

Discussion Paper No. 964

**A CHARACTERIZATION OF STRATEGY-PROOF  
SOCIAL CHOICE FUNCTIONS  
FOR ECONOMIES WITH PURE PUBLIC GOODS\***

by

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**3rd DRAFT**

May 1991

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\*This paper was written while both authors were visiting GREMAQ, Université des Sciences Sociales de Toulouse. We are thankful for its hospitality and good research atmosphere. Barberà's work is supported by the Instituto de Estudios Fiscales and by research grants PB 86-0613 and PB 89-0294 from the Secretaría de Estado de Universidades e Investigación, Spain. Jackson acknowledges the support of NSF grant SES 8921409. We thank Jacques Crémer for helpful conversations.

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## ABSTRACT

We characterize strategy-proof social choice functions when individuals have strictly quasi-concave, continuous and satiated utility functions on convex subsets of  $\mathbb{R}^l$ , representing preferences for the provision of  $l$  pure public goods.

When specialized to the case  $l=1$ , these assumptions amount to requiring that preferences are single peaked, and for such a domain there exists a wide class of strategy-proof social choice functions. These were studied by Moulin (1980) under additional assumptions. Our first results characterize the complete class, after an appropriate extension of the single-peakedness condition. The new characterization retains the flavor of Moulin's elegant representation theorem.

For the general  $l$ -dimensional case, previous results have shown that there is no efficient, strategy-proof, nondictatorial social choice function, even within the domain restrictions under consideration [Border and Jordan (1983), Zhou (1991)]. In fact, Zhou's powerful result indicates that nondictatorial strategy-proof s.c.f.'s will have a range of dimension one. This allows us to conclude with a complete characterization of all strategy-proof s.c.f.'s on  $\mathbb{R}^l$ , because restrictions of preferences from our admissible class to one dimensional subsets satisfy the slightly generalized notion of single-peakedness that is used in our characterization for the case  $l=1$ .

We feel that a complete knowledge of the class of strategy-proof mechanisms, in this as well as in other contexts, is an important step in the analysis of the trade-offs between strategy-proofness and other performance criteria, like efficiency.

## 1. INTRODUCTION

Consider a society which must decide on the level of provision of  $l$  pure public goods. If no restriction is placed on the admissible preferences of agents, then any mechanism which takes these preferences into account for reaching a decision must either be trivial or manipulable. This is the spirit of the Gibbard-Satterthwaite theorem<sup>1</sup>. Yet, in many instances the preferences of individuals will belong to some restricted class. In our  $l$ -pure public goods case it is quite natural, though not completely general, to assume that the preferences of individuals are convex, continuous and have a single alternative preferred to all others (an ideal point).

For the case  $l=1$ , this assumption amounts to requiring that preferences are single peaked, and for this restricted domain there exists a wide class of strategy-proof mechanisms. Moulin (1980) studied such mechanisms in the case where all levels of public good are attainable and the mechanisms are restricted to operate on limited information: specifically, they are required to operate on the basis of the agents' ideal points alone. Our first results in this paper extend Moulin's characterization to the general case where the set of attainable public good levels is arbitrary and all aspects of the agents' preferences may be taken into account. It turns out that allowing for the use of additional information does not enlarge the set of strategy-proof mechanisms and that Moulin's elegant representation theorem can be essentially retained.

For the general  $l$ -dimensional case, previous results have shown that there is no efficient, strategy-proof, nondictatorial mechanism, even within the domain restriction under consideration (Border and Jordan [1983]; Zhou [1991]). In fact, Zhou's powerful result indicates that nondictatorial strategy-proof mechanisms will have a limited range, which must be of dimension one. Why, then, bother to characterize strategy-proof mechanisms at all? We take the view that it is worthwhile to provide full characterizations of strategy-proof mechanisms in interesting domains, because we know that there are trade-offs between the

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<sup>1</sup> Yet, Gibbard (1973) and Satterthwaite's (1975) formulation and proofs do not automatically reach as far as the spirit of the result does. New proofs and new techniques are needed as soon as we introduce a natural restriction like continuity of preferences on a subset of  $\mathbb{R}^l$ . See Barberà and Peleg (1989), Zhou (1989).

desirable characteristics of mechanisms, and more specifically between efficiency and strategy-proofness. Knowing exactly how strategy-proof mechanisms look like will help in analyzing these trade-offs, and eventually in choosing mechanisms which compromise between different performance criteria.

The paper proceeds as follows. First, we characterize strategy-proof mechanisms on arbitrary closed subsets of  $\mathbb{R}$  using a slightly generalized notion of single-peakedness. Next, we apply Zhou's (1991) result to the  $l$ -dimensional case, and thus restrict attention to mechanisms with one-dimensional ranges. Since preferences in our admissible class are (weakly) single peaked on a one-dimensional subset of  $\mathbb{R}^l$ , this allows us to conclude with a complete characterization of all strategy-proof mechanisms on  $\mathbb{R}^l$ .

## 2. Strategy Proofness and Single-Peakedness on Subsets of the Real Line.

We begin by stating some general definitions, and then specialize them to our case.

