

A CHARACTERIZATION OF RIEMANNIAN MANIFOLDS OF CONSTANT CURVATURE

RICHARD HOLZSAGER

As in [2], consider the parallel bodies of a hypersurface in a Riemannian manifold. That is, suppose M is a submanifold of codimension 1 with oriented normal bundle in a manifold \bar{M} . Define a homotopy $h: M \times R \rightarrow \bar{M}$, by letting $h(x, t) = \gamma_x(t)$, where γ_x is the geodesic through x whose tangent at x is the positive (with respect to the orientation on the normal bundle of M) unit normal vector. In other words, $h_t(M)$ is obtained by translating M distance t along orthogonal geodesics.

If M is a compact hypersurface (with or without boundary), it makes sense to consider the area (or volume) $A_M(t)$ of the singular hypersurface M_t . If M and \bar{M} are C^∞ , so is $A_M: R \rightarrow R$. In [1], we showed that surfaces of constant curvature c are characterized by the fact that for any hypersurface (i.e., curve) A_M satisfies the differential equation $A'' + cA = 0$. This result is now generalized to higher dimensions.

Theorem. *For an n -dimensional C^∞ Riemannian manifold \bar{M} , there is a differential equation $A^{(n)} + a_1 A^{(n-1)} + \dots + a_n A = 0$ (a_i constant) satisfied by A_M for every hypersurface M if and only if \bar{M} has constant sectional curvature. The relation between the equation and the curvature is $c = a_2 / \binom{n+1}{3}$.*

Remark. It is impossible for an equation of order m less than n to be satisfied by every A_M . To show this choose some $x \in M$ and an orthogonal base T_1, \dots, T_n for the tangent space at x . Define a coordinate system ϕ_m about x by $\phi_m(r_1, \dots, r_n) = \exp_y X$, where $y = \exp_x \sum_{m+2}^n r_i T_i$, and X is the parallel translation of $\sum_1^{m+1} r_i T_i$ to y along $\exp_x t \sum_{m+2}^n r_i T_i$. Let U be a small neighborhood of $(1, 0, \dots, 0)$ in an m -sphere S^m , and V a small neighborhood of the origin in an $(n - m - 1)$ -dimensional Euclidean space R^{n-m-1} . For small values of t , ϕ_m will imbed $(tU) \times V$ in \bar{M} , $\phi_m((tU) \times V)$ forms a family of "parallel" hypersurfaces and $A(t) = t^m \int \sqrt{g} \circ \phi \circ (t \times \text{id}) d \text{Vol}$, integral over $U \times V$, where g is the determinant of the metric tensor on \bar{M} with respect to ϕ . Then $A^{(m)}(0) = m! \text{Vol}(U \times V)$, $A^{(i)}(0) = 0$ for $i < m$. Thus A cannot satisfy an equation of order m .

Proof of the theorem. Assume the equation is satisfied by every A_M . Let $\phi = \phi_{n-2}$ be as in the remark (i.e., build a coordinate system using tubes about the geodesic through T_n). Let

$$A_0(t) = \lim A(t)/\text{Vol}(U \times V) = t^{n-2}\sqrt{g}(\phi(1, 0, \dots, 0)) ,$$

limit taken as U and V converge down to $(1, 0, \dots, 0) \in S^{n-2}$ and $0 \in R$ respectively. A_0 will also satisfy the equation, giving for $t = 0$

$$(1) \quad \binom{n}{2} (n-2)! T_1 T_1 \sqrt{g} + (n-1)(n-2)! a_1 T_1 \sqrt{g} + (n-2)! a_2 \sqrt{g} = 0 ,$$

where we write T_i for $\partial/\partial x_i$ throughout the coordinate system.

Let us also write D_i for covariant differentiation with respect to T_i . Then, as in [2], $T_1 \sqrt{g} = \sum_{i=1}^n \gamma_{ii} \sqrt{g}$, where $D_1 T_i = \sum_j \gamma_{ij} T_j$ and $T_1 T_1 \sqrt{g} = (\sum_{i,j=2}^n \gamma_{ii} \gamma_{jj} + \sum_i T_i \gamma_{ii}) \sqrt{g}$. By definition of ϕ , $\sum_{i=1}^{n-1} r_i D_i (\sum_{j=1}^{n-1} r_j T_j) = 0$ at any point of the form $(tr_1, \dots, tr_{n-1}, r_n)$. In particular, this implies $D_i T_j = 0$ for all $i, j \leq n-1$ at x . Also, $D_n T_i = 0$ for all i , so $D_1 T_n = D_n T_1 = 0$. Consequently $\gamma_{ii}(x) = 0$ for all i . Thus

