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A characterization of Smyth complete quasi-metric spaces via Caristi's fixed point theorem

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Abstract

We obtain a quasi-metric generalization of Caristi's fixed point theorem for a kind of complete quasi-metric spaces. With the help of a suitable modification of its proof, we deduce a characterization of Smyth complete quasi-metric spaces which provides a quasi-metric generalization of the well-known characterization of metric completeness due to Kirk. Some illustrative examples are also given. As an application, we deduce a procedure which allows to easily show the existence of solution for the recurrence equation of certain algorithms.

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1 Introduction and preliminaries

We start by recalling several notions and properties of the theory of quasi-metric spaces. Our basic references are [1] and [2].

By a quasi-metric on set X we mean a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$: (i) $x = y \Leftrightarrow d(x, y) = d(y, x) = 0$; (ii) $d(x, z) \leq d(x, y) + d(y, z)$.

A quasi-metric space is a pair (X, d) such that X is a set and d is a quasi-metric on X.

Given a quasi-metric d on X, the function d^{-1} defined by $d^{-1}(x, y) = d(y, x)$ is also a quasi-metric on X, called the conjugate of d, and the function d^s defined by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a metric on X.

Each quasi-metric d on X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x, r) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If τ_d is a T_1 topology on X, we say that (X, d) is a T_1 quasi-metric space.

Note that a quasi-metric space (X, d) is T_1 if and only if for each $x, y \in X$, condition $d(x, y) = 0$ implies $x = y$.

There exist many different notions of Cauchy net, Cauchy sequence and quasi-metric completeness in the literature (see, *e.g.*, [1–3]). For our purposes, here we will consider the following ones.

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A net $(x_\alpha)_{\alpha \in \Lambda}$ in a quasi-metric space (X, d) is called left K-Cauchy if for each $\varepsilon > 0$ there is $\alpha_\varepsilon \in \Lambda$ such that $d(x_\alpha, x_\beta) < \varepsilon$ whenever $\alpha_\varepsilon \leq \alpha \leq \beta$. The notion of a left K-Cauchy sequence is defined in the obvious manner.

We say that a quasi-metric space (X, d) is complete if every left K-Cauchy net is convergent for $\tau_{d^{-1}}$, and say that it is sequentially complete if every left K-Cauchy sequence is convergent for $\tau_{d^{-1}}$. (Note that our notion of (sequential) completeness of (X, d) coincides with the usual notion of right K-(sequential) completeness of (X, d^{-1}).)

A quasi-metric space (X, d) is Smyth complete provided that every left K-Cauchy net in (X, d) is convergent for τ_{d^s} (compare Definition 8 in [4], [5], p.454, *etc.*).

The following well-known result is a consequence of Definition 8 and Theorem 9 in [4] (see also [6], p.323, [7], p.347).

Proposition 1 *A quasi-metric space (X, d) is Smyth complete if and only if every left* K-Cauchy *sequence in (X, d) is convergent for τ_{d^s}.*

The following implications are also known and easy to check:

$$\text{Smyth complete} \quad \Rightarrow \quad \text{complete} \quad \Rightarrow \quad \text{sequentially complete.}$$

However, the converse implications do not hold, in general. For instance, the Sorgenfrey quasi-metric space (see, *e.g.*, [5], p.463 or Example 1.1.6 in [1]) provides a distinguished example of a complete T_1 quasi-metric space which is not Smyth complete, while Stoltenberg presented in Example 2.4 of [8] an example of a sequentially complete T_1 quasi-metric space which is not complete.

On the other hand, Caristi proved in 1976 the following important and well-known generalization of the Banach contraction principle.

Theorem 1 ([9]) *Let T be a self-mapping of a complete metric space (X, d). If there is a lower semicontinuous function $\varphi : X \to [0, \infty)$ satisfying*

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx)$$

for all $x \in X$, then T has a fixed point in X.

Kirk showed in [10] that the validity of Caristi's fixed point theorem in a metric space characterizes its completeness. More exactly, he proved the following.

Theorem 2 ([10]) *For a metric space (X, d), the following conditions are equivalent:*
  (1) *(X, d) is complete.*
  (2) *If T is a self-mapping of X such that there is a lower semicontinuous function $\varphi : X \to [0, \infty)$ satisfying $d(x, Tx) \leq \varphi(x) - \varphi(Tx)$ for all $x \in X$, then T has a fixed point in X.*

Extensions and generalizations of Theorems 1 and 2 to partial metric spaces, cone metric spaces, quasi-metric spaces and probabilistic metric spaces have been obtained by several authors (see, *e.g.*, [11–20]). In particular, Cobzaş ([15], Theorem 2.3) proved, among other interesting results, the following quasi-metric generalization of Caristi's fixed point theorem.

Theorem 3 ([15]) *Let T be a self-mapping of a sequentially complete T_1 quasi-metric space (X, d). If there is a function $\varphi : X \to [0, \infty)$ which is lower semicontinuous for $\tau_{d^{-1}}$ and satisfies*

$$d(x, Tx) \le \varphi(x) - \varphi(Tx)$$

for all $x \in X$, then T has a fixed point in X.