- (i)  $I = \{1, 2, \dots, n\}$  is a set of *agents*.
- (ii)  $A$  is a set of *alternatives*.
- (iii)  $U$  is a set of *admissible utility functions on  $A$* .<sup>2</sup>
- (iv) A *social choice function* on  $(U, A)$  is a function  $f: U^n \rightarrow A$ .
- (v)  $A_f$  is the *range* of  $f$ .

$n$ -tuples of utility functions  $(u_1, u_2, \dots, u_n)$  are called *preference profiles*. It is convenient to denote them in several ways. A complete  $n$ -tuple may be denoted by  $\tilde{u}$  or  $\tilde{u}'$  and then  $u_i$  or  $u'_i$  will naturally stand for an  $i$ -th component. To distinguish between the preferences of one subset  $J$  of agents and those of the rest, we write  $(u_J, u_{-J})$ .

A social choice function  $f$  on  $(U, A)$  is *strategy-proof* iff  $u_i[f(u_i, u_{-i})] \geq u_i[f(u'_i, u_{-i})]$  for all  $i \in I$ ,  $u_i, u'_i \in U$  and  $u_{-i} \in U^{n-1}$ .

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<sup>2</sup> The use of utility functions rather than preference preorders is not restrictive here, given that we only want to consider continuous preferences on  $\mathbb{R}^l$ .

If a social choice function  $f$  is not strategy-proof, then there exist  $i, u_i, u'_i$  and  $u_{-i}$  such that  $u_i[f(u'_i, u_{-i})] > u_i[f(u_i, u_{-i})]$ . We then say that  $f$  is *manipulable* (at  $(u_i, u_{-i})$ , by  $i$ , via  $u'_i$ ).

In this paper we concentrate on the case where  $A$  is a closed, convex set of  $\mathbb{R}^l$ . Although most of this section is devoted to the case  $l = 1$ , we first present a result about the range of strategy-proof social choice functions which applies for any  $l$ .

**LEMMA 1.** Let  $A$  be a closed, convex subset of  $\mathbb{R}^l$ . Let  $U$  include the set of continuous, strictly quasi-concave utility functions with a unique maximal point in  $A$ . If  $f: U^n \rightarrow A$  is strategy-proof, then  $A_f$  is closed.

*Proof.* Suppose the contrary. Then  $\exists$  a sequence  $x^k \rightarrow x^\circ$ , where  $x^k \in A_f \forall k, x^\circ \notin A_f, x^\circ \in A$ . Consider  $\tilde{u} \in U^n$  defined by  $u_i(y) = \|y - x^\circ\|$  (where  $\|y - x\|$  represents the standard Euclidean metric). Let  $a = f(\tilde{u})$  where  $a \in A_f$ . Since  $x^k \rightarrow x^\circ, \exists K$  s.t.  $\|x^K - x^\circ\| < \|a - x^\circ\|$ . Since  $x^K \in A_f, \exists \hat{u}$  s.t.  $f(\hat{u}) = x^K$ .

By strategy-proofness,  $f(\hat{u}_1, u_{-1})$  is no closer to  $x^\circ$  than  $a$ .

By strategy-proofness,  $f(\hat{u}_1, \hat{u}_2, u_{-1,2})$  is no closer to  $x^\circ$  than  $f(\hat{u}_1, u_{-1,2})$  and thus no closer than  $a$ .

By the same reasoning  $f(\hat{u})$  is no closer to  $x^\circ$  than  $a$ . This contradicts the fact that  $f(\hat{u}) = x^K$ .

Thus our supposition is wrong, and  $A_f$  is closed.

The rest of this section is devoted to the case where  $A = \mathbb{R}$ . In order to specify the set of admissible preferences, we need the following definitions.

**DEFINITION 1.** For any  $B \subset A$ , a utility function  $u: A \rightarrow \mathbb{R}$  is *weakly single peaked* on  $B$  iff there exist alternatives  $t_1(u), t_2(u) \in B$  with  $t_1(u) \leq t_2(u)$  —the "peaks" of  $u$  on  $B$ —, such that  $u(t_1(u)) = u(t_2(u))$ , and for all  $x, y \in B, x, y \notin \{(t_1(u), t_2(u))\}$

$$\left. \begin{aligned} x < y < t_i(u) &\Rightarrow u(x) < u(y) < u(t_i(u)) \\ t_i(u) < x < y &\Rightarrow u(t_i(u)) > u(x) > u(y) \end{aligned} \right\} \text{ for } i \in \{1, 2\}$$

If  $t_1(u) = t_2(u)$  in the above definition, we say that  $u$  is *single peaked on B*, and we denote its unique peak by  $t(u)$ . We will assume that agents have continuous single peaked preferences on  $A$ .  $S$  denotes the set of all such functions.

Our objective is to characterize all strategy-proof social choice functions on  $(A, S)$ . Before we do that, we restrict attention to a particular subclass of preferences.

Given that individual utilities are single-peaked on  $A$ , the restriction of any such  $u \in S$  to  $A_f$ , the range of  $f$ , will be weakly single-peaked on  $A_f$ , for any strategy-proof  $f$ . Let  $S_f$  be the subset of single-peaked utility functions on  $A$  that are also single-peaked on  $A_f$ . Let  $f^*$  be the restriction of  $f$  to  $S_f^n$ . Clearly, if  $f$  is strategy-proof, then  $f^*$  must also be.

