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A CHARACTERIZATION OF SOME IDEMPOTENT ABELIAN GROUPOIDS

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In this paper we prove some theorems (announced in [1]) on idempotent abelian groupoids having a representation as idempotent reducts of abelian groups and sums of direct systems of such reducts. For the notion of such sums, see [3].

Let $(G;\oplus)$ be an idempotent abelian groupoid, i. e. $x\oplus x = x$, $x\oplus y = y\oplus x$. Let us call the operation

$$s_n(x_1, x_2, \ldots, x_n) = \big(\ldots\big((x_1\oplus x_2)\oplus x_3\big)\oplus \ldots \oplus x_{n-1}\big)\oplus x_n$$

the *simple iteration* for $n = 2, 3, \ldots$ (see [2]).

The identity and the transposition interchanging 1 and 2, will be called the *trivial permutations* of indices $1, 2, \ldots, n$.

Let $S_{(k)}$ be the group of all permutations of indices $1, 2, \ldots, k$ for which all numbers $3, 4, \ldots, k-1$ are fixpoints.

THEOREM 1. *If $(G;\oplus)$ is an idempotent abelian groupoid and m is the smallest natural number such that $s_m(x_1, x_2, \ldots, x_m)$ admits a non-trivial permutation σ, then $\sigma\in S_{(m)}$.*

Proof. Let us assume that the equality

$$(+)\qquad s_m(x_1, x_2, \ldots, x_m) = s_m(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$$

holds for some non-trivial permutation $\sigma = (i_1, i_2, \ldots, i_m)$ and that m is the smallest number for which such a σ exists. If $\{i_1, i_2\} = \{1, 2\}$, then s_{m-1} admits a non-trivial permutation of its variables, namely

$$s_{m-1}(x_2, x_3, \ldots, x_m) = s_m(x_2, x_2, x_{i_3}, \ldots, x_{i_m}) = s_{m-1}(x_2, x_{i_3}, \ldots, x_{i_m}),$$

which contradicts the assumption that m is minimal. If $\{i_1, i_2\}\cap\{1, 2\} = \emptyset$, then by the idempotency and commutativity of $\oplus$ the iteration s_{m-1} admits a non-trivial transposition.

Further, if $i_m = m$, then putting $x_m = x_{m-1}(x_1, x_2, \ldots, x_{m-1})$ in the equality $(+)$ we get

$$\begin{aligned} s_{m-1}(x_1, x_2, \ldots, x_{m-1}) &= s_{m-1}(x_{i_1}, x_{i_2}, \ldots, x_{i_{m-1}}) \oplus s_{m-1}(x_1, x_2, \ldots, x_{m-1}) \\ &= s_{m-1}(x_{i_1}, x_{i_2}, \ldots, x_{i_{m-1}}) \oplus s_{m-1}(x_{i_1}, x_{i_2}, \ldots, x_{i_{m-1}}) \\ &= s_{m-1}(x_{i_1}, x_{i_2}, \ldots, x_{i_{m-1}}), \end{aligned}$$

which also contradicts the definition of m.

Without loss of generality we may assume that $i_1 \geqslant 3$, $i_2 = 2$, $i_m \leqslant m-1$. If $i_1 \neq m$, then the iteration s_m admits a transposition $(2, i_1)$ for $3 \leqslant i_1 \leqslant m-1$. Hence s_m admits a non-trivial permutation of variables with x_m as a fixpoint, contradicting $i_m \neq m$. If $i_m \neq 1$, then by $(+)$ the operation $s_m(x_m, x_2, x_{i_3}, \ldots, x_{i_{m-1}}, x_{i_m})$ admits the transposition of variables x_1, x_2 with fixed x_{i_m}, which is impossible.

Thus $\sigma = (m, 2, i_3, \ldots, i_{m-1}, 1)$. Let $i_k \neq k$ for some k and $3 \leqslant k \leqslant m-1$. By $(+)$ and commutativity of $\oplus$ we have

$$\begin{aligned} s_m(x_1, x_2, \ldots, x_{m-1}, x_m) &= s_m(x_m, x_2, x_{i_3}, \ldots, x_{i_k}, \ldots, x_{i_{m-1}}, x_1) \\ &= s_m(x_2, x_m, x_{i_3}, \ldots, x_{i_k}, \ldots, x_{i_{m-1}}) = s_m(x_1, x_2, x_{i_3}, \ldots, x_m). \end{aligned}$$

Putting $x_1 = x_2$, we get a contradiction with the minimality of m. Thus $\sigma \in S_{(m)}$.

