

A CHARACTERIZATION OF PSEUDO-EINSTEIN REAL HYPERSURFACES IN A QUATERNIONIC PROJECTIVE SPACE

By

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0. Introduction.

Let HP^n be a quaternionic projective space, $n \geq 3$, with metric G of constant quaternionic sectional curvature 4, and let M be a connected real hypersurface of HP^n . Let ξ be a unit local normal vector field on M and $\{I, J, K\}$ a local basis of the quaternionic structure of HP^n (cf. [4]). Then $U_1 = -I\xi, U_2 = -J\xi, U_3 = -K\xi$ are unit vector fields tangent to M . We call them *structure vectors*. Now we put $f_i(X) = g(X, U_i)$, for arbitrary $X \in TM$, $i = 1, 2, 3$, where TM is the tangent bundle of M and g denotes the Riemannian metric induced from the metric G . We denote D and D^\perp the subbundles of TM generated by vectors perpendicular to structure vectors, and structure vectors, respectively. There are many theorems from the point of view of the second fundamental tensor A of M (cf. [1], [8] and [9]). It is known that if M satisfies $g(AD, D^\perp) = 0$ then there is a local basis of quaternionic structure such that structure vectors are principal vectors. Berndt classified the real hypersurfaces which satisfy this condition (cf. [1]). On the other hand we know some results on real hypersurfaces of HP^n in terms of the Ricci tensor S of M (cf. [3] and [8]). If the Ricci tensor satisfies that $SX = aX + b\sum_{i=1}^3 f_i(X)U_i$ for some smooth functions a and b on M , then M is called a pseudo-Einstein real hypersurface of HP^n . This notion comes from the problem for the real hypersurfaces in complex projective space CP^n . Kon studied it under the assumption that they have constant coefficients (cf. [5]) and Cecil and Ryan gave a complete classification (cf. [2]). In [8] Martinez and Perez studied pseudo-Einstein real hypersurfaces of HP^n , $n \geq 3$ under the condition that a and b are constant. Using Berndt's classification we show that we do not need the assumption. The main purpose of this paper is to provide a characterization of pseudo-Einstein real hypersurface in HP^n by using an estimate of the length of the Ricci tensor S , which is a quaternionic version of a result of Kimura and

Maeda (cf. [5]).

THEOREM 1. *Let M be a real hypersurface of HP^n ($n \geq 3$) with $f_i(SU_i) = \alpha$ for $i = 1, 2, 3$, α is a function on M . Then the following holds:*

$$(0.1) \quad \|S\|^2 \geq 3\alpha^2 + \frac{1}{4(n-1)}(\rho - 3\alpha)^2,$$

where $\|S\|$ is the length of the Ricci tensor S of M and ρ is the scalar curvature of M . The equality of (0.1) holds if and only if M is an open subset of one of the following:

- (a) a geodesic hypersphere,
- (b) a tube of radius r over a totally geodesic HP^k , $1 \leq k \leq n-2$, $0 < r < \pi/2$ and $\cot^2 r = (4k+2)/(4n-4k-2)$.

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1. Preliminaries.

Let M be a connected real hypersurface of HP^n , $n \geq 3$, and let ξ be a unit normal vector field on M . The Riemannian connection $\tilde{\nabla}$ in HP^n and ∇ in M are related by the following formulas for arbitrary vector fields X and Y on M :

$$(1.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\xi,$$

$$(1.2) \quad \tilde{\nabla}_X \xi = -AX,$$

where A is the second fundamental tensor of M in HP^n . We put

$$(1.3) \quad \begin{aligned} IX &= \phi_1 X + f_1(X)U_1, \\ JX &= \phi_2 X + f_2(X)U_2, \\ KX &= \phi_3 X + f_3(X)U_3, \end{aligned}$$

for any vector field X tangent to M , where $\phi_1 X, \phi_2 X$, and $\phi_3 X$, are the tangential parts of IX, JX and KX respectively, ϕ_i are tensors of type $(1, 1)$, f_i are 1-forms for $i = 1, 2, 3$. Then they satisfy