Our first theorem proves that, even if  $f^*$  is defined on the set of single peaked utility profiles, it can only depend on the peaks on  $A_f$  of each of the preferences in the profile.

**THEOREM 1. ("Tops only").** *If the restriction of a social choice function  $f$  to  $S_f^n$ ,  $f^* : S_f^n \rightarrow A$ , is strategy-proof, then, for any  $\underline{u}, \underline{u}' \in S_f^n$ ,  $[(\forall i \in I) t(u_i) = t(u'_i)] \rightarrow f(\underline{u}) = f(\underline{u}')$ .*

The proof of the theorem follows from several lemmas and is to be found in the Appendix. We now elaborate on its interest and proceed to make use of its implications. The result is interesting on its own, because it shows that, even if the class of social choice functions that operate only on the basis of the tops of agents' preferences is a very small part of the set of all conceivable social choice functions, there is no need to look for more elaborate forms of

processing utility information if one cares for strategy-proofness.<sup>3</sup> Actually, this is also true in many contexts different from the present one (see, for example, Barberà, Sonnenschein and Zhou [1991]). It is nevertheless necessary to prove it in each specific context, because it is not true for all domain restrictions.

Given Theorem 1, every strategy-proof  $f: S_f^n \rightarrow A$  can be identified with a  $g_f: (A_f^n) \rightarrow A_f$ , such that  $\forall (t_1, \dots, t_n \in A_f^n) g_f(t_1, t_2, \dots, t_n) \equiv f(u_1, u_2, \dots, u_n)$ , for any choice of  $u_i$ 's such that  $t(u_i) = t_i$ . The class of  $g_f$  functions with range  $\mathbb{R}$  was characterized by Moulin (1980). The careful reader will be able to adapt the proof of his elegant theorem 2 to our present context, to show that:

**THEOREM 2.** *The restriction of a social choice function  $f$  to  $S_f^n$ ,  $f^*: S_f^n \rightarrow A$ , is strategy-proof iff there exist extended real numbers  $a_C \in A_f \cup \{-\infty, +\infty\}$  for each  $C \subseteq I$  (including  $\emptyset$ ), such that  $(\forall (u_1, u_2, \dots, u_n) \in S_f^n) f(u_1, u_2, \dots, u_n) = \min_{C \subseteq I} (\max_{i \in C} [a_C, t(u_i)])$ .*

Notice that  $a_I$  can be identified with the lower bound of the range,  $a_\emptyset$  with its upper bound, and that  $i$  is a dictator when we set  $a_i = a_I$  and all other  $a_C$ 's  $C \neq I$ , equal to  $a_\emptyset$ . As the original result may not be as well known as it deserves, we think it is worthwhile to provide some examples of rules which belong to this class.

**EXAMPLE 1.** (Anonymous rules with range  $A$ ). For  $\lambda_1, \lambda_2, \dots, \lambda_{n-1} \in A \cup \{-\infty, +\infty\}$  and for all  $(u_1, u_2, \dots, u_n) \in S$ , let  $f(u_1, u_2, \dots, u_n) = m(t(u_1), t(u_2), \dots, t(u_n), \lambda_1, \lambda_2, \dots, \lambda_{n-1})$ , where  $m$  is the median of these  $2n-1$  extended real numbers (with at least the  $t(\cdot)$ 's being real).

These are the median voting rules with phantom voters discussed in Moulin (1980).

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<sup>3</sup> Sprumont (1989) had reported a similar but weaker result for anonymous functions in Moulin's context.

**EXAMPLE 2.** For  $n=2$ , let  $a_1 < a_2$ ,  $a_\emptyset = +\infty$ ,  $a_{12} = -\infty$ , and  $f(u_1, u_2) = \min [\max (a_1, t(u_1)), \max (a_2, t(u_2)), \max (t(u_1), t(u_2))]$ .

This rule is strategy-proof, and may be described in either one of the following ways:

- (a)  $f$  picks the median of  $a_1$ ,  $a_2$ ,  $t(u_2)$  and  $t(u_1)$ , where  $t(u_1)$  is counted twice, *or*  
 (b)  $f$  is a rule where agent 1 dictates whenever his peak lies between  $a_1$  and  $a_2$ . Otherwise,  $f$  selects the median of  $(\bar{a}, t(u_1), t(u_2))$ , where  $\bar{a}$  is the  $a_i$  closest to  $t(u_1)$ .

Our previous statements have limited attention to the restriction of strategy-proof social choice functions to profiles where all preferences are single-peaked on the range. This includes the case  $A = A_f$  where our statement is unconditional, since we have already assumed that preferences are single-peaked on  $A$ , the set of conceivable alternatives. In general, however, an agent's preferences on the range  $A_f$  may only be weakly single-peaked, given that they are single peaked on  $A$  and  $A_f$  is closed (see Definition 1 and Lemma 1). Thus, a utility function  $u_i \in S$  may have two peaks on  $A_f$  (but no more than two).

We use the notation  $t_f(u_i) = \{t_1(u_i), t_2(u_i)\}$ , where  $t_1(u_i) \leq t_2(u_i)$ , to represent the peaks of  $u_i$  on  $A_f$ . If  $t_1(u_i) = t_2(u_i)$ , then  $u_i$  is still single peaked when restricted to the range of  $f$ .

