Then $\tilde{F}_{k}$ is positive and harmonic on $\mathbf{H} \backslash B(k)$, is zero on $\mathbf{R}$ and near $\infty$, so takes its maximum on $B(k)$. For $z, w \in B(k),|z-\bar{w}| \sim 1$, so the second term in the last line above is uniformly bounded (depending only on the cone) whereas the first term (the potential) is like $\gamma(k)^{-1}$ on $B(k)$. By replacing $B(k)$ by a subset whose capacity is smaller by a fixed constant we may arrange for the series to still diverge, but also

$$
\begin{equation*}
\frac{1}{2} \gamma(k)^{-1} \leqq \tilde{F}_{k}(z) \leqq 2 \gamma(k)^{-1} \tag{5.1}
\end{equation*}
$$

for $z \in B(k)$.
We also need to estimate $\tilde{F}_{k}(z)$ for $z$ far away from $B(k)$. Let $\varrho$ denote the
hyperbolic metric on $\mathbf{H}$. Note that if $w \in B(k)$ and $\varrho(z, A(k)) \geqq 1$, then

$$
\tilde{F}_{k}(z) \sim\left\|\mu_{k}\right\| \log \left|\frac{z-\bar{w}}{z-w}\right| .
$$

Since $\|\bar{\mu}\|=1$, if either $|z| \geqq 2^{n}$ or $\operatorname{Im}(z) \leqq 2^{-n}$ we have $\widetilde{F}_{k}(z) \leqq C 2^{-n}$ for some absolute constant $C$. Now let

$$
F_{k}(z)=\tilde{F}_{k}\left(2^{k-2} z\right)
$$

Then $F_{k}$ is positive and harmonic on $\mathbf{H} \backslash A(k)$, is approximately $\gamma(k)^{-1}$ on $A(k)$ and for $z \in A(n), F_{k}(z) \sim 2^{-|n-k|}$. For $|z|=1, F_{k}(z) \sim 2^{-k}$.

Now choose an integer $m$ such that

$$
\begin{equation*}
1 \leqq \sum_{k=1}^{m} \gamma(k) \leqq 2 \tag{5.2}
\end{equation*}
$$

This can be done since each term in the series is less than $|\log (\operatorname{diam}(B(k)))|<1$ and the series diverges. Define $F$ by

$$
F(z)=\sum_{k=1}^{m} 2^{k-m} \gamma(k) F_{k}(z)
$$

Then $F$ is positive and harmonic on $D=\mathbf{H} \backslash \bigcup_{1}^{m} A(k)$, is zero on $\mathbf{R}$, is greater than $2^{k-m-1}$ on each $A(k)$ (since $F_{k}(z) \geqq \gamma(k)^{-1} / 2$ there). Next we want to find an upper bound for $F$ on each $A(k)$.

So fix a $1 \leqq n \leqq m$ and a $z \in A(n)$. Write

$$
\begin{aligned}
F(z) & =\sum_{k=1}^{n-2} 2^{k-m} \gamma(k) F_{k}(z)+\sum_{k=n-1}^{n+1} \gamma(k) 2^{k-m} F_{k}(z)+\sum_{k=n+2}^{m} 2^{k-m} \gamma(k) F_{k}(z) \\
& =I_{1}+I_{2}+I_{3} .
\end{aligned}
$$

By (5.1), each term of $I_{2}$ is bounded by $C 2^{n-m}$ so $I_{2}$ is also bounded by this. Next,

$$
I_{1} \leqq \sum_{1}^{n-2} 2^{k-m} \gamma(k) F_{k}(z) \leqq C \sum_{1}^{n-2} \gamma(k) 2^{k-m+k-n} \leqq C 2^{n-m} \sum_{1}^{m} \gamma(k) \leqq C 2^{n-m}
$$

since $2 k-n \leqq n$ and by (5.2). Finally,

$$
I_{3} \leqq \sum_{n+2}^{m} 2^{k-m} \gamma(k) F_{k}(z) \leqq C \sum_{n+2}^{m} 2^{k-m+n-k} \gamma(k) \leqq C 2^{n-m} .
$$