Since complete and Smyth complete non-T_1 quasi-metric spaces provide efficient tools in several areas as asymmetric functional analysis, domain theory, theoretical computer science, complexity analysis of algorithms defined by recurrence equations, *etc.* (see, *e.g.*, [1, 4, 5, 7, 21, 22] and their references), it seems natural to discuss the question of generalizing Theorem 3 to (non-necessarily T_1) quasi-metric spaces. In this direction, we shall give an example of a sequentially complete quasi-metric space for which Theorem 3 does not hold. We shall show that, nevertheless, Theorem 3 remains valid for complete quasi-metric spaces. A suitable and slight modification of the proof of that result will be used to deduce a characterization of Smyth complete quasi-metric spaces which provides a generalization to the quasi-metric framework of Kirk's characterization of metric completeness. As an application, we obtain a procedure which allows to easily deduce the existence of solution for the recurrence equation of certain algorithms.

2 Results and examples

In order to simplify the terminology and the statements of our results, we shall use the following notions.

A self-mapping T of a quasi-metric space (X, d) will be called a d-Caristi mapping (resp. a d^s-Caristi mapping) on (X, d) if there is a function $\varphi : X \to [0, \infty)$ which is lower semicontinuous for $\tau_{d^{-1}}$ (resp. for τ_{d^s}) and satisfies $d(x, Tx) \le \varphi(x) - \varphi(Tx)$ for all $x \in X$.

Clearly, every d-Caristi mapping is a d^s-Caristi mapping. The following example shows that the converse is not true in general.

Example 1 Let d be the quasi-metric on the set $\mathbb{N}$ of all positive integer numbers, given by $d(x, x) = 0$ for all $x \in \mathbb{N}$ and $d(x, y) = 1/x$ for all $x, y \in \mathbb{N}$ with $x \ne y$. Clearly $(\mathbb{N}, d)$ is a T_1 quasi-metric space such that τ_d, and hence τ_{d^s} is the discrete topology on $\mathbb{N}$. Define $T : \mathbb{N} \to \mathbb{N}$ as $Tx = 2x$ for all $x \in \mathbb{N}$. Then $d(x, Tx) = 1/x = \varphi(x) - \varphi(Tx)$, where $\varphi : \mathbb{N} \to [0, \infty)$ is defined as $\varphi(x) = 2/x$ for all $x \in \mathbb{N}$. Since τ_{d^s} is the discrete topology on $\mathbb{N}$, φ is lower semicontinuous for τ_{d^s} and thus T is a d^s-Caristi mapping on $(\mathbb{N}, d)$. Finally, suppose that T is also a d-Caristi mapping. Then there exists a function $\varphi : \mathbb{N} \to [0, \infty)$ which is lower semicontinuous for $\tau_{d^{-1}}$ and satisfies $d(x, 2x) = 1/x \le \varphi(x) - \varphi(2x)$ for all $x \in \mathbb{N}$. We easily deduce that $\varphi(1) \ge 1 + \varphi(2^x)$ for all $x \in \mathbb{N}$, which contradicts that φ is a lower semicontinuous function for $\tau_{d^{-1}}$ because the sequence $(2^n)_{n \in \mathbb{N}}$ converges to 1 for $\tau_{d^{-1}}$.

Our next example, based on Example 2.1 in [5], shows that condition T_1 cannot be removed in Theorem 3.

Example 2 Let $(\mathcal{A}, d)$ be the non-T_1 quasi-metric space such that $\mathcal{A}$ is the family of all nonempty countable subsets of the set $\mathbb{R}$ of all real numbers, and d is the quasi-metric on $\mathcal{A}$ defined as $d(A, B) = 0$ if $A \subseteq B$, and $d(A, B) = 1$ otherwise. Let $(A_n)_{n \in \mathbb{N}}$ be a left K-Cauchy

sequence in $(\mathcal{A}, d)$. Assume, without loss of generality, that $d(A_n, A_m) = 0$ whenever $n \leq m$, *i.e.*, $A_n \subseteq A_m$ whenever $n \leq m$. Since $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$ and $d(A_n, \bigcup_{n \in \mathbb{N}} A_n) = 0$ for all $n \in \mathbb{N}$, we deduce that $(\mathcal{A}, d)$ is sequentially complete. Now let

$$\Lambda = \{A \in \mathcal{A} : A \text{ is a nonempty finite subset of } \mathbb{R} \text{ consisting of irrational numbers}\}$$

ordered by inclusion. Then the net $(A)_{A \in \Lambda}$ is left K-Cauchy in $(\mathcal{A}, d)$ (see Example 2.1 in [5]) but it does not converge for $\tau_{d^{-1}}$ because the elements of $\mathcal{A}$ are countable subsets of $\mathbb{R}$. We conclude that $(\mathcal{A}, d)$ is not complete.

However, we have the following extension of Theorem 3 whose proof is based on a classical technique used by Kirk [10], which is inspired in the partial order of Brøndsted [23, 24].

Theorem 4 *Every d-Caristi mapping on a complete quasi-metric space (X, d) has a fixed point in X.*

Proof Let (X, d) be a complete quasi-metric space and let $T : X \to X$ be a d-Caristi mapping on (X, d). Then there exists a function $\varphi : X \to [0, \infty)$ which is lower semicontinuous for $\tau_{d^{-1}}$ and satisfies

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx)$$

for all $x \in X$. As in the classical metric case, define a binary relation $\preceq$ on X by

$$x \preceq y \quad \Longleftrightarrow \quad d(x, y) \leq \varphi(x) - \varphi(y)$$

for all $x, y \in X$. Clearly $\preceq$ is a partial order on X. Note also that $x \preceq Tx$ for all $x \in X$.