The notation  $t_f(u_i) \geq t_f(v_i)$  indicates that  $t_1(u_i) \geq t_1(v_i)$  and  $t_2(u_i) \geq t_2(v_i)$ .

**DEFINITION 2.** A *tie-breaking rule* for agent  $i$  is a function  $g^i: S^n \rightarrow A$ , such that  $g^i(u) \in t_f(u_i)$ . Thus, a tie-breaking rule for agent  $i$ ,  $g^i$ , is a function which selects one of agent  $i$ 's peaks.

**LEMMA 2.** A tie breaking rule for agent  $i$ ,  $g^i$ , is strategy-proof if and only if

$$(*) (\forall u, v \in S^n) [g^i(u_{-i}, v_i) > g^i(v)] \Rightarrow [\exists j \text{ s.t. } u_j \neq v_j \text{ and } t_f(v_j) \leq t_f(v_i) \leq t_f(u_j)].$$

Before we prove Lemma 2, we make a few remarks about its meaning. A strategy-proof tie breaking rule is almost tops only. The condition (\*) shows that  $g^i$  only depends on where the tops of each agent fall relative to the tops of agent  $i$ . The exception occurs when agent  $j$  has the



same two peaks as agent  $i$ : that is, when  $t_f(v_j) = t_f(v_i) = t_f(u_j)$ . In this case (and only in this case)  $g^i$  may break ties based on other information about agent  $j$ 's utility. The reason this can be done is that in this case agent  $j$  does not care which point is chosen and thus cannot manipulate the outcome by changing utilities. If on the other hand, an agent has a strict preference over the two points, then any other utility with the same top must have the same strict preference and thus must lead to the same outcome.

We now proceed to prove the Lemma.

*Proof of Lemma 2.* In order to prove that if  $g^i$  is strategy-proof, then  $(*)$  holds, it will suffice to show that:

$$(**) (\forall u, v, \tilde{v} \in S^n) [g^i(u_j, v_{-j}) > g^i(v)] \Rightarrow [u_j \neq v_j \text{ and } t_f(v_j) \leq t_f(v_i) \leq t_f(u_j)].$$

It is clear that if  $(**)$  is satisfied, then so is  $(*)$ : just change agents' utilities from  $v_j$  to  $u_j$ , one at a time. Since  $g^i(u_{-j}, v_j) > g^i(v)$ , there is an increase for some change.

To verify that  $(**)$  is satisfied, first notice that  $g^i(u_j, v_{-j}) > g^i(v)$  implies that  $t_f(v_i) = \{g^i(v), g^i(u_j, v_{-j})\}$ . Since  $g^i$  is strategy-proof it follows that  $t_1(v_j) \leq g^i(v)$ . Otherwise,  $v_j(g^i(u_j, v_{-j})) < v_j(g^i(v))$ , which contradicts strategy-proofness. This implies that  $t_f(v_j) \leq t_f(v_i)$ . By similar reasoning,  $t_2(v_j) \geq g^i(u_j, v_{-j})$ , which implies that  $t_f(v_i) \leq t_f(u_j)$ . Thus, we have established  $(**)$ .

Now we prove the converse: if  $g^i$  satisfies  $(*)$ , then it is strategy-proof. First, notice that  $i$  cannot manipulate  $g^i$  since it always chooses one of  $i$ 's peaks. Thus we consider  $j \neq i$ . If  $g^i(v) \in t_f(v_j)$ , the agent  $j$  is at his peak and cannot manipulate  $g^i$ . So assume that  $g^i(v) \notin t_f(v_j)$ . If  $t_2(v_j) < g^i(v)$ , then it follows from  $(*)$  that  $j$  can only raise the choice of  $g^i$ , which is not an improvement.

Similarly, if  $g^i(\underline{v}) < t_1(\underline{v}_j)$ , then it follows from (\*) that  $j$  can only lower the choice of  $g^i$ , which is not an improvement.

**THEOREM 3.** *A social choice function  $f : S^n \rightarrow A$  is strategy-proof if and only if there exist strategy-proof tie-breaking functions  $g^i$  for each  $i \in I$  as characterized by (\*) in Lemma 2, and extended real numbers*

$$a_C \in A_f \cup \{-\infty, +\infty\} \text{ for each } C \subseteq I \text{ such that } (\forall \underline{u} \in S^n) f(\underline{u}) = \min_{C \subseteq I} (\max_{i \in C} [a_C, g^i(\underline{u})]).$$

*Proof of Theorem 3.* It is easily checked that any  $f$  written in this form is strategy-proof. Proving the converse involves extending Theorem 2 to all of  $S^n$ . This is accomplished by first showing that  $f$  can be written in the same form, but with tie-breaking rules determining which peak of each agent is used in the calculation. Next, we show that these tie-breaking rules must be strategy-proof for certain  $\underline{u}$ 's. Precisely, a tie breaking rule must be strategy-proof at a profile for which it may make a difference which of an agent's peaks are chosen. The proof is completed by verifying that the choice of each  $g^i$  is irrelevant at other points, and that we can extend any  $g^i$  which is strategy-proof at these decisive profiles to be strategy-proof everywhere.