Thus $F(z) \sim 2^{n-m}$ for $z \in A(n)$. Also note if $|z|=1$ then,

$$
\begin{aligned}
F(z) & \geqq \sum_{1}^{m} 2^{k-m} \gamma(k) F_{k}(z) \geqq C^{-1} \sum_{1}^{m} 2^{k-m-k} \gamma(k) \\
& \geqq C^{-1} 2^{-m} \sum_{1}^{m} \gamma(k) \geqq C^{-1} 2^{-m} .
\end{aligned}
$$

by our choice of $m$. Let $J=\left(x-2^{-m}, x+2^{-m}\right)$. Since the harmonic measure in $\mathbf{H}$ of this interval looks like $2^{-m}$ at $i$ a more careful version of the above argument gives a $M>0$ such that

$$
\omega(z, J, \mathbf{H}) \leqq M F(z), \quad|z|=1
$$

Note that if $z \in A(n)$ then $\omega(z, J, \mathbf{H}) \sim 2^{n-m}$. Since the same is true for $F$, we may also assume $M$ satisfies $F(z) \leqq M \omega(z, J, \mathbf{H})$ for $z \in A(n)$. Now let

$$
u(z)=\omega(z, J, \mathbf{H})-M^{-1} F(z)
$$

Then $u$ is harmonic on $D$, equals 1 on $J$ and 0 on $\mathbf{R} \backslash J$ and is positive on every $A(k)$. Thus by the maximum principle,

$$
\omega(z, J, D) \leqq u(z)
$$

for $z \in D$ and in particular, for $|z|=1$,

$$
\omega(z, J, D) \leqq \omega(z, J, \mathbf{H})-M^{-1} F(z) \leqq\left(1-M^{-2}\right) \omega(x, J, \mathbf{H})
$$

The same argument works for any interval $J$ with $|J| \leqq 2^{-m}$ if replace the coefficient $2^{k-m}$ in the definition of $F$ by $|J| 2^{k}$.

Now we define a new domain $D_{2}$ by $D_{2}=\mathbf{H} \backslash \bigcup_{1}^{m_{2}} A(k)$ where $m_{2}$ has been chosen so

$$
1 \leqq \sum_{k=m+2}^{m_{2}} \gamma(k) \leqq 2
$$

The argument above can be easily modified to show that if $J_{2}=\left(x-2^{-m_{2}}, x+2^{-m_{2}}\right)$

$$
\omega\left(z, J_{2}, D_{2}\right) \leqq\left(1-M^{-2}\right) \omega\left(z, J_{2}, D\right)
$$

for $|z-x|=2^{-m-1}$ and hence everywhere in $D \cap\left\{|z-x|>2^{-m-1}\right\}$ (the inequality elsewhere on $\partial D$ is obvious). By the maximum principle,

$$
\omega\left(i, J_{2}, D_{2}\right) \leqq\left(1-M^{-2}\right) \omega\left(i, J_{2}, D\right) \leqq\left(1-M^{-2}\right)^{2} \omega\left(i, J_{2}, \mathbf{H}\right)
$$

The obvious induction argument gives us intervals $\left\{J_{n}\right\}$ shrinking down to $x$ such that

$$
\omega\left(i, J_{n}, \Omega\right) \leqq\left(1-M^{-2}\right)^{n}\left|J_{n}\right| .
$$

Doing this for every $x \in E$ and using Vitali's covering theorem, we see that for every $\delta>0$ we can cover almost all of $E$ by intervals $\left\{J_{j}\right\}$ such that

$$
\sum_{j} \omega\left(J_{j}\right) \leqq \delta \sum_{j}\left|J_{j}\right| \leqq 2 \delta|E|
$$

and hence $\omega(E)=0$ as desired.
Now we turn to the proof of the other direction of Lemma 4.2: if $|E|>0$ and every point of $E$ has a cone for which the Wiener series converges, then $E$ has positive harmonic measure in $\Omega$. For $x \in E$ and $k>0$ an integer we define $A(x, k)$, $\gamma(x, k), F_{x, k}(z)$ as before and set

$$
F_{x}(z)=\sum_{k=1}^{m} 2^{k} \gamma(x, k) F_{x, k}(z)
$$

By dividing $E$ up into a countable number of pieces, we may assume the same $\varepsilon$ and $\theta$ work for all points of $E$. We may also assume