In the sequel we shall denote by $\mathcal{G}_m$ the class of all idempotent and abelian groupoids $(G; \oplus)$ satisfying the identities

$$(*) \qquad s_m(x_1, x_2, \ldots, x_{m-1}, x_m) = s_m(x_m, x_2, \ldots, x_{m-1}, x_1),$$

$$(**) \qquad s_{m-1}(x, y, y, \ldots, y) = x \quad \text{for some } m \geqslant 3.$$

LEMMA 1. *If $(G; +, -, 0)$ is an abelian group for which $2^{m-2}x = x$ $(m \geqslant 4)$, then the groupoid $(G; \oplus)$, where $x \oplus y = 2^{m-3}(x+y)$, belongs to the class $\mathcal{G}_m$.*

Proof. Let $(G; +, -, 0)$ be an abelian group in which $2^{m-2}x = x$ holds for some $m \geqslant 4$. Then the operation $x \oplus y = 2^{m-3}(x+y)$ is idempotent, commutative and

$$s_k(x_1, x_2, \ldots, x_k) = d^{k-1}x_1 + d^{k-1}x_2 + d^{k-2}x_3 + \ldots + d^2 x_{k-1} + dx_k,$$

where $d = 2^{m-3}$.

The groupoid $(G; \oplus)$ satisfies $(*)$ and $(**)$. Indeed, $(G; \oplus)$ satisfies $(*)$, because

$$2^{(m-3)(m-1)} - 2^{m-3} = 2^{m-3}[2^{(m-3)(m-2)} - 1] \equiv 0 \pmod{(2^{m-2}-1)}.$$

Since $2^{m-3}-1$ and $2^{m-2}-1$ are relatively prime, we infer that

$$\sum_{k=1}^{m-2} 2^{k(m-3)} = 2^{m-3}\frac{2^{(m-2)(m-3)}-1}{2^{m-3}-1} \equiv 0 \pmod{(2^{m-2}-1)}.$$

Hence, in view of the congruence

$$2^{(m-3)(m-2)} \equiv 1 \pmod{(2^{m-3}-1)},$$

we have

$$s_{m-1}(x, y, y, \ldots, y) = 2^{(m-3)(m-2)}x + \sum_{k=1}^{m-2} 2^{k(m-3)}y = x.$$

This completes the proof of the lemma.

THEOREM 2. *An idempotent abelian groupoid* $(G;\, \oplus)$ *belongs to the class* $\mathcal{G}_m$ *if and only if it can be represented in the form described in Lemma* 1.

Proof. In view of Lemma 1 it suffices to prove the necessity of the condition. Let 0 be a fixed element of the groupoid $(G;\, \oplus) \in \mathcal{G}_m$. If we put

$$x_1 + x_2 = s_{m-1}(x_1, x_2, 0, 0, \ldots, 0),$$

then $x_1 + x_2$ is commutative and, by (**), we have $x+0 = x$.

By (*) we get

$$\begin{aligned}(x_1+x_2)+x_3 &= s_{m-1}\big(s_{m-1}(x_1, x_2, 0, 0, \ldots, 0), x_3, 0, 0, \ldots, 0\big)\\ &= s_{m-2}\big(s_m(x_1, x_2, 0, 0, \ldots, 0, x_3), 0, 0, \ldots, 0\big)\\ &= s_{m-2}\big(s_m(x_3, x_2, 0, 0, \ldots, 0, x_1), 0, 0, \ldots, 0\big) = (x_3+x_2)+x_1.\end{aligned}$$

Further, we have

$$2x = s_{m-2}(x, 0, 0, \ldots, 0),$$

$$\begin{aligned}2^2x &= s_{m-2}\big(s_{m-2}(x, 0, \ldots, 0), 0, \ldots, 0\big)\\ &= s_{m-3}\big(s_{m-1}(x, 0, \ldots, 0), 0, \ldots, 0\big) = s_{m-3}(x, 0, \ldots, 0)\end{aligned}$$

and, generally,

$$2^kx = s_{m-1-k}(x, 0, 0, \ldots, 0).$$

Hence for $k = m-3$ we have $2^{m-3}x = x\oplus 0$ and

$$2^{m-2}x = s_{m-2}(x\oplus 0, 0, \ldots, 0) = s_{m-1}(x, 0, \ldots, 0) = x.$$

Thus we see that $(G;\, +, 0)$ is an abelian group satisfying $2^{m-2}x = x$ and $x\oplus y = 2^{m-3}(x+y)$, q.e.d.