We now proceed formally.

$$\text{Let } A_f(\underline{u}) = \{a \mid \exists \underline{v} \in S_f^n \text{ s.t. } f(\underline{v}) = a \text{ and } t_1(\underline{v}_i) \in t_f(u_i) \forall i\}.$$

The above set is obtained from profile  $\underline{u}$  in the following way. First get a new profile  $\underline{v}$  by changing every  $u_i$  that has two peaks for a new  $v_i$  with only one top alternative, chosen in such a way that the unique top of  $v_i$  is one of the tops of  $u_i$ . Then compute the outcome  $f(\underline{v})$ . This

outcome may of course depend on our choice of  $v_i$  for each  $u_i$ , and  $A_f(\underline{u})$  is the set of alternatives that we may attain by these choices.

*Step 1.*  $f(\underline{u}) \in A_f(\underline{u})$  for all  $\underline{u} \in S^n$ .

Consider  $\underline{u} \in S^n$  and let  $f(\underline{u}) = a$ . For each  $i$  construct  $v_i \in S_f$  (which is thus single-peaked on  $A_f$ ) as follows. Find which of  $\{t_1(u_i), t_2(u_i)\}$  is closest to  $a$ . If it is  $t_2(u_i)$ , then let  $v_i$  be by  $v_i(b) = u_i(b)$  if  $b \geq t_2(u_i)$  and  $v_i(b) = u_i(b) - 1$ , otherwise. If  $t_1(u_i)$  is the closest peak to  $a$ , then let  $v_i$  be defined by  $v_i(b) = u_i(b)$  if  $b \leq t_1(u_i)$  and  $v_i(b) = u_i(b) - 1$ , otherwise.

(Notice that the discontinuity is not important, since it occurs outside the range of  $f$  and we could easily provide continuous functions having the same consequences as the ones we construct here).

We verify that  $f(\underline{v}) = a$  and thus  $f(\underline{u}) \in A_f(\underline{u})$ . Notice that by the construction of  $v$ ,

$$(1) \quad [b \in A_f, b \neq a \text{ and } v_i(b) \geq v_i(a)] \rightarrow [u_i(b) > u_i(a)].$$

Now change agents' utilities, one at a time, from  $u_i$  to  $v_i$ . We show that through these changes,  $f$  must always give  $a$ . By strategy-proofness, if  $f$  changes to some  $b \neq a$ , then  $v_i(b) \geq v_i(a)$ .

By (1), this implies that  $u_i(b) > u_i(a)$ , which contradicts the fact that  $f$  is strategy-proof.

*Step 2.* There exist tie breaking rules  $g^i$  for each  $i \in I$  and extended real numbers  $a_C \in A_f \cup \{-\infty, +\infty\}$  for each  $C \subseteq I$  such that  $(\forall \underline{u} \in S) f(\underline{u}) = \min_{C \subseteq I} (\max_{i \in C} [a_C, g^i(\underline{u})])$ .

This follows directly from Step 1 and Theorem 2. Actually, the  $g^i(\underline{u})$  functions are just

indicating which one among the two peaks of  $u_i$  must be chosen in order to compute  $f(\underline{u})$  as the image of a profile  $\underline{v}$ , with each of the  $v_i$ 's being strictly single peaked.

Define the tie breaking rule  $\hat{g}^i$  as follows

$$\hat{g}^i(\underline{u}) = \begin{cases} t_1(u_i) & \text{if } f(\underline{u}) \leq t_1(u_i) \\ t_2(u_i) & \text{if } f(\underline{u}) \geq t_2(u_i) \end{cases}$$

*Step 3.*  $\hat{g}^i$  is strategy-proof.

Clearly  $i$  cannot manipulate  $\hat{g}^i$  since it picks from  $i$ 's tops.

Consider  $j \neq i$  and suppose that  $\hat{g}^i(\underline{u}) \neq \hat{g}^i(u_{-j}, u'_j)$  for some  $u'_j \in S$ . Without loss of generality assume that  $t_1(u_i) = \hat{g}^i(\underline{u}) < \hat{g}^i(u_{-j}, u'_j) = t_2(u_i)$ .

From the definition of  $\hat{g}^i$ , it follows that  $f(\underline{u}) \leq t_1(u_i) < t_2(u_i) \leq f(u_{-j}, u'_j)$ .

Since  $f$  is strategy-proof it follows that  $t_f(u_j) \leq t_f(u_i) \leq t_f(u'_j)$ .

Thus by  $(**)$  in the proof of Lemma 2,  $\hat{g}^j$  is strategy-proof.

*Step 4.*  $f(\underline{u}) = \min_{C \subseteq I} \left( \max_{i \in C} [a_C, \hat{g}^i(\underline{u})] \right)$ .

This is established by showing that  $\min_{C \subseteq I} \left( \max_{i \in C} [a_C, \hat{g}^i(\underline{u})] \right) = \min_{C \subseteq I} \left( \max_{i \in C} [a_C, g_i(\underline{u})] \right)$  and

applying step 2. Pick some  $j$ , and replace  $g^j$  by  $\hat{g}^j$ . Consider any  $\underline{u} \in S^n$ .