$$
\sum_{k=1}^{\infty} \gamma(x, k) \leqq A<\infty
$$

for some fixed $A$ and every $x \in E$. Furthermore, it is easy to see that $E$ has positive harmonic measure in $\Omega$ iff it does in $\Omega_{\lambda}=\Omega \cup\{\operatorname{Im}(z)>\lambda\}$ for all small $\lambda>0$ (use Lemma 3.2 and the maximum principle. Thus we need only show that $E$ has positive harmonic measure in $\Omega=\Omega_{\lambda}$ for some $\lambda$. Therefore we may assume that $\partial \Omega \subset$ $\{0 \leqq \operatorname{Im}(z) \leqq \lambda\}$. This also means

$$
\sum_{k=1}^{\infty} \gamma(x, k) \leqq \delta, \quad x \in E
$$

may be assumed as small as we wish. Since $\gamma(x, k)$ is small, we have inequalities $F_{x, k}(z) \sim \gamma(x, k)^{-1}$ for $z \in A(x, k)$ as before.

Note that $F_{x}(z) \geqq M^{-1} P_{x}(z)$ for $z \in A(x, k)$ and some fixed $M>0$ where $P_{x}(z)$ is the Poisson kernel on $\mathbf{H}$ with a pole at $x$. This holds because $P_{x}(z) \sim 2^{k}$ on $A(x, k)$ and $F_{x}(z) \geqq 2^{k} \gamma(x, k) F_{x, k}(z) \geqq 2^{k-1}$ there. Now define $u_{x}(z)=P_{x}(z)-M F_{x}(z)$ and let $v(z)=\omega(z, E, \mathbf{H})$. Then by (5.1)

$$
u(z)=\int_{E} u_{x}(z) d x=v(z)-M \int_{E} F_{x}(z) d x
$$

Then $u$ is harmonic on $\Omega$ and is 0 on $\mathbf{R} \backslash E$. Note that $u_{x}(z)$ is negative if $z \in A(x, k)$ by our choice of $M$ and this happens iff $x \in I(z)$ where $I(z)$ is an interval with $|I(z)| \sim$ $\operatorname{Im}(z)$ and $\operatorname{dist}(z, I(z)) \sim \operatorname{Im}(z)$. The constants in these " $\sim$ "'s depend only on the fixed $\varepsilon$ and $\theta$ we are considering. Thus if $z \in A(x, k)$ for some $x$ and $k$

$$
u(z) \leqq \int_{E \backslash I(z)} P_{x}(z) d x \leqq \omega(z, E \backslash I(z), \mathbf{H}) \leqq 1-\eta
$$

for some $\eta>0$ which only depends on $\varepsilon$ and $\theta$.
On the other hand $u(i)$ must be very close to $\omega(i, E, \mathbf{H})$. This is because for each $x \in E$

$$
F_{x}(i) \leqq \sum_{k=1}^{m} 2^{k} \gamma(x, k) F_{x, k}(i) \leqq C \sum_{k=1}^{m} 2^{k-k} \gamma(x, k) \leqq C \delta
$$

by our earlier remarks. Thus $u(i) \geqq v(i)-C \delta$ with $\delta \rightarrow 0$ as $\lambda \rightarrow 0$. Choose a compact $E_{0} \subset E$ of positive measure such that $v$ has non-tangential limit 1 uniformly on $E_{0}$ (we can do this since $v$ has non-tangential limit 1 a.e. on $E$ ). Let $D$ be the union of all the cones under consideration with vertices in $E_{0} . D$ is a Lipschitz domain with $E_{0}$ in its boundary and $E_{0}$ has positive harmonic measure in $D$ since it has positive length (by the F. and M. Riesz theorem). Let $F_{1}=\partial \Omega \cap D$ and $F_{2}=$ $\partial \Omega \cap H \backslash D$. Since $u(z) \leqq v(z)$ for $z \in F_{2}$ and $u(z) \leqq 1-\eta$ for $z \in F_{1}$ we have by the maximum principle

$$
u(z) \leqq v(z)+\sup _{F_{1}}(1-v)-\eta \omega\left(z, F_{1}, \Omega_{\lambda}\right) .
$$

Using $z=i, \delta \rightarrow 0$ and $v(i)-C \delta \leqq u(i)$ we have

$$
\omega\left(i, F_{1}, \Omega_{\lambda}\right) \leqq \eta^{-1}\left(\sup _{F_{1}}(1-v)+C \delta\right) .
$$

$\eta$ is fixed and the right-hand side goes to 0 as $\lambda$ does, so $\omega\left(z, F_{1}, \Omega_{\lambda}\right) \rightarrow 0$ as $\lambda \rightarrow 0$.