$$(1.4) \quad \phi_i^2 X = -X + f_i(X)U_i,$$

$$(1.5) \quad f_i(U_i) = 1, \quad f_i(U_{i+1}) = f_i(U_{i+2}) = 0,$$

$$(1.6) \quad \phi_i U_i = 0, \quad \phi_i U_{i+1} = -\phi_{i+1} U_i = U_{i+2},$$

$$(1.7) \quad \phi_{i+1} \phi_{i+2} X = \phi_i X + f_{i+2}(X) U_{i+1},$$

$$(1.8) \quad \phi_{i+2} \phi_{i+1} X = -\phi_i X + f_{i+1}(X) U_{i+2},$$

for $i = 1, 2, 3$, where we take the index i modulo 3. From the expression of the curvature tensor of HP^n (cf. [4]), we have the following Gauss and Codazzi equations:

$$(1.9) \quad \begin{aligned} R(X, Y)Z &= g(Y, Z)X - g(X, Z)Y \\ &+ \sum_{i=1}^3 (g(\phi_i Y, Z)\phi_i X - g(\phi_i X, Z)\phi_i Y - 2g(\phi_i X, Y)\phi_i Z) \\ &+ g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

$$(1.10) \quad (\nabla_X A)Y - (\nabla_Y A)X = \sum_{i=1}^3 (f_i(X)\phi_i Y - f_i(Y)\phi_i X - 2g(\phi_i X, Y)U_i).$$

We denote by S the Ricci tensor of type $(1, 1)$ on M . Then by (1.9) we have

$$(1.11) \quad SX = (4m + 7)X - 3 \sum_{i=1}^3 f_i(X)U_i + (\text{trace } A)AX - A^2 X,$$

We use the following lemma.

LEMMA 2. *Let M be a real hypersurface of HP^n , $n \geq 2$. Then $g(AD, D^\perp) = 0$ if and only if there exists a local basis $\{I, J, K\}$ of quaternionic structure, such that the corresponding $U_1 = -I\xi, U_2 = -J\xi, U_3 = -K\xi$ are principal vectors.*

We know that Berndt classified the real hypersurface with the above condition (cf. [1]).

PROPOSITION 3. *Let M be a real hypersurface of HP^n , $n \geq 2$, satisfying $g(AD, D^\perp) = 0$. Then M is congruent to an open subset of one of the following:*

- (a) *a geodesic hypersphere,*
- (b) *a tube of radius r over a totally geodesic HP^k , $1 \leq k \leq n-2, 0 < r < \pi/2$,*
- (c) *a tube of radius r over a totally geodesic CP^n , $0 < r < \pi/4$.*

The geodesic hypersphere of HP^n has two distinct principal curvatures. Conversely Martinez and Perez proved the following proposition in [8].

PROPOSITION 4. *Let M be a real hypersurface of $HP^n, n \geq 3$, with at most two distinct principal curvatures at each point of M . Then M is an open subset of the geodesic hypersphere of HP^n .*

We show the following result, which was proved by Martinez and Perez (cf. [8]) under the additional assumption that a and b are constant.

PROPOSITION 5. *Let M be a pseudo-Einstein real hypersurface of $HP^n, n \geq 3$. Then M is an open subset of one of the following:*

(a) *a geodesic hypersphere,*

(b) *a tube of radius r over a totally geodesic $HP^k, 1 \leq k \leq n-2, 0 < r < \pi/2$ and $\cot^2 r = (4k+2)/(4n-4k-2)$.*