*Case 1*  $f(\underline{u}) \geq t_2(u_j)$ .

In this case it must be that  $\max_{i \in C} (a_C, g^i(\underline{u})) \geq t_2(u_j)$  for all  $C \subseteq I$ . Since in this case  $\hat{g}^i(\underline{u}) = t_2(u_j) \geq g^i(\underline{u})$ , it follows that for any  $C$  containing  $j$ :  $\max(a_C, \hat{g}^i(\underline{u})) = \max(a_C, g^i(\underline{u}))$  and so

there is no change.

Case 2  $f(\underline{u}) \leq t_1(u_j)$

In this case  $\hat{g}^j(\underline{u}) = t_1(u_j) \leq g^j(\underline{u})$ . Thus the expression obtained by replacing  $g^j(\underline{u})$  with  $\hat{g}^j(\underline{u})$  is no higher than  $f(\underline{u})$ . Since  $\max[a_C, \hat{g}^j(\underline{u})] \geq t_1(u_j)$  for all  $C$  containing  $j$ , the new expression is not lower than  $f(\underline{u})$ . This complete the proof of Theorem 3.

Notice that if  $A_f$  is connected, then all preferences are single-peaked and there is no need for tie-breaking rules. However, there might be interesting cases where the range is not connected: for example, if units of the public good are indivisible, then alternatives are a subset of the integers.

### 3.- Strategy-Proofness with Continuous, Strictly Convex Preferences on Subsets of $\mathbb{R}^n$ .

We now turn to the general case, where  $A$  is a closed convex subset of  $\mathbb{R}^l$ , of dimension at least two<sup>4</sup>. The set of admissible utilities for agents is given by  $U$ , the set of continuous, strictly quasi-concave utility functions with domain  $A$  and a unique maximizer in  $A$ .

We rely on the following result.

**THEOREM 4.** (Zhou, 1991). *Any strategy-proof social choice function on  $U^n$  with range of dimension greater than one is dictatorial.*

There is little to add about dictatorial rules. Even with no substantial domain restrictions (assume continuity of preferences for convenience), one can define a strategy-proof social choice function by any choice of a compact subset  $T \subset A$ , and of an agent  $i \in I$ , and letting the

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<sup>4</sup> Dimension of  $A$  refers to the number of vectors in the basis of the smallest affine subspace containing  $A$ . For our purposes, all that is important is that dimension  $A = 1$ , then all point in  $A$  lie on a line, while if dimension  $A \geq 2$  then they do not. If dimension  $A = 1$  then the analysis of the previous section applies.

function have as outcome, for any preference profile,  $i$ 's best element on  $T$  (with adequate provision for the case of multiple utility maximizers).

We thus concentrate on non-dictatorial rules, i.e., rules whose range must be of dimension one, some subset of a line. Our preceding results for the one-dimensional case turn out to be the right ones to use for characterizing these rules, after the following remarks.

*Remark 1.* Let  $B \subset A$  be a one dimensional subset of  $A$ . Then for some  $j \in \{1, \dots, l\}$  and for all  $b, b' \in B$ ,  $[b \neq b'] \Leftrightarrow [b_j \neq b'_j]$ . Thus we can use the  $j$ th component of elements in  $B$  to identify each one of them, uniquely. We'll say that  $j$  is a nondegenerate component of  $B$ . We'll denote by  $B_j$  the projection of  $B$  on the  $j$ -th coordinate axis, i.e., the set of all values which are the  $j$ -th component  $b_j$  for some  $b \in B$ . For such  $j$ 's, we denote by  $B\text{-proj}^{-1}(b_j)$ , the unique element of  $B$  having this component.

*Remark 2.* Let  $A \subset \mathbb{R}^l$ , and let  $u \in U$ . Given a closed one dimensional subset  $B \subset A$ , let  $j$  be a nondegenerate component of  $B$ , and define a continuous utility function  $u^j: B_j \rightarrow \mathbb{R}$  so that, for all  $x \in B$ ,  $u^j(x_j) = u(x)$ . Then since  $u$  is strictly quasi-concave on  $A$ ,  $u^j$  is weakly single-peaked on  $B$ .

We can now state our next theorem, which provides a full characterization of strategy-proof social choice functions within our domain.

**THEOREM 5.** *A social choice function  $f: U^n \rightarrow A$  is strategy-proof iff it is either dictatorial or has a one dimensional range  $A_f$ . In the latter case, for a given nondegenerate component  $j$  of*

$A_f$  the social choice function can be expressed for all profiles  $\underline{u} \in U$  as

$$f(\underline{u}) = A_f\text{-proj}^{-1} \left[ \min_{C \subseteq I} \left( \max_{i \in C} [a_C, g_i(\underline{u}^j)] \right) \right],$$

where, as in Theorem 3,  $g_i$  is a strategy-proof tie breaking rule (as characterized by (\*) in Lemma 2) and  $a_C$ 's are extended real numbers in  $A_f \cup \{-\infty, +\infty\}$  for each  $C \subseteq I$ , with  $a_i < a_\emptyset$  for at least two different agents.

We close with an example illustrating Theorem 5.