On the other hand $E_{0}$ has positive harmonic measure in $D$, say $>v$ (with respect to $i$ ), and hence in $\mathbf{H} \backslash F_{2}$. Therefore if $\lambda$ is chosen so small that $\omega\left(i, F_{1}, \Omega_{\lambda}\right) \leqq v / 2$ we must have by the maximum principle

$$
\omega\left(i, E_{0}, \Omega_{\lambda}\right) \geqq \omega\left(i, E_{0}, \mathbf{H} \backslash F_{2}\right)-\omega\left(i, F_{1}, \Omega_{\lambda}\right) \geqq v / 2 .
$$

Since $E_{0} \subset E$, this completes the proof of Lemma 4.2.

## 6. Remarks

In this final section I would like to make a few remarks about the Martin boundary of $\Omega$. A minimal harmonic function $\varphi$ on $\Omega$ is a positive harmonic function with the property that if $\psi$ is another positive harmonic function on $\Omega$ such that $\psi \leqq \varphi$ then $\psi=\lambda \varphi$ for some $\lambda>0$. For example, on the unit disk the minimal harmonic functions are just the Poisson kernels corresponding to different points on the unit circle. On a general domain, just as on the disk, any positive harmonic function has a unique integral representation in terms of the minimal harmonic functions and, in particular, every bounded harmonic function $u$ can be represented as

$$
u=\int_{\Delta_{1}} K_{\lambda} f(x) d \mu(x)
$$

where $\Delta_{1}$ is the set of minimal functions, $\mu$ is the measure representing the constant function 1 and $f \in L^{\infty}(\mu)$. Moreover, $\mu$ a.e. minimal function is unbounded in every neighborhood of exactly one point of $\partial \Omega$ (although this is not true for every minimal function). Thus there is a "projection" $P: \Delta_{1} \rightarrow \partial \Omega$ defined $\mu$ a.e. on $\Delta_{1}$.

Poissonian domains are simply those for which there is a full measure subset of $\Delta_{1}$ on which $P$ is 1-1 (see e.g. [23]). What we wish to point out here is that for domains in $\mathbf{R}^{2}$ there is always a set of full measure in $\Delta_{1}$ on which $P$ is at most 2 to 1. This is because given three subsets of $\Delta_{1}, X_{1}, X_{2}, X_{3}$, we can form the three harmonic functions $u_{1}, u_{2}, u_{3}$ corresponding to the characteristic functions of the sets. Since $\sum u_{i} \leqq 1$, the domains $\Omega_{i}=\left\{u_{i}>1 / 2\right\}$ are disjoint subdomains on $\Omega$. If $E \subset \partial \Omega_{1} \cap \partial \Omega_{2}$ is a set where $\omega_{1}$ and $\omega_{2}$ are mutually continuous then we know $\Omega_{1}$ and $\Omega_{2}$ look like Lipschitz domains on near $E$. From this it is easy to see $\omega_{3}(E)=0$. Thus we can find sets $E_{i} \subset \partial \Omega_{1} \cap \partial \Omega_{2} \cap \partial \Omega_{3}$ such that $\omega_{i}\left(E_{i}\right)=0$ and $\omega_{j}\left(\partial \Omega_{j} \backslash E_{i}\right)=1$ for $j \neq i$. Then let $F_{i}=X_{i} \cap P^{-1}\left(P\left(X_{i}\right) \backslash E_{i}\right)$. Then $F_{i} \subset X_{i}$ has the same measure as $X_{i}$ but $P$ is at most 2 to 1 on $F_{1} \cup F_{2} \cup F_{3}$.

We should point out that it is possible for 3 disjoint planar domains to have the same boundary, e.g., the so-called "Wada lakes" (see [16, pages 143-144]). Also, the " 2 to 1 " nature of harmonic measures on planar domains had previously been observed by John Garnett and Peter Jones using the Green's lines of $\Omega$ and

Moore's triod theorem. (A Green's line is a path orthogonal to every level line of Green's function, i.e., a line of steepest descent. A triod is any continuum consisting of three Jordan arcs with a common endpoint. Moore's theorem states that any disjoint collection of triods in the plane must be countable).