We shall prove that every (nonempty) linearly ordered subset of the partially ordered set $(X, \preceq)$ has an upper bound. Indeed, let A be a (nonempty) linearly ordered subset of X. We show that the net $(x_x)_{x \in A}$ is a left K-Cauchy net in (X, d) where we have defined $x_x := x$ for all $x \in A$. To this end, put $r = \inf_{x \in A} \varphi(x)$. Given an arbitrary $\varepsilon > 0$, choose $x \in A$ such that $\varphi(x) < r + \varepsilon$. Thus, for any $y, z \in A$ with $x \preceq y \preceq z$, we obtain

$$d(y, z) \leq \varphi(y) - \varphi(z) \leq \varphi(x) - \varphi(z) < r + \varepsilon - r = \varepsilon.$$

Consequently, $(x_x)_{x \in A}$ is a left K-Cauchy net in (X, d), and hence it converges, for $\tau_{d^{-1}}$, to some $p \in X$. Fix $x \in A$ and let $\varepsilon > 0$ be arbitrary. Then there is $y \in A$ such that $d(z, p) < \varepsilon$ and $\varphi(p) - \varphi(z) < \varepsilon$ whenever $z \in A$ and $y \preceq z$. Choose $z_0 \in A$ with $x \preceq z_0$ and $y \preceq z_0$. Hence

$$d(x, p) \leq d(x, z_0) + d(z_0, p) < \varphi(x) - \varphi(z_0) + \varepsilon$$

$$< \varphi(x) - \varphi(p) + 2\varepsilon.$$

Since ε is arbitrary, we deduce that $d(x, p) \leq \varphi(x) - \varphi(p)$, *i.e.*, $x \preceq p$, so p is an upper bound of A. It follows from Zorn's lemma that $(X, \preceq)$ has a maximal element, say a. Since $a \preceq Ta$, we conclude that $a = Ta$, so a is a fixed point of T. The proof is finished. $\square$

Of course, Caristi's fixed point theorem is a consequence of Theorem 4 when (X, d) is a metric space. Next we present two examples of complete quasi-metric spaces (X, d) with appropriate d-Caristi mappings, for which Caristi's fixed point theorem cannot be applied to the metric space (X, d^s).

Example 3 Let $X = \mathbb{N} \cup \{\infty\}$. Define a nonnegative real-valued function d on $X \times X$ by $d(\infty, \infty) = 0$, $d(x, y) = |1/x - 1/y|$ if $x, y \in \mathbb{N}$, $d(x, \infty) = 1/x$ and $d(\infty, x) = 1$ for all $x \in \mathbb{N}$. It is easily seen that (X, d) is a complete T_1 quasi-metric space (in fact, note $(X, \tau_{d^{-1}})$ is a compact topological space). Define $T : X \to X$ as $T\infty = \infty$, and $Tx = x^2$ for all $x \in \mathbb{N}$. Now define $\varphi : X \to [0, \infty)$ as $\varphi(\infty) = 0$, and $\varphi(x) = 1/x$ for all $x \in \mathbb{N}$. Then φ is clearly a lower semicontinuous function for $\tau_{d^{-1}}$. Since $d(\infty, T\infty) = d(1, T1) = 0$, and for every $x \in X \setminus \{1, \infty\}$,

$$d(x, Tx) = \frac{1}{x} - \frac{1}{x^2} = \varphi(x) - \varphi(Tx),$$

we conclude that T is a d-Caristi mapping on (X, d). Hence, we can apply Theorem 4 to this case. In fact, T has 1 and ∞ as fixed points. However, we cannot apply Caristi's fixed point theorem to the metric space (X, d^s) because it is not complete. Indeed, $(x)_{x \in \mathbb{N}}$ is a Cauchy sequence in (X, d^s) that does not converge for τ_{d^s}.

In the above example the metric space (X, d^s) is not complete. Now, we give an example of a complete quasi-metric space (X, d) where the metric space (X, d^s) is complete and there is a d-Caristi mapping on (X, d) which is not a Caristi mapping for the metric space (X, d^s).

Example 4 As in Example 3, let $X = \mathbb{N} \cup \{\infty\}$. Define a nonnegative real-valued function d on $X \times X$ by $d(x, y) = 0$ if $x \leq y$, and $d(x, y) = y$ if $y < x$ (here, $\leq$ denotes the usual order on X). It is routine to check that (X, d) is a complete quasi-metric space (note that every net in X converges to ∞ for $\tau_{d^{-1}}$). Define $T : X \to X$ as $Tx = x + 1$ for all $x \in \mathbb{N}$ and $T\infty = \infty$. Then $d(x, Tx) = 0$ for all $x \in X$, so that T is trivially a d-Caristi mapping on (X, d). Hence, we can apply Theorem 4. Finally, suppose that there exists a lower semicontinuous function, for τ_{d^s}, $\varphi : X \to [0, \infty)$, such that $d^s(x, Tx) \leq \varphi(x) - \varphi(Tx)$ for all $x \in X$. Then

$$d^s(x, x + 1) = x + 1 \leq \varphi(x) - \varphi(x + 1)$$

for all $x \in \mathbb{N}$. We deduce that $\varphi(1) = \infty$, a contradiction. Hence, we cannot apply the classical Caristi fixed point theorem in this case.