**EXAMPLE 3.** A society is deciding on the provision of  $l$  public goods subject to a budget constraint. Thus they choose  $x \in \mathbb{R}^l$  such that  $p \cdot x \leq I$  for some  $p \in \mathbb{R}_{++}^l$  and  $I \geq 0$ . Individual utilities are continuous, strictly quasi-concave, and increasing. Since utilities are increasing society will want the budget constraint to hold with equality, so we set  $A = \{x \in \mathbb{R}^l \mid p \cdot x = I\}$ .

The restriction of utilities to  $A$  is the set of all continuous and strictly quasi-concave utilities defined on  $A_1$  with unique maximizers in  $A$ .

If  $l = 2$  and society desires an anonymous, unanimous, strategy-proof social choice function, then the range of  $f$  must be all of  $A$  ( $A_f = A$ ). We can then apply theorem 5 to get a characterization of such social choice functions. In this case all utilities are single peaked on  $A_f$  and so we can ignore tie-breaking rules. Either axis will serve as nondegenerate component of  $A_f$ , so we choose the first axis.  $f(\underline{u})$  can then be expressed as the  $x \in A$  whose projection onto the first axis is the median of  $\{t(u_1^1), t(u_1^2), \dots, t(u_1^n), \lambda_1, \dots, \lambda_{n-1}\}$ , where  $\lambda_1, \dots, \lambda_{n-1}$  are real numbers with  $0 \leq \lambda_k \leq \frac{I}{p_1}$ . It is interesting to note that this social choice function is also efficient (subject to the budget constraint).

If we go further and allow coalitions to enforce a status-quo, then we put restrictions on the placement of the  $\lambda$ 's. For instance, if any individual can enforce a status-quo then all  $\lambda$ 's must coincide with the projection of the status-quo onto the first axis.

For  $l \geq 3$ , society will not have such nicely behaved social choice functions available. As shown by Zhou (1989), in this example unanimity and strategy-proofness are not compatible with having a non-dictatorial choice. Unanimity requires that  $A_f = A$ , while strategy-proofness and no dictator imply that  $A_f$  is of dimension one. Theorem 5 above tells us that there are strategy-proof and non-dictatorial social choice functions with one dimensional  $A_f \subset A$ . For the anonymous case, these amount to median voting (with phantoms) over a line segment in  $A$ . The restriction to a line segment may be viewed as restricting tradeoffs between various public goods to be in fixed proportions. This loss of flexibility, and thus efficiency, is the cost of having a non-manipulable choice.



## Appendix

The proof of Theorem 1 follows from several definitions and lemmas.

DEFINITION. Let  $A \subset B \subset \mathbb{R}$ . We say that  $A$  is *connected relative to*  $B$  iff

$$[x \in A, y \in A, z \in B; x < z < y] \Rightarrow z \in A.$$

Notice that the relation "being connected relative to" is transitive.

DEFINITION. Let  $f$  be a social choice function, and  $u_J$  be a list of preferences for the agents in  $J \subset I$ . Then  $\sigma_f(u_J)$  is the range of  $f(u_J, \cdot)$ , i.e.  $\sigma_f(u_J) = \{x \in A_f \mid \exists u_{-J} \ni f(u_J, u_{-J}) = x\}$ .

When there is no ambiguity about the relevant function  $f$ , we may omit the reference to it and just write  $\sigma(\cdot)$ .

NOTATION. Before we proceed into the proof, we introduce a useful piece of notation. Given two profiles  $\underline{u}$  and  $\underline{u}'$ , we denote by  $(\underline{u}, \underline{u}', t)$  the profile that obtains by attributing the same preferences as in  $\underline{u}'$  to the first  $t$  agents, and the same preferences as in  $\underline{u}$  to the remaining  $n-t$  agents. Thus, for example,  $(\underline{u}, \underline{u}', 0) = \underline{u}$ , and  $(\underline{u}, \underline{u}', n) = \underline{u}'$ .

**Lemma A-1.** Let  $(u_J, u_{-J})$  be such that for some  $a \in A_f$ ,  $\operatorname{argmax}_{x \in \sigma(u_J)} u_i(x) = a$  for all  $i \in J$ . Then,

$$f(u_J, u_{-J}) = a.$$

*Proof.* Since  $\operatorname{argmax}_{x \in \sigma(u_J)} u_i(x) = a$ ,  $a \in \sigma(u_J)$ . Thus, there exists  $\tilde{u}_{-J}$  such that  $f(u_J, \tilde{u}_{-J}) = a$ .

Let  $\underline{u} = (u_J, u_{-J})$  and  $\underline{u}' = (u_J, \tilde{u}_{-J})$ . Consider the sequence of profiles  $(\underline{u}', \underline{u}, t)$  for  $t = 0, 1, \dots, n$ . We have that  $f(\underline{u}', \underline{u}, 0) = f(\underline{u}') = a$ . If  $f(\underline{u}) \neq a$ , there must be a first  $t$  such that

$f(\tilde{u}', \tilde{u}, t-1) = a$  and  $f(\tilde{u}', \tilde{u}, t) \neq a$ . But this  $t$  must correspond to an agent not in  $J$ , and this agent prefers  $a$  above all other alternatives when his preferences are as in  $(\tilde{u}, \tilde{u}', t)$ .