The same result should be true in higher dimensions, but I don't know how to prove it because it is unknown under what conditions two disjoint domains in $\mathbf{R}^{n}$ can have mutually absolutely continuous harmonic measures. If mutual absolute continuity of the measures implied that they look " $(n-1)$ dimensional" then the 2 to 1 property would also hold. In 2 dimensions harmonic measure for any domain $\Omega$ always gives full measure to a set of dimension $\leqq 1$ [19] (in fact to a set of sigma finite $\Lambda_{1}$ measure) and it had been conjectured that for domains in $\mathbf{R}^{n}$ the same would be true with sets of dimension $n-1$. Tom Wolff has shown this is false by building a domain for which $\omega(E)=0$ for every $E \subset \partial \Omega$ with $\Lambda_{n-1+\varepsilon}(E)=0$ and some $\varepsilon$ [28]. It should be possible to build such a domain so that the complement of its closure also has this property, but it is not clear whether the measures will be mutually absolutely continuous (probably they will not).

We can prove something weaker than 2 to 1 by using the estimate mentioned in Section 3 from [14]. Suppose $\Omega_{1}, \ldots, \Omega_{m}$ are disjoint domains in $\mathbf{R}^{n}$ and $x \in \bigcap_{j} \partial \Omega_{j}$. Then the estimates imply

$$
\prod_{j=i}^{m} \omega_{j}(B(x, r)) \leqq C r^{m(n-2)} \exp \left(-\int_{r}^{1} \sum_{j=1}^{m} \alpha_{j}(t) \frac{d t}{t}\right)
$$

where $\alpha_{i}$ are the characteristic constants for the Dirichlet problem on the domains $\Omega_{j}(t)$. If $S_{j}(t)$ denotes the $n-1$ area of $\Omega_{j}(t)$ (normalized so the sphere has area 1), [14] contains the estimates (dropping the $j ’ s$ ) $\alpha(\Omega) \geqq \varphi_{\infty}(S)$

$$
\varphi_{\infty}(S)= \begin{cases}\frac{1}{2} \log \left(\frac{1}{4 S}\right)+\frac{3}{2}, & 0<S \leqq \frac{1}{4} \\ 2(1-S), & \frac{1}{4} \leqq S<1\end{cases}
$$

independent of the dimension. They also give estimates depending on the dimension, which I will repeat only for $n=3$. We have $\alpha(\Omega) \geqq \varphi_{3}(S)$ where

$$
\varphi_{3}(S)=\max \left(\varphi_{\infty}(S), \frac{1}{2} j_{0}\left(\frac{1}{S}-\frac{1}{2}\right)^{1 / 2}-\frac{1}{2}\right)
$$

where $j_{0}=2.4048 \ldots$ is the first zero of Bessel's function of order 0 . The two dimensional version of this harmonic measure estimate is called Tsuji's estimate [26] (see [3] for the history of such estimates).

For a bounded domain in $\mathbf{R}^{n}$ and a.e. $(\omega)$ point $x \in \partial \Omega$ we have an estimate $\Omega(B(x, r)) \geqq C r^{n}$ for some $C>0$ (since $\partial \Omega$ has finite $n$ measure). (In fact a result of Bourgain improves this to $r^{n-\varepsilon_{n}}$ [9].) Therefore if $x$ is a common boundary point
of $\Omega_{1}, \ldots, \Omega_{m}$ where this estimate holds for every domain we must have

$$
m n \geqq m(n-2)+\sum_{j=1}^{m} \alpha_{j}
$$

Using the fact that the domains are disjoint, we see that $\sum_{j} S_{i}(t) \leqq 1$. Since the functions $\varphi_{n}$ are convex we get

$$
\begin{gathered}
m n \geqq m(n-2)+m \varphi_{n}\left(\frac{1}{m}\right) \\
2 \geqq \varphi_{n}\left(\frac{1}{m}\right) .
\end{gathered}
$$

Plugging in the formulas for $\varphi_{n}$ we see that $m \leqq 4$ for $n=3\left(\varphi_{3}(1 / 4) \approx 1.749\right.$ and $\left.\varphi_{3}(1 / 5) \approx 2.045\right)$ and $m \leqq 10$ for $n=\infty\left(\varphi_{\infty}(1 / 10) \approx 1.958\right.$ and $\left.\varphi_{\infty}(1 / 11) \approx 2.005\right)$. Thus $\Delta_{1}$ always has a finite to 1 projection onto the topological boundary (a.e. with respect to harmonic measure). This is essentially just a restatement of results in [14].

