105. A Characterization of Smooth Banach Spaces

By Bruce CALVERT*) and Simeon REICH**)

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Our main purpose in this note is to show how (nonlinear) accretive operators can be used to characterize smooth (reflexive) Banach spaces. More precisely, we show that if a Banach space E is not smooth, then there is accretive $A \subset E \times E$ that satisfies the range condition such that $\mathrm{cl}\,(R(A))$ does not have the minimum property (see the definitions below). On the other hand, it is also true that if a reflexive space E is smooth and $A \subset E \times E$ is an accretive operator that satisfies the range condition, then $\mathrm{cl}\,(R(A))$ has the minimum property. Consequently, a reflexive Banach space E is smooth if and only if $\mathrm{cl}\,(R(A))$ has the minimum property for all accretive $A \subset E \times E$ that satisfy the range condition (Theorem 1). In fact, the same result is true if A is restricted to be of the form $I-T$, where T is nonexpansive. This provides an answer to a question of Pazy [3]. In addition, we characterize (finite-dimensional) smooth Banach spaces by using an invariance criterion for nonlinear semigroups (Theorem 2).

Let E be a real Banach space, and let I denote the identity operator. Recall that a subset A of $E \times E$ with domain $D(A)$ and range $R(A)$ is said to be *accretive* if $|x_1 - x_2| \leq |x_1 - x_2 + r(y_1 - y_2)|$ for all $[x_i, y_i] \in A$, $i=1, 2$, and $r>0$. The resolvent $J_r : R(I+rA) \to D(A)$ of A is defined by $J_r = (I+rA)^{-1}$. We denote the closure of a subset D of E by $\mathrm{cl}\,(D)$, its closed convex hull by $\mathrm{clco}\,(D)$ and its distance from a point x in E by $d(x, D)$. We shall say that A satisfies the *range condition* if $R(I+rA) \supset \mathrm{cl}\,(D(A))$ for all $r>0$. In this case, $-A$ generates a nonexpansive nonlinear semigroup $S : [0, \infty) \times \mathrm{cl}\,(D(A)) \to \mathrm{cl}\,(D(A))$ by the exponential formula: $S(t)x = \lim_{n\to\infty} (I+(t/n)A)^{-n}x$. A closed subset D of E is said to have the *minimum property* [3] if $d(O, \mathrm{clco}\,(D)) = d(O, D)$.

Recall that the norm of E is said to be Gâteaux differentiable (and E is said to be *smooth*) if $\lim_{t\to 0} (|x+ty| - |x|)/t$ exists for each x and y in $U = \{x \in E : |x| = 1\}$. The duality map from E into the family of nonempty subsets of its dual E^* is defined by $J(x) = \{x^* \in E^* : (x, x^*) = |x|^2 = |x^*|^2\}$. It is single-valued if and only if E is smooth. An

*) Department of Mathematics, The University of Auckland, Auckland, New Zealand.

**) Department of Mathematics, The University of Southern California, Los Angeles, California 90007.

operator $A \subset E \times E$ is accretive if and only if for each $x_i \in D(A)$ and each $y_i \in Ax_i$, $i=1, 2$, there exists $j \in J(x_1-x_2)$ such that $(y_1-y_2, j) \geq 0$.

Theorem 1. *A reflexive Banach space E is smooth if and only if* cl $(R(A))$ *has the minimum property for all accretive* $A \subset E \times E$ *that satisfy the range condition.*

Proof. Suppose that E is not smooth. Then there are points x and y on the unit sphere of E such that $\lim_{t \to 0} (|x+ty|-|x|)/t$ does not exist. Let F be the plane determined by x, y and the origin. F is a two-dimensional space that is not smooth. We may assume that $(0, c)$ is a point of norm 1 where the norm of F is not differentiable. Let $t_1 < 0 < t_2$, and let $g: [t_1, t_2] \to R$ be a real-valued function such that the slope of each one of the chords of its graph equals a tangential slope at $(0, c)$. We now define a set $A \subset F \times F$ by

$$A(a, b) = \begin{cases} (t_1, g(t_1)) & \text{if } a < 0 \\ \{(t, g(t)) : t_1 \leq t \leq t_2\} & \text{if } a = 0 \\ (t_2, g(t_2)) & \text{if } a > 0. \end{cases}$$

It is not difficult to check that A is accretive. It is, in fact, m-accretive (that is, $R(I+rA)=F$ for all positive r):

$$J_r(a, b) = \begin{cases} (a-rt_1, b-rg(t_1)) & \text{if } a < rt_1 \\ (0, \mathrm{b}-rg(a/r)) & \text{if } rt_1 \leq a \leq rt_2 \\ (a-rt_2, b-rg(t_2)) & \text{if } a > rt_2 \end{cases}$$

for all $(a, b) \in F$ and $r > 0$. If one chord of the graph of g passes through the origin, although $g(0) \neq 0$ (this is possible because there is more than one tangent at $(0, c)$), then we obtain $d(O, R(A)) = |(O, g(0))| > 0 = d(O, \text{clco}(R(A)))$. In other words, as a subset of $E \times E$, A is an accretive operator that satisfies the range condition but does not possess the minimum property.