For the notion of sum of direct system of algebras and P-functions (partition functions) which we use in the sequel, see [3].

LEMMA 2. *If an idempotent abelian groupoid satisfies* $(*)$ *for some* $m \geqslant 3$, *then the operation* $f(x, y) = s_{m-1}(x, y, y, \ldots, y)$ *is its* P-*function.*

Proof. Indeed,

$$\begin{aligned} f\big(f(x, y), z\big) &= s_{m-1}\big(s_{m-1}(x, y, \ldots, y), z, \ldots, z\big) \\ &= s_{m-2}\big(s_m(x, y, \ldots, y, z), z, \ldots, z\big) \\ &= s_{m-2}\big(s_m(z, y, \ldots, y, x), z, \ldots, z\big) \\ &= s_{m-2}\big(s_{m-2}(y \oplus z, y, \ldots, y) \oplus x, z, \ldots, z\big) \\ &= s_{m-1}\big(s_{m-2}(y \oplus z, y, \ldots, y), z, \ldots, z\big) \\ &= s_{m-1}\big(s_{m-2}(y, z, y, \ldots, y), x, y \oplus z, z, \ldots, z\big) \\ &= s_{m-1}\big(s_{m-3}(y \oplus z, y, \ldots, y), x, y \oplus z, z, \ldots, z\big) \\ &= s_{m-1}\big(s_{m-3}(y, z, y, \ldots, y), x, y \oplus z, y \oplus z, z, \ldots, z\big) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= s_{m-1}(x, y \oplus z, y \oplus z, \ldots, y \oplus z) \end{aligned}$$

and

$$\begin{aligned} & f\big(x, f(y, z)\big) \\ &= s_{m-1}\big(x, s_{m-1}(y, z, \ldots, z), s_{m-1}(y, z, \ldots, z), \ldots, s_{m-1}(y, z, \ldots, z)\big) \\ &= s_{m-1}\big(s_{m-2}(y \oplus z, z, \ldots, z), x, s_{m-1}(y, z, \ldots, z), \ldots, s_{m-1}(x, z, \ldots, z)\big) \\ &= s_{m-1}\Big(s_{m-2}\big(s_{m-1}(y, z, \ldots, z), z, \ldots, z\big), x, y \oplus z, s_{m-1}(y, z, \ldots, z), \ldots, \\ &\qquad s_{m-1}(y, z, \ldots, z)\Big) \\ &= s_{m-1}\big(s_{m-2}(y, z, \ldots, z), x, y \oplus z, s_{m-1}(y, z, \ldots, z), \ldots, s_{m-1}(y, z, \ldots, z)\big) \\ &= s_{m-1}\big(s_{m-3}(y \oplus z, z, \ldots, z), x, y \oplus z, s_{m-1}(y, z, \ldots, z), \ldots, s_{m-1}(y, z, \ldots, z)\big) \\ &= s_{m-1}\Big(s_{m-3}\big(s_{m-1}(y, z, \ldots, z), z, \ldots, z\big), x, y \oplus z, y \oplus z, s_{m-1}(y, z, \ldots, z), \ldots, \\ &\qquad s_{m-1}(y, z, \ldots, z)\Big) \\ &= s_{m-1}\big(s_{m-3}(y, z, \ldots, z), x, y \oplus z, y \oplus z, s_{m-1}(y, z, \ldots, z), \ldots, \\ &\qquad s_{m-1}(y, z, \ldots, z)\big) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= s_{m-1}(x, y \oplus z, y \oplus z, \ldots, y \oplus z). \end{aligned}$$

Hence operation f satisfies identities (1)-(3) from [3] for P-function.

From Lemma 2 and Theorem 2 of [3] we have

THEOREM 3. *An idempotent abelian groupoid satisfies* $(*)$ *for some* $m \geqslant 3$ *if and only if it is the sum of a direct system of groupoids belonging to class* $\mathscr{G}_m$.

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