Of course the estimates we have used are not sharp. The estimates on the characteristic constants involve replacing each domain $\Omega_{j}(t)$ by a spherical cap of the same area. But three disjoint domains on $S^{2}$, for example, need not be spherical caps. We would get $m \leqq 2$ for $n=3$ if we knew $\alpha_{1}+\alpha_{2}+\alpha_{3} \geqq 6$ for any three disjoint domains on $S^{2}$. Unfortunately, this is not true. Think of $\mathbf{R}^{3}=(x, y, z)$ in the polar coordinates $(r, \theta, z)$ and let $\Omega_{i} i=1,2,3$ be the domains corresponding to $0<\theta<2 \pi / 3,2 \pi / 3<\theta<4 \pi / 3$ and $4 \pi / 3<\theta<2 \pi$. On $\Omega_{1} u(x, y, z)=\operatorname{Im}(x+i y)^{3 / 2}$ is a positive harmonic function which vanishes on $\partial \Omega_{1}$ and is homogeneous of degree $3 / 2$. Thus the characteristic constant $\alpha$ of $\Omega_{1}(t)$ is equal to $3 / 2$ for all $t$. Similarly for $\Omega_{2}$ and $\Omega_{3}$ since they are just rotations of $\Omega_{1}$. Therefore $\alpha_{1}+\alpha_{2}+\alpha_{3}=$ $9 / 2=4 \frac{1}{2}$.

We could also get $m=2$ from following improvement of Bourgain's theorem: For any $\Omega \subset \mathbf{R}^{3} \omega(B(x, r)) \geqq r^{\beta}$ for a.e. ( $\omega$ ) $x \in \partial \Omega$, all $r>0$ small enough and $\beta \leqq \varphi_{3}(1 / 3)+1 \approx 2.4011$. This is probably false since the critical $\beta$ for this estimate is conjectured to be 2.5 . However if we knew that the example in the last paragraph was extremal, i.e., $\alpha_{1}+\alpha_{2}+\alpha_{3} \supseteqq 9 / 2$ for any three disjoint domains on the sphere, then we would only need the previous estimate with $\beta=2.5-\varepsilon$ for any $\varepsilon>0$. Moreover, we may also assume that the harmonic measures for $\Omega_{1}, \Omega_{2}$ and $\Omega_{3}$ are pairwise mutually absolutely continuous on a set $E$ and that $x$ is a point of density of this set. The estimate should certainly be true with this additional hypothesis.

I will finish the paper with one last conjecture concerning harmonic measure in $\mathbf{R}^{\mathbf{2}}$. Tom Wolff has proven that

$$
F=\left\{x \in \partial \Omega: \limsup _{r \rightarrow 0} \frac{\omega(B(x, r) \cap \partial \Omega)}{r}>0\right\}
$$

has sigma finite length and full harmonic measure for any planar domain $\Omega$ (unpublished). We have also seen in Section 5 that if $E \subset \partial \Omega$ has positive length and if every point of $E$ is a vertex of a single cone with convergent series (as in (1.1) but with only one cone) then

$$
\lim _{r \rightarrow 0} \frac{\omega(B(x, r) \cap \partial \Omega)}{r}<\infty
$$

$\Lambda_{1}$ a.e. on $E$. It seems possible that the converse is also true, i.e., if $0<\Lambda_{1}(E)<\infty$ and no point of $E$ has a such a "convergent cone" then for $\omega$ a.e. $x \in E$

$$
\limsup _{r \rightarrow 0} \frac{\omega(B(x, r) \cap \partial \Omega)}{r}=\infty
$$

so that there exists $F \subset E$ with $\omega(F)=\omega(E)$, but $\Lambda(F)=0$. This is consistent with what is known in the simply connected case. It also has the following consequence which is of interest in its own right: if $\Omega \subset \mathbf{R}^{2}$ is a domain and $E \subset \partial \Omega$ is Besicovitch irregular then there exists $F \subset E$ with $\omega(F)=\omega(E)$ and $\Lambda_{1}(F)=0$. Peter Jones has pointed out that this is true in the case when $E=\partial \Omega$ satisfies a capacitary "thickness" condition: there exists $\varepsilon>0$ such that for every $x \in \partial \Omega$ and $0<r<r_{0}, \operatorname{cap}\left(r^{-1}(B(x, r / 4) \cap \partial \Omega)\right) \geqq \varepsilon$.

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