Observe that the aforementioned example of Stoltenberg and Example 4 (or Example 3) above show that Theorems 3 and 4 are independent of each other.

Although we do not know whether the converse of Theorem 4 holds, *i.e.*, if Kirk's theorem can be generalized to complete quasi-metric spaces, we are going to show that it is possible to obtain such a generalization for Smyth complete quasi-metric spaces. To this end, the following essentially well-known fact (see, *e.g.*, Proposition 1.2.4 in [1]) will be useful.

Proposition 2 *Let $(x_n)_{n \in \mathbb{N}}$ be a left K-Cauchy sequence in a quasi-metric space (X, d). If $(x_n)_{n \in \mathbb{N}}$ has a subsequence convergent to $x \in X$ for τ_{d^s}, then $(x_n)_{n \in \mathbb{N}}$ converges to x for τ_{d^s}.*

Theorem 5 *A quasi-metric space* (X, d) *is Smyth complete if and only if every* d^s*-Caristi mapping on* (X, d) *has a fixed point in* X.

Proof Suppose that (X, d) is a Smyth complete quasi-metric space, and let T be a d^s-Caristi mapping on (X, d). Then there exists a function $\varphi : X \to [0, \infty)$ which is lower semicontinuous for τ_{d^s} and satisfies $d(x, Tx) \le \varphi(x) - \varphi(Tx)$ for all $x \in X$. Exactly as in the proof of Theorem 4, we construct a left K-Cauchy net in (X, d), which converges for τ_{d^s} to an element $p \in X$ by Smyth completeness of (X, d). Finally, we deduce that p is a fixed point of T again as in the proof of Theorem 4 and taking into account that φ is now lower semicontinuous for τ_{d^s}.

Conversely, it will be enough to prove, by Proposition 1, that every left K-Cauchy sequence in (X, d) converges for τ_{d^s}. Assume the contrary. Then there exists a left K-Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ in (X, d) which is not convergent for τ_{d^s}. For each $k \in \mathbb{N}$, there exists $n_k \ge k$ such that $d(x_{n_k}, x_n) < 2^{-(k+1)}$ for all $n \ge n_k$. Therefore $d(x_{n_k}, x_{n_{k+1}}) < 2^{-(k+1)}$ for all $k \in \mathbb{N}$. Put $y_k := x_{n_k}$ for all $k \in \mathbb{N}$. Then, by Proposition 2, we can suppose, without loss of generality, that $y_k \ne y_j$ whenever $k \ne j$, and that the sequence $\{y_k : k \in \mathbb{N}\}$ does not have any convergent subsequence for τ_{d^s}.

We want to show that the self-mapping T of X given by $Ty_k = y_{k+1}$ for all $k \in \mathbb{N}$, and $Tx = y_1$ for all $x \notin \{y_k : k \in \mathbb{N}\}$, is a d^s-Caristi mapping. To this end, construct a function $\varphi : X \to [0, \infty)$ as follows: $\varphi(y_k) = 2^{-k}$ for all $k \in \mathbb{N}$, and $\varphi(x) = d^s(x, y_1) + 1/2$ whenever $x \notin \{y_k : k \in \mathbb{N}\}$. Since, for each $k \in \mathbb{N}$, $\varphi(y_k) < \varphi(x)$ whenever $x \notin \{y_k : k \in \mathbb{N}\}$, and the function $x \to d^s(x, y_1)$ is continuous for τ_{d^s}, we immediately deduce that φ is lower semicontinuous for τ_{d^s}. Moreover, we have

$$d(y_k, Ty_k) = d(y_k, y_{k+1}) < 2^{-(k+1)} = \varphi(y_k) - \varphi(Ty_k)$$

for all $k \in \mathbb{N}$, and

$$d(x, Tx) = d(x, y_1) \le d^s(x, y_1) = \varphi(x) - \varphi(Tx)$$

for all $x \notin \{y_k : k \in \mathbb{N}\}$, so T is a d^s-Caristi mapping on (X, d). However, T has no fixed point. This contradiction concludes the proof. □

As in the metric case, we are going to deduce a multivalued version of Theorem 5.

Given a quasi-metric space (X, d), we denote by $\mathcal{P}_0(X)$ the collection of all nonempty subsets of X. A multivalued mapping $T : X \to \mathcal{P}_0(X)$ will be called d^s-Caristi on (X, d) if there is a function $\varphi : X \to [0, \infty)$ which is lower semicontinuous for τ_{d^s} and satisfies the following condition: For each $x \in X$, there exists $y_x \in Tx$ such that $d(x, y_x) \le \varphi(x) - \varphi(y_x)$.

As usual, we say that a point $z \in X$ is a fixed point of $T : X \to \mathcal{P}_0(X)$ if $z \in Tz$.