Conversely, we assume now that E is a reflexive smooth Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, and $d = d(O, R(A))$. Let x and y belong to cl $(D(A))$, and let $t > s > 0$. Since $((1/s)(J_t y - J_s x) + ((1/s)-(1/t))(x-J_t y) + (1/t)(x-y), j) \geq 0$ for some j in $J(J_s x - J_t y)$, it follows that $|J_s x - J_t y| \leq (1-(s/t))|x-J_t y| + (s/t)|x-y|$. This inequality, in turn, implies that $(x-J_s x, j) \geq |x-J_t y|(s/t)(|x-J_t y|-|x-y|)$ for all j in $J(x-J_t y)$. Hence $((x-J_s x)/s, j_t) \geq |x-J_t y|^2/t^2 - |x-J_t y||x-y|/t^2$ for all $j_t \in J((x-J_t y)/t)$. Suppose that a subsequence of $\{j_t\}$ converges weakly to j and that a subsequence of $(x-J_s x)/s$ converges weakly to z as t an s tend to infinity. $((x-J_s)/s, j) \geq d^2$, $|j| = d$, $|z| = d$, and $(z, j) = d^2$. In other words, j belongs to $J(z)$. But the duality map J is single-valued because E is smooth. Consequently, the weak $\lim_{t \to \infty} J((x-J_t y)/t)$ exists and is independent of x and y. The result now follows by combining this fact with the proof of [5, Theorem 1].

Theorem 1 remains true if A is restricted to be of the form $I-T$, where T is nonexpansive. This fact provides an answer to a question of Pazy [3]. It can be seen by considering the Yosida approximation $A_1=I-J_1$ of A:

$$A_1(a,\ b)=\begin{cases}(t_1,\ g(t_1)) & \text{if } a<t_1\\ (a,\ g(a)) & \text{if } t_1\le a\le t_2\\ (t_2,\ g(t_2)) & \text{if } a>t_2.\end{cases}$$

It can also be shown [6] that in the second part of the proof of Theorem 1 it suffices to assume that E^* is strictly convex. The first part of the proof is related to an example in [2].

Theorem 2. *A finite-dimensional Banach space E is smooth if and only if for all accretive operators $A\subset E\times E$ that satisfy the range condition, any closed convex subset C of* cl $(D(A))$ *that is invariant under the semigroup S generated by $-A$ is also invariant under the resolvent J_r, $r>0$.*

Proof. If E is not smooth, we consider the operator A constructed in the first part of the proof of Theorem 1. The exponential formula shows that

$$S(t)(a,\ b)=\begin{cases}(a-t_1t,\ b-g(t_1)t) & \text{if } a<0,\ 0\le t\le a/t_1\\ (0,\ b-tg(0)) & \text{if } a=0,\ t\ge 0\\ (a-t_2t,\ b-g(t_2)t) & \text{if } a>0,\ 0\le t\le a/t_2.\end{cases}$$

We assume that $g(0)=0$ and let $(a,b)\in F$ with $a<0$. Then the half-line $y-b=(g(t_1)/t_1)(x-a)$, $x\le 0$, is invariant under S. We claim that if g is not linear (this is possible because there is more than one tangent at $(0,c)$), then this half-line is not invariant under J_r. Indeed, if it were invariant, we would obtain for $rt_1\le x\le 0$,

$$y-rg\left(\frac{x}{r}\right)-b=\frac{g(t_1)}{t_1}(-a),\ \frac{g(t_1)}{t_1}(x-a)-rg\left(\frac{x}{r}\right)=\frac{g(t_1)}{t_1}(-a),$$

and $g(t_1)/t_1=g(s)/s$ for all $t_1\le s\le 0$. In other words, g would be linear for ≤ 0.

Conversely, we observe that if C is invariant under S, then it is also invariant under $(I+(r/t)(I-S(t)))^{-1}$. Let $x\in C$ and $y_t=(I+(r/t)(I-J_t))^{-1}x$. Since E is finite-dimensional and smooth, the uniqueness part of [4, Theorem 2.1] shows that $\lim_{t\to 0+} y_t=\lim_{t\to 0+}(I+(r/t)(I-S(t)))^{-1}x$ belongs to C. We also have

$$((x-J_rx)/r-(x-y_t)/r,\ J(J_rx-J_ty_t))\ge 0,$$
$$|J_rx-J_ty_t|^2\le(y_t-J_ty_t,\ J(J_rx-J_ty_t)),$$

and $|J_rx-y_t|\le(2t/r)\,|x-y_t|$. Hence $\lim_{t\to 0+} y_t=J_rx$ and the proof is complete.

The first part of the proof of Theorem 2 is related to an observation in [1]. In the second part of the proof it is sufficient to assume that E is reflexive with a uniformly Gâteaux differentiable norm.

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