$$(2) \quad \binom{n}{2} \sum_i T_i \gamma_{ii} + a_2 = 0 \quad \text{at } x .$$

$$\begin{aligned} T_i \gamma_{ii} &= \sum_j (T_i \gamma_{ij}) \langle T_i, T_j \rangle = T_1 \left(\sum_j \gamma_{ij} \langle T_i, T_j \rangle \right) - \sum_j \gamma_{ij} T_1 \langle T_i, T_j \rangle \\ &= T_1 \langle T_i, D_1 T_i \rangle = \langle D_1 T_i, D_1 T_i \rangle + \langle T_i, D_1 D_1 T_i \rangle = \langle T_i, D_1 D_i T_1 \rangle , \end{aligned}$$

so (2) becomes

$$(3) \quad \binom{n}{2} \sum_i \langle T_i, D_1 D_i T_1 \rangle + a_2 = 0 .$$

At $\phi(x_1, \dots, x_n)$, $\sum_{j,k=1}^{n-1} x_j x_k D_j T_k = 0$. Applying D_i ($i = 1, \dots, n-1$) at $\phi(x_1, 0, \dots, 0)$ gives $2 x_1 D_i T_1 + x_1^2 D_i D_1 T_1 = 0$. Dividing by x_1 and applying D_1 give $2 D_1 D_i T_1 + D_i D_1 D_1 + x_1 D_1 D_i D_1 T_1 = 0$, so $D_i D_1 T_1 = -2 D_1 D_i T_1$ at x . Therefore the sectional curvature determined by T_1 and T_i at x for $i = 1, \dots, n-1$ is $R(1, i) = -3 \langle D_1 D_i T_1, T_i \rangle$. Also, $R(1, n) = -\langle D_1 D_n T_1, T_n \rangle$, since $D_1 T_1$ vanishes along $\phi(0, \dots, 0, t)$, so $D_n D_1 T_1 = 0$ at x . Now (3) becomes

$$(4) \quad R(1, n) + \frac{1}{3} \sum_2^{n-1} R(1, i) = a_2 / \binom{n}{2} .$$

The roles played in this whole argument by T_n and T_i ($i = 2, \dots, n-1$) may be switched, adding $\frac{2}{3}(R(1, i) - R(1, n))$ to the left side without changing the

right. Thus $R(1, i) = R(1, n)$, so $\frac{n+1}{3}R(1, n) = a_2 / \binom{n}{2}$, or $R(1, n) = a_2 / \binom{n+1}{3}$. Since x, T_1, T_n were arbitrary, this finishes the proof in one direction.

Now assume \bar{M} has constant curvature. For any tangent V to M at x , V has a canonical extension along the orthogonal geodesic ($V(h_t(x)) = dh_t(V)$), so if T is the unit normal vector, then $D_T V$ makes sense. Note that if W is another tangent to M at x , $\langle D_T V, W \rangle = \langle D_T W, V \rangle$. To see this, a coordinate system ϕ in \bar{M} about x is said to be allowable if it is obtained by taking a coordinate system ψ in M about x and setting $\phi(r_1, \dots, r_n) = h_{r_1}(\psi(r_2, \dots, r_n))$. If T, V, W are extended to have constant components in an allowable coordinate system, then $[V, W] = [T, V] = [T, W] = \langle T, V \rangle = \langle T, W \rangle = 0$, so

$$\begin{aligned} \langle D_T V, W \rangle &= \langle D_V T, W \rangle = -\langle T, D_V W \rangle = -\langle T, D_W V \rangle \\ &= \langle D_W T, V \rangle = \langle D_T W, V \rangle. \end{aligned}$$

Further, applying T to the relation $\langle D_T V, W \rangle = \langle D_T W, V \rangle$ gives $\langle D_T D_T V, W \rangle = \langle D_T D_T W, V \rangle$.

Since $\langle D_T V, W \rangle$ is symmetric and bilinear in V and W , it is possible to choose an orthonormal base T_2, \dots, T_n for the tangent space to M at x such that $\langle D_T T_i, T_j \rangle = 0$ for $i \neq j$, and also an allowable coordinate system so that at x $\partial/\partial x_i = T_i$ for $i = 1, \dots, n$ (where we now write T_1 for T). If V is a linear combination of T_2, \dots, T_n at any point of the coordinate neighborhood and R is the curvature tensor, then $\langle R(T_1, V)T_1, V \rangle = \langle D_V D_1 T_1, V \rangle - \langle D_1 D_V T_1, V \rangle = -\langle D_1 D_V T_1, V \rangle = -\langle D_1 D_1 V, V \rangle$ since $D_1 T_1$ is identically 0. If c is the sectional curvature, then since $\langle T_1, V \rangle = 0$ and $\langle T_1, T_1 \rangle = 1$, $c = -\langle D_1 D_1 V, V \rangle / \langle V, V \rangle$. Thus, as quadratic forms on the span of T_2, \dots, T_n at any point, $\langle D_1 D_1 V, V \rangle$ is equal to $\langle -cV, V \rangle$. The symmetric bilinear forms $\langle D_1 D_1 V, W \rangle$ and $\langle -cV, W \rangle$ are equal, so $\langle D_1 D_1 T_i, T_j \rangle = \langle -cT_i, T_j \rangle$ for $i, j \geq 2$. Since also