Corollary *A quasi-metric space* (X, d) *is Smyth complete if and only if every* d^s*-Caristi multivalued mapping on* (X, d) *has a fixed point.*

Proof Suppose that (X, d) is Smyth complete, and let $T : X \to \mathcal{P}_0(X)$ be a d^s-Caristi multivalued mapping. Then there is a function $\varphi : X \to [0, \infty)$ which is lower semicontinuous for τ_{d^s} and satisfies that for each $x \in X$ there exists $y_x \in Tx$ such that $d(x, y_x) \le \varphi(x) - \varphi(y_x)$. Define a self-mapping f on X as follows: $fx = y_x$ for all $x \in X$. Obviously f is a d^s-Caristi

mapping on (X, d), so, by Theorem 5, there is $z \in X$ such that $z = fz$. Therefore $z = y_z$. Since $y_z \in Tz$, we conclude that z is a fixed point of T.

Conversely, suppose that every d^s-Caristi multivalued mapping on (X, d) has a fixed point. Then every d^s-Caristi mapping on (X, d) has a fixed point, so (X, d) is Smyth complete by Theorem 5. $\qquad\square$

Note that if (X, d) is a quasi-metric space and T is a self-mapping of X such that $d(x, Tx) = 0$ for all $x \in X$, then T is a d^s-Caristi mapping on (X, d). If, in addition, (X, d) is Smyth complete, then T has a fixed point by Theorem 5. Our next example illustrates this situation.

Example 5 Let Σ be a nonempty alphabet. Denote by Σ^∞ the set of all finite and infinite words (sequences) over Σ, and denote by ϕ the empty word. For each $x, y \in \Sigma^\infty$, we define $x \sqcap y$ as the longest common prefix of x and y, and for each $x \in \Sigma^\infty$, we denote by $\ell(x)$ the length of x. Then $\ell(x) \in [1, \infty]$ whenever $x \neq \phi$ and $\ell(\phi) = 0$. Now, for each $x, y \in \Sigma^\infty$, let $d(x, y) = 0$ if x is a prefix of y, and $d(x, y) = 2^{-\ell(x \sqcap y)}$ otherwise. Then d is a quasi-metric on Σ^∞ [6, 25]. In fact, the quasi-metric space (Σ^∞, d) is Smyth complete [5], Example 3.1. Define $T : \Sigma^\infty \to \Sigma^\infty$ as follows: For each $x \in \Sigma^\infty$, Tx is an element of Σ^∞ such that x is a prefix of Tx with $\ell(Tx) = \ell(x) + 1$. Then $d(x, Tx) = 0$ for all $x \in \Sigma^\infty$. By Theorem 5, T has a fixed point. In fact, $Tx = x$ if and only if $\ell(x) = \infty$.

Observe that if (X, d) is a non-Smyth complete quasi-metric space such that (X, d^s) is complete, we can apply Caristi's fixed point theorem to (X, d^s). However, by Theorem 5, there exists a d^s-Caristi mapping on (X, d) without fixed point. We conclude this section with an example illustrating this fact.

Example 6 Let d be the quasi-metric on $\mathbb{R}$ given by $d(x, y) = y - x$ if $x \leq y$, and $d(x, y) = 1$ if $x > y$. Then $(\mathbb{R}, d)$ is the Sorgenfrey quasi-metric space. Since $d^s(x, y) \geq 1$ for all $x, y \in \mathbb{R}$ with $x \neq y$, we deduce that the metric space $(\mathbb{R}, d^s)$ is complete and τ_{d^s} is the discrete topology on $\mathbb{R}$. As we indicated in Section 1, $(\mathbb{R}, d)$ is not Smyth complete (indeed, note that the sequence $((n-1)/n)_{n \in \mathbb{N}}$ is left K-Cauchy but it does not converge for τ_{d^s}). Define $T : \mathbb{R} \to \mathbb{R}$ as $Tx = 0$ for all $x > 0$, $T0 = -1$, and $Tx = x/2$ for all $x < 0$. Although T has no fixed point, we show that it is a d^s-Caristi mapping on $(\mathbb{R}, d)$. To this end, define $\varphi : \mathbb{R} \to [0, \infty)$ as $\varphi(x) = 3$ for all $x > 0$, $\varphi(0) = 2$, and $\varphi(x) = -x$ for all $x < 0$. Obviously φ is lower semicontinuous for τ_{d^s}. Moreover, for $x > 0$, we obtain

$$d(x, Tx) = d(x, 0) = 1 = \varphi(x) - \varphi(Tx).$$

For $x = 0$, we obtain

$$d(x, Tx) = d(0, -1) = 1 = \varphi(x) - \varphi(Tx),$$

and for $x < 0$,

$$d(x, Tx) = d\left(x, \frac{x}{2}\right) = -\frac{x}{2} = \varphi(x) - \varphi(Tx).$$

Hence T is a d^s-Caristi mapping on (X, d) without fixed point. Finally, observe that for $x = -1$ one has

$$d^s(x, Tx) = 1 > \frac{1}{2} = \varphi(x) - \varphi(Tx).$$

3 An application

In this section we shall apply Theorem 5 to obtaining a general fixed point theorem in the setting of the complexity space, from which we shall deduce, in a unified and fast way, the existence of solution for a large class of algorithms defined by recurrence equations that includes Hanoi, Largetwo (average case), and Quicksort (worst case), (see, *e.g.*, [26] for a detailed study of these algorithms).