PROOF. Suppose that M is a pseudo-Einstein real hypersurface of $HP^n, n \geq 3$. Let $H = A^2 - (\text{trace} A)A$. From the assumption, we have

$$(1.12) \quad HX = (4n+7-a)X, \quad HZ = (4n+4-(a+b))Z,$$

for arbitrary $X \in D, Z \in D^\perp$. If $b = -3$, from (1.12) we get

$$(A^2 - (\text{trace} A)A - (4n+7-a))X = 0,$$

for any $X \in TM$. Hence M has at most two distinct principal curvatures at each point of M , so that, by Proposition 4, M is an open subset of the geodesic hypersphere. If $b \neq -3$, D and D^\perp are invariant under H . $4n+4-(a+b)$ is an eigenvalue of multiplicity three of H . On the other hand, if X is an eigenvector of A , clearly X is an eigenvector of H . If $\{X_1, \dots, X_{4n-1}\}$ is an orthonormal basis of eigenvectors of A , then it is also a basis of eigenvectors for H . There must be three X_i for $i = 1, 2, 3$, by suitable choice of indices, so that $HX_i = (4n+4-(a+b))X_i$ for $i = 1, 2, 3$. Then X_1, X_2 and X_3 span the distribution D^\perp . Thus $g(AD, D^\perp) = 0$ and, by Proposition 3, we get the result.

2. Proof of Theorem 1.

We first remark that the real hypersurface M is pseudo-Einstein if and only if

$$(2.1) \quad g(SX, Y) = \lambda g(X, Y) \quad \text{for any } X, Y \in D,$$

λ is a function on M , and

$$(2.2) \quad U_1, U_2 \text{ and } U_3 \text{ are eigenvectors of } S \text{ with the same eigenvalue } \alpha.$$

We can rewrite the condition (2.1) to get

$$(2.3) \quad g(SX, Y) = \rho_0 g(X, Y) \quad \text{for any } X, Y \in D$$

and

$$\rho_0 = \frac{1}{4(n-1)} \left(\rho - \sum_{i=1}^3 f_i(SU_i) \right).$$

This equation (2.3) is equivalent to

$$\begin{aligned} g\left(SX - \sum_{i=1}^3 f_i(X)SU_i, Y - \sum_{j=1}^3 f_j(Y)U_j\right) \\ = \rho_0 g\left(X - \sum_{k=1}^3 f_k(X)U_k, Y - \sum_{l=1}^3 f_l(Y)U_l\right) \end{aligned}$$

for any tangent vector fields X, Y on M . Consequently we obtain

$$\begin{aligned} g(SX, Y) - \rho_0 g(X, Y) \\ = \sum_{i=1}^3 (\rho_0 f_i(X) f_i(Y) - f_i(SX) f_i(Y) - f_i(X) f_i(SY)) \\ + \sum_{j,k=1}^3 f_j(X) f_k(Y) f_j(SU_k) = 0. \end{aligned}$$

We define the tensor T by

$$\begin{aligned} T(X, Y) &= g(SX, Y) - \rho_0 g(X, Y) \\ &+ \sum_{i=1}^3 (\rho_0 f_i(X) f_i(Y) - f_i(SX) f_i(Y) - f_i(X) f_i(SY)) \\ &+ \sum_{j,k=1}^3 f_j(X) f_k(Y) f_j(SU_k). \end{aligned}$$

Using (1.4), (1.5), (1.6), (1.7) and (1.8), we calculate the length of T to get

$$\|T\|^2 = \|S\|^2 - \frac{1}{4(n-1)} \left(\rho - \sum_{i=1}^3 f_i(SU_i) \right)^2 - 2 \sum_{j=1}^3 \|SU_j\|^2 + \sum_{k,l=1}^3 (f_k(SU_l))^2.$$

We know the inequality

$$\sum_{j=1}^3 \|SU_j\|^2 \geq \sum_{k,l=1}^3 (f_k(SU_l))^2 \geq 3\alpha^2$$

holds on any real hypersurface M of HP^n . From the assumption, the equality holds if and only if U_1, U_2 and U_3 are eigenvectors of S with the same eigenvalue α . We assert that the equality (0.1) holds if and only if M is pseudo-Einstein. By Proposition 5, we have proved our theorem.

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