He will then manipulate via  $\tilde{u}_t$  contradicting  $f$ 's strategy-proofness.

**Lemma A-2.** For any  $J \subset I$ , and  $u_J \in S_f^{\#J}$ ,  $\sigma_f(u_J)$  is closed.

*Proof.* This follows from Lemma 1 in the text, noting that with  $u_J$  fixed,  $f$  is strategy-proof for  $i \notin J$ .

**Lemma A-3.** For any  $J \subset I$  and  $u_J \in S_f^{\#J}$ ,  $\sigma(u_J)$  is connected relative to  $A_f$ .

*Proof.* Consider the case  $J = \{1\}$ . Choose any  $x, y \in \sigma(u_1)$  and any  $z \in A_f$  such that  $x < z < y$ . We must prove that  $z \in \sigma(u_1)$ . If  $z = t(u_1)$ , then  $z \in \sigma(u_1)$ . Otherwise, let  $(\tilde{u}_1, \tilde{u}_{-1})$  be a profile such that  $f(\tilde{u}_1, \tilde{u}_{-1}) = z$ . Such a profile exists, since  $z \in A_f$ . If  $z \notin \sigma(u_1)$ ,  $f(u_1, \tilde{u}_{-1}) \neq z$ .

But then agent 1 can manipulate at  $(u_1, \tilde{u}_{-1})$  via  $\tilde{u}_1$ .

Thus, suppose  $z \neq t(u_1)$ . By single peakedness, either  $u_1(z) > u_1(x)$  or  $u_1(z) > u_1(y)$ . W.l.o.g., say that  $u_1(z) > u_1(x)$ . Then  $t(u_1) > z > x$ . If  $z \notin \sigma(u_1)$ , by Lemma A-2 there exists a neighborhood of  $z$ ,  $N(z)$ , such that  $N(z) \cap \sigma(u_1) = \emptyset$ . Therefore, we can choose a preference  $\bar{u}$  such that  $t(\bar{u}) = z$  and  $\bar{u}(x) > \bar{u}(w)$  for all  $w \in \sigma(u_1)$ ,  $w > z$ . Consider the profile  $\hat{\tilde{u}}$ , where  $\hat{u}_i = \bar{u}$  for all  $i \neq 1$  and  $\hat{u}_1 = u_1$ . By Lemma A-1  $x \leq f(\hat{\tilde{u}}) \neq z$ . Since  $u_1(z) > f(\hat{\tilde{u}})$ , agent 1 could manipulate  $f$  at  $\hat{\tilde{u}}$  via  $\bar{u}$ , a contradiction.

We have proven then that  $\sigma(u_1)$  is connected relative to  $A_f$ . To show that  $\sigma(u_1, u_2)$  is connected relative to  $\sigma(u_1)$ , notice that  $f(u_1, \cdot)$  is an  $n-1$  agent strategy-proof social choice

function on  $\sigma(u_1)$ , and that the restrictions of  $S_f$  to  $\sigma(u_1)$  are single-peaked, because  $\sigma(u_1)$  is connected relative to  $A_f$ . Therefore, reasoning identical to the above will show that  $\sigma(u_1, u_2)$  is connected relative to  $\sigma(u_1)$ , and thus relative to  $A_f$  by the transitivity of the "being connected relative to" relation. We iterate the procedure to get all the force of Lemma A-3.

**Lemma A-4.** For all  $i \in I$ ,  $\tilde{u}_i \in S_f$ ,  $u_{-i} \in S_f^{n-1}$ ,  $[f(\tilde{u}_i, u_{-i}) < t(\tilde{u}_i)] \rightarrow [f(u_i, u_{-i}) \leq f(\tilde{u}_i, u_{-i})]$  and  $[f(\tilde{u}_i, u_{-i}) > t(\tilde{u}_i)] \rightarrow [f(u_i, u_{-i}) \geq f(\tilde{u}_i, u_{-i})]$ , for all  $u_i$ .

*Proof.* By Lemma A-3,  $\sigma(u_{-i})$  is a connected set in  $A_f$ . By strategy-proofness (Lemma A-1 with  $J = I - \{i\}$ ),  $f(\tilde{u}_i, u_{-i})$  is the maximal element of  $\tilde{u}_i$  on  $\sigma(u_{-i})$  and this must coincide with the point in  $\sigma(u_{-i})$  which is closest to  $t(\tilde{u}_i)$ .

We can now proceed to prove Theorem 1.

Let  $f(\tilde{u}) = x$ . Let  $J$  be the set of agents for which  $t(u_j) = t(\tilde{u}_j) \neq x$ . For  $j \in J$ , we get from Lemma A-4 and strategy-proofness that  $f(u_j, u_{-j}) = f(\tilde{u}_j, u_{-j}) = x$ . We can thus proceed to change all preferences  $u_j$  into  $\tilde{u}_j$ , for  $j \in J$ , while keeping the outcome equal to  $x$ . Now,  $f(\tilde{u}_j, u_j) = x$ , and thus  $x \in \sigma(\tilde{u}_j)$ . But then, since  $t(\tilde{u}_j) = x$  for all  $j \in J$ , Lemma A-1 guarantees that  $f(\tilde{u}) = x$ .

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