Let us recall that the so-called complexity space was introduced by Schellekens in [27] to the development of a topological foundation for the complexity analysis of algorithms and programs. Further contributions to the study of this space and its applications may be found in [7, 22, 28–30], *etc.*

The complexity space is the quasi-metric space $(\mathcal{C}, d_{\mathcal{C}})$, where

$$\mathcal{C} = \left\{ f : \mathbb{N} \to (0, \infty] : \sum_{n=1}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty \right\},$$

and $d_{\mathcal{C}}$ is the quasi-metric on $\mathcal{C}$ given by

$$d_{\mathcal{C}}(f, g) = \sum_{n=1}^{\infty} 2^{-n} \max\left(\frac{1}{g(n)} - \frac{1}{f(n)}, 0 \right)$$

for all $f, g \in \mathcal{C}$. (We adopt the convention that $1/\infty = 0$.)

The set $\{ f \in \mathcal{C} : f(n) < \infty \text{ for all } n \in \mathbb{N} \}$ is denoted by $\mathcal{C}_0$.

The elements of $\mathcal{C}$ are called complexity functions. According to Schellekens [27], p.540, given two complexity functions f and g, the numerical value $d_{\mathcal{C}}(f, g)$ (the complexity distance from f to g) can be interpreted as the relative progress made in lowering the complexity by replacing any program P with complexity function f by any program Q with complexity function g. Therefore, condition $d_{\mathcal{C}}(f, g) = 0$, with $f \neq g$, can be read as the program P is at least as efficient as the program Q because $d_{\mathcal{C}}(f, g) = 0$ if and only if $f(n) \leq g(n)$ for all $n \in \mathbb{N}$. Obviously, the metric $(d_{\mathcal{C}})^s$ is not able to give this information since in the case that $d_{\mathcal{C}}(f, g) = 0$, with $f \neq g$, we deduce that $d_{\mathcal{C}}(g, f) = (d_{\mathcal{C}})^s(f, g)$, and thus the last measure does not indicate that program is more efficient. However, we know that the program with complexity function f is more efficient than the one with complexity function g (see [27], p.541).

Now let c and a be positive real constants and $h \in \mathcal{C}_0$. Define

$$\mathcal{C}_{cah} = \left\{ f \in \mathcal{C} : f(1) = c \text{ and } f(n) \geq af(n-1) + h(n) \text{ for all } n \geq 2 \right\}.$$

Observe that $\mathcal{C}_{cah} \neq \emptyset$ since the complexity function f_1 defined by $f_1(1) = c$ and $f_1(n) = \infty$ for all $n \geq 2$ clearly belongs to $\mathcal{C}_{cah}$.

The restriction of the quasi-metric $d_{\mathcal{C}}$ to $\mathcal{C}_{cah}$ will be denoted by $d_{\mathcal{C}_{cah}}$.

The following auxiliary results will be useful in the proof of the main result of this section (Theorem 6 below).

Lemma 1 *Let $(f_k)_{k \in \mathbb{N}}$ be a sequence in $\mathcal{C}$ such that $\lim_{k \to \infty}(d_{\mathcal{C}})^s(f, f_k) = 0$ for some $f \in \mathcal{C}$, and let $m \in \mathbb{N}$.*

  (a) *If $f(m) < \infty$, then $f_k(m) < \infty$ eventually, and $\lim_{k \to \infty} f_k(m) = f(m)$.*

  (b) *$f(m) = \infty$ if and only if $\lim_{k \to \infty} f_k(m) = \infty$.*

Proof Since $\lim_{k \to \infty}(d_{\mathcal{C}})^s(f, f_k) = 0$, for each $\varepsilon > 0$, there is $k_{\varepsilon} \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} 2^{-n} \left| \frac{1}{f(n)} - \frac{1}{f_k(n)} \right| < \varepsilon$$

for all $k \geq k_{\varepsilon}$. In particular

$$2^{-m} \left| \frac{1}{f(m)} - \frac{1}{f_k(m)} \right| < \varepsilon \tag{1}$$

for all $k \geq k_{\varepsilon}$.

Suppose that $f(m) < \infty$. Taking $\varepsilon = 2^{-m}/f(m)$, it follows from (1) that $f_k(m) < \infty$ for all $k \geq k_{\varepsilon}$. Hence $\lim_{k \to \infty} f_k(m) = f(m)$ by Proposition 2 of [7]. Thus, we have shown (a).

If $f(m) = \infty$, relation (1) gives $2^{-m}/\varepsilon < f_k(m)$ for all $k \geq k_{\varepsilon}$. Since $\varepsilon > 0$ is chosen arbitrarily, we deduce that $\lim_{k \to \infty} f_k(m) = \infty$. Conversely, if $\lim_{k \to \infty} f_k(m) = \infty$, again it follows from (1) that $1/f(m) = 0$, *i.e.*, $f(m) = \infty$. Thus, we have shown (b). □

Lemma 2 ([22]) *The quasi-metric space $(\mathcal{C}, d_{\mathcal{C}})$ is Smyth complete.*

Lemma 3 *Let c and a be positive real constants and $h \in \mathcal{C}_0$. Then the quasi-metric space $(\mathcal{C}_{cah}, d_{\mathcal{C}_{cah}})$ is Smyth complete.*

Proof We first show that $\mathcal{C}_{cah}$ is a closed subset of the metric space $(\mathcal{C}, (d_{\mathcal{C}})^s)$. Indeed, let $(f_k)_{k \in \mathbb{N}}$ be a sequence in $\mathcal{C}_{cah}$ and $f \in \mathcal{C}$ such that $\lim_{k \to \infty}(d_{\mathcal{C}})^s(f, f_k) = 0$. We shall show that $f(1) = c$ and $f(m) \geq af(m-1) + h(m)$ whenever $m \geq 2$.