$$\begin{aligned} \langle D_1 D_1 T_i, T_1 \rangle &= T_1 \langle D_1 T_i, T_1 \rangle - \langle D_1 T_i, D_1 T_1 \rangle = T_1 \langle D_i T_1, T_1 \rangle \\ &= \frac{1}{2} T_1 T_i \langle T_1, T_1 \rangle = 0 = \langle -cT_i, T_1 \rangle, \end{aligned}$$

it follows that $D_1 D_1 T_i = -cT_i$ for $i = 2, \dots, n$.

Next note that $c\langle T_i, T_j \rangle + \langle D_1 T_i, D_1 T_j \rangle$ is constant along $h_t(x)$ ($i, j \geq 2$), since

$$\begin{aligned} T_1(c\langle T_i, T_j \rangle + \langle D_1 T_i, D_1 T_j \rangle) \\ = c\langle D_1 T_i, T_j \rangle + c\langle T_i, D_1 T_j \rangle + \langle D_1 D_1 T_i, D_1 T_j \rangle + \langle D_1 T_i, D_1 D_1 T_j \rangle = 0. \end{aligned}$$

But at x , $\langle D_1 T_i, T_j \rangle = 0$ for $j \neq i$, so $D_n T_i$ is a multiple of T_i for $i = 2, \dots, n$, so $c\langle T_i, T_j \rangle + \langle D_1 T_i, D_1 T_j \rangle = 0$ at x and consequently at $h_t(x)$. Thus

$$\begin{aligned} T_1 T_i \langle T_i, T_j \rangle &= \langle D_1 D_1 T_i, T_j \rangle + 2 \langle D_1 T_i, D_1 T_j \rangle + \langle T_i, D_1 D_1 T_j \rangle \\ &= -4c \langle T_i, T_j \rangle, \end{aligned}$$

for $i, j \geq 2, i \neq j$. This second order equation, together with the initial conditions $\langle T_i, T_j \rangle = T_1 \langle T_i, T_j \rangle = 0$ at x , implies that $\langle T_i, T_j \rangle$ is identically 0 along $h_t(x)$. Therefore $|dh_t(T_2 \wedge \cdots \wedge T_n)| = \prod_2^n |T_i|$ at $h_t(x)$. Now

$$\begin{aligned} T_1 T_1 |T_i| &= -c |T_i| + (\langle T_i, T_i \rangle \langle D_1 T_i, D_1 T_i \rangle - \langle D_1 T_i, T_i \rangle^2) / |T_i|^3 \\ &= -c |T_i| \end{aligned}$$

($D_1 T_i$ being a multiple of T_i). This means that $|T_i|$ is a linear combination of $\sin \sqrt{c} t$ and $\cos \sqrt{c} t$ or of $\sinh \sqrt{-c} t$ and $\cosh \sqrt{-c} t$ or of 1 and x , depending on whether c is positive, negative or 0. Therefore $|dh_t(T_2 \wedge \cdots \wedge T_n)|$ is a linear combination of $\sin^i \sqrt{c} t \cos^{n-i-1} \sqrt{c} t$ or of $\sinh^i \sqrt{-c} t \cosh^{n-i-1} \sqrt{-c} t$ or of $1, \dots, x^{n-1}$. In any of these cases there is a (unique) differential equation of order n with constant coefficients satisfied by any such combination. The same equation would hold for h_t applied to the unit $(n-1)$ -vector at any y in M , and therefore also for A_M , since integration over M will commute with differentiation by t .

Added in proof. More general results have been announced in the author's paper, *Riemannian manifolds of finite order*, Bull. Amer. Math. Soc. **78** (1972) 200-201.

References

- [1] R. A. Holzinger & H. Wu. *A characterization of two-dimensional Riemannian manifolds of constant curvature*, Michigan Math. J. **17** (1970) 297-299.
- [2] H. Wu. *A characteristic property of the Euclidean plane*, Michigan Math. J. **16** (1969) 141-148.

AMERICAN UNIVERSITY