To this end, we distinguish the following cases.

Case 1. $m = 1$. Then $f_k(1) = c$ for all $k \in \mathbb{N}$, so by Lemma 1(b), $f(1) < \infty$. Then $f(1) = c$ by Lemma 1(a).

Case 2. $m > 1$ and $f(m) = \infty$. Then $f(m) \geq af(m-1) + h(m)$, obviously.

Case 3. $m > 1$ and $f(m) < \infty$. Then, by Lemma 1(a), there is $k_0 \in \mathbb{N}$ such that $f_k(m) < \infty$ for all $k \geq k_0$, and $\lim_{k \to \infty} f_k(m) = f(m)$. From this equality and the fact that $f_k \in \mathcal{C}_{cah}$, we deduce the existence of $k_1 \geq k_0$ such that for each $k \geq k_1$,

$$1 + f(m) \geq f_k(m) \geq af_k(m-1) + h(m). \tag{2}$$

Consequently, $f_k(m-1) < \infty$ for all $k \geq k_1$, and by Lemma 1(b), $f(m-1) < \infty$ (otherwise, $\lim_k f_k(m-1) = \infty$, which contradicts (2)). Therefore, we also have $\lim_{k \to \infty} f_k(m-1) = f(m-1)$, by Lemma 1(a).

Now choose an arbitrary $\varepsilon > 0$. Then there exists $k_{\varepsilon} \in \mathbb{N}$ such that

$$\left| f_k(m-1) - f(m-1) \right| < \varepsilon \quad \text{and} \quad \left| f_k(m) - f(m) \right| < \varepsilon$$

for all $k \geq k_\varepsilon$. Hence

$$\varepsilon + f(m) > f_k(m) \geq af_k(m-1) + h(m) \geq a\big(f(m-1) + \varepsilon\big) + h(m)$$

for all $k \geq k_\varepsilon$. Thus $\varepsilon + f(m) > a(f(m-1) + \varepsilon) + h(m)$ for any $\varepsilon > 0$, so $f(m) \geq af(m-1) + h(m)$. Consequently, $f \in \mathcal{C}_{cah}$, and hence $\mathcal{C}_{cah}$ is closed in the metric space $(\mathcal{C}, (d_{\mathcal{C}})^s)$. Then $(\mathcal{C}_{cah}, d_{\mathcal{C}_{cah}})$ is Smyth complete by Lemma 2. $\qquad\square$

Theorem 6 *Let c and a be positive real constants with $a \geq 1$, let $h \in \mathcal{C}_0$, and let Ψ be the mapping on $\mathcal{C}_{cah}$ defined as*

$$\Psi(f)(n) = \begin{cases} c & \text{if } n = 1, \\ f(n-1) + h(n) & \text{if } n \geq 2. \end{cases} \tag{3}$$

Then the following hold:
 (A) *Ψ is a self-mapping on $\mathcal{C}_{cah}$.*
 (B) *For each $f \in \mathcal{C}_{cah}$,*

$$d_{\mathcal{C}_{cah}}(f, \Psi f) = \varphi(f) - \varphi(\Psi f),$$

   where $\varphi : \mathcal{C}_{cah} \to [0, \infty)$ is the lower semicontinuous function for $\tau_{(d_{\mathcal{C}_{cah}})^s}$ given by

$$\varphi(f) = \frac{a+1}{2ac} - \sum_{n=1}^{\infty} 2^{-n} \frac{1}{f(n)}$$

   for all $f \in \mathcal{C}_{cah}$.
 (C) *Ψ has a fixed point in $\mathcal{C}_{cah}$.*

Proof (A) Let $f \in \mathcal{C}_{cah}$. Then $\Psi f(1) = c$ by definition of Ψ. We also have $\Psi f(2) = af(1) + h(2) = a\Psi f(1) + h(2)$.
   Now let $n > 2$. Then

$$\begin{aligned} \Psi f(n) &= af(n-1) + h(n) \\ &\geq a\big[af(n-2) + h(n-1)\big] + h(n) \\ &= a\Psi f(n-1) + h(n). \end{aligned}$$

We conclude that $\Psi f \in \mathcal{C}_{cah}$.
   (B) We first observe that, in fact, $\varphi(f) \geq 0$ for all $f \in \mathcal{C}_{cah}$. Indeed, since $a \geq 1$, we have $f(n) \geq f(n-1)$ for all $n \geq 2$, and thus $f(n) \geq f(2) \geq ac$ for all $n \geq 2$. Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} 2^{-n} \frac{1}{f(n)} &= \frac{1}{2c} + \sum_{n=2}^{\infty} 2^{-n} \frac{1}{f(n)} \\ &\leq \frac{1}{2c} + \frac{1}{2ac} = \frac{a+1}{2ac}. \end{aligned}$$

Let now $f \in \mathcal{C}_{cah}$ and $(f_k)_{k \in \mathbb{N}}$ be a sequence in $\mathcal{C}_{cah}$ such that $\lim_{k \to \infty} (d_{\mathcal{C}_{cah}})^s(f, f_k) = 0$. Since

$$\varphi(f) - \varphi(f_k) = \sum_{n=1}^{\infty} 2^{-n} \frac{1}{f_k(n)} - \sum_{n=1}^{\infty} 2^{-n} \frac{1}{f(n)}$$

$$\leq \sum_{n=1}^{\infty} 2^{-n} \left| \frac{1}{f_k(n)} - \frac{1}{f(n)} \right| \leq 2(d_{\mathcal{C}_{cah}})^s(f, f_k),$$

we deduce that $\varphi(f) \leq \liminf_{k \to \infty} \varphi(f_k)$. Therefore φ is lower semicontinuous for $\tau_{(d_{\mathcal{C}_{cah}})^s}$.

Furthermore, for each $f \in \mathcal{C}_{cah}$, we have $f \geq \Psi f$, and hence

$$d_{\mathcal{C}_{cah}}(f, \Psi f) = \sum_{n=1}^{\infty} 2^{-n} \max\left(\frac{1}{\Psi f(n)} - \frac{1}{f(n)}, 0 \right) = \sum_{n=1}^{\infty} 2^{-n} \left(\frac{1}{\Psi f(n)} - \frac{1}{f(n)} \right)$$

$$= \sum_{n=1}^{\infty} 2^{-n} \frac{1}{\Psi f(n)} - \sum_{n=1}^{\infty} 2^{-n} \frac{1}{f(n)} = \varphi(f) - \varphi(\Psi f).$$

(C) From (B) we deduce that Ψ is a $(d_{\mathcal{C}_{cah}})^s$-Caristi mapping on $(\mathcal{C}_{cah}, d_{\mathcal{C}_{cah}})$. Then Ψ has a fixed point by Lemma 3 and Theorem 5. □

It follows from Theorem 6 that those algorithms defined by recurrence equations, whose associated functional is a mapping Ψ of type (3), admit a solution. We conclude the paper by applying this fact to deduce the existence of solution for the three algorithms mentioned at the beginning of this section.

Example 7 The algorithm Hanoi solves the celebrated Towers of Hanoi problem. The running time of computing of this algorithm is the solution of the recurrence equation $S : \mathbb{N} \to (0, \infty)$ given by

$$S(n) = \begin{cases} c & \text{if } n = 1, \\ 2S(n-1) + d & \text{if } n \geq 2, \end{cases}$$

with $c, d > 0$ (see, *e.g.*, [26]). The functional Ψ_S naturally associated to S is defined as

$$\Psi_S(f)(n) = \begin{cases} c & \text{if } n = 1, \\ 2f(n-1) + d & \text{if } n \geq 2. \end{cases}$$

Clearly Ψ_S is a mapping of type (3) for $a = 2$, and $h \in \mathcal{C}_0$ defined as $h(n) = d$ for all $n \in \mathbb{N}$. By Theorem 6, there exists $f_S \in \mathcal{C}_{cah}$ such that $f_S = \Psi f_S$. Hence f_S is a solution of the recurrence equation S.

Example 8 The algorithm Largetwo is a typical example of average case behavior whose running time of computing is the solution of the recurrence equation $S : \mathbb{N} \to (0, \infty)$ given by

$$S(n) = \begin{cases} c & \text{if } n = 1, \\ S(n-1) + 2 - 1/n & \text{if } n \geq 2, \end{cases}$$

with $c > 0$ (see, *e.g.*, [26]). The functional Ψ_S naturally associated to S is defined as

$$\Psi_S(f)(n) = \begin{cases} c & \text{if } n = 1, \\ f(n-1) + 2 - 1/n & \text{if } n \geq 2. \end{cases}$$

Clearly Ψ_S is a mapping of type (3) for $a = 1$, and $h \in \mathcal{C}_0$ defined as $h(n) = 2 - 1/n$ for all $n \in \mathbb{N}$. By Theorem 6, there exists $f_S \in \mathcal{C}_{cah}$ such that $f_S = \Psi f_S$. Hence f_S is a solution of the recurrence equation S.

Example 9 The running time of computing of the well-known algorithm Quicksort is, for the worst case, the solution of the recurrence equation $S : \mathbb{N} \to (0, \infty)$ given by

$$S(n) = \begin{cases} c & \text{if } n = 1, \\ S(n-1) + bn & \text{if } n \geq 2, \end{cases}$$

with $c, b > 0$ (see, *e.g.*, [26]). The functional Ψ_S naturally associated to S is defined as

$$\Psi_S(f)(n) = \begin{cases} c & \text{if } n = 1, \\ f(n-1) + bn & \text{if } n \geq 2. \end{cases}$$

Clearly Ψ_S is a mapping of type (3) for $a = 1$, and $h \in \mathcal{C}_0$ defined as $h(n) = bn$ for all $n \in \mathbb{N}$. By Theorem 6, there exists $f_S \in \mathcal{C}_{cah}$ such that $f_S = \Psi f_S$. Hence f_S is a solution of the recurrence equation S.

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