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A CHARACTERIZATION OF THE INTERVAL FUNCTION OF A (FINITE OR INFINITE) CONNECTED GRAPH

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Abstract. By the interval function of a finite connected graph we mean the interval function in the sense of H. M. Mulder. This function is very important for studying properties of a finite connected graph which depend on the distance between vertices. The interval function of a finite connected graph was characterized by the present author.

The interval function of an infinite connected graph can be defined similarly to that of a finite one. In the present paper we give a characterization of the interval function of each connected graph.

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The letters f–n will be reserved for denoting integers here. By a graph we will mean an undirected graph without loops or multiple edges. A graph will be referred to as finite or infinite if its vertex set is finite or infinite, respectively.

Let G be a connected graph with a vertex set $V(G)$ and an edge set $E(G)$, and let $d_G(u, v)$ denote the distance between u and v in G, where $u, v \in V(G)$. By the *interval function* I_G of G we mean the mapping of $V(G) \times V(G)$ into the power set of $V(G)$ defined as follows:

$$I_G(x, y) = \{w \in V(G);\ \ d(x, y) = d(x, w) + d(w, y)\}$$

for all $x, y \in V(G)$.

This function is very important for studying properties of a connected graph which depend on the distance between vertices. The interval functions of finite connected graphs were widely studied by Mulder [3].

The following notion will be important for us.

Let W be a nonempty set. We will say that J is a *geometric function* on W if J is a mapping of $W \times W$ into the power set of W such that the following statements are fulfilled for all $u, v, x, y \in W$:

if $v \in J(u,x)$ and $y \in J(v,x)$, then $v \in J(u,y)$ and $y \in J(u,x)$;
$x \in J(u,x)$;
$J(u,u) = \{u\}$;
$J(u,x) = J(x,u)$.

Our term *geometric function* was inspired by the terminology of Bandelt, van de Vel and Verheul [2] and Bandelt and Chepoi [1], namely by their term *geometric interval space*: if J is a geometric function on a nonempty set W in our sense, then J together with W form a geometric interval space in the sense of [1] and [2]. Note that every geometric function on a finite nonempty set is a transit function in the sense of Mulder [4].

Let G be a graph, and let J be a geometric function on a nonempty set W. We will say that J is associated with G if $W = V(G)$ and

$$E(G) = \{\{u,v\};\ u,v \in V(G) \text{ such that } u \neq v \text{ and } J(u,v) = \{u,v\}\}.$$

It is easy to show that if G is a connected graph, then I_G is a geometric function associated with G.

The following lemma will be presented without proof. Its proof is easy.

Lemma 1. *Let G be a finite graph, and let J be a geometric function associated with G. Then G is connected and J satisfies the following Axiom* (Z)*:*

$$\text{(Z)} \qquad \textit{if } u \neq x, \textit{ then there exists } v \in J(u,x) \textit{ such that } \{u,v\} \in E(G) \quad (\textit{for all } u,x \in V(G)).$$

The following theorem was proved by the present author in [5]. (A different proof of a slight modification of this theorem was given in [6]). Using the terminology of [1] and [2], we may say that this theorem gives a necesary and sufficient condition for a finite geometric interval space to be graphic.

Theorem 0. *Let G be a finite connected graph, and let J be a geometric function associated with G. Then J is the interval function of G if and only if J satisfies the following Axioms* (X) *and* (Y)*:*

$$\text{(X)} \quad \textit{if } \{u,x\},\{v,y\} \in E(G),\ u,v \in J(x,y) \textit{ and } x \in J(u,v), \textit{ then } y \in J(u,v) \quad (\textit{for all } u,v,x,y \in V(G));$$

$$\text{(Y)} \qquad \textit{if } \{u,x),\{v,y\} \in E(G) \textit{ and } x \in J(u,v), \textit{ then either } v \in J(x,y) \textit{ or } x \in J(u,y) \textit{ or } y \in J(u,v) \ (\textit{for all } u,v,x,y \in V(G)).$$

Let G be a graph, and let J be a geometric function associated with G. Let $u_0, \ldots, u_m \in V(G)$, $m \geqslant 0$. We will say that $(u_0, \ldots, u_m)$ is a path in J if $u_j \in J(u_i, u_k)$ for each $0 \leqslant i \leqslant j \leqslant k \leqslant m$, and if $m \geqslant 1$, then $\{u_0, u_1\}, \ldots, \{u_{m-1}, u_m\} \in E(G)$. Obviously, if $(u_0, u_1, \ldots, u_m)$ is a path in J and $m \geqslant 1$, then both $(u_m, \ldots, u_1, u_0)$ and $(u_1, \ldots, u_m)$ are paths in J. Since J is a geometric function associated with G, we see that every path in J is a path in G. Let $P(J)$ denote the set of all paths in J.

Observation 1. Let J be a geometric interval function associated with a graph G, let $0 \leqslant m \leqslant n$, let $u_0, \ldots, u_n$, $v \in V(G)$ and $(u_0, \ldots, u_n) \in P(J)$. If $u_n \in J(u_m, v)$ and $\{u_n, v\} \in E(G)$, then $(u_m, \ldots, u_n, v) \in P(J)$. If $u_0 \in J(v, u_m)$ and $\{v, u_0\} \in E(G)$, then $(v, u_0, \ldots, u_m) \in P(J)$.

Observation 2. Let J be a geometric interval function associated with a graph G, let $0 \leqslant m \leqslant n$, let $u_0, \ldots, u_n \in V(G)$ and

$$(u_0, \ldots, u_m), (u_m, \ldots, u_n) \in P(J).$$

If $u_m \in J(u_0, u_n)$, then $(u_0, \ldots, u_m, \ldots, u_n) \in P(J)$.

Let G be a graph and let J be a geometric function associated with G. Consider $y \in V(G)$. By a $*$–y *slide* in J we will mean an infinite sequence $(v_0, v_1, v_2, \ldots)$ of vertices of G such that

$$v_{i+1} \in J(v_i, y) \text{ and } \{v_i, v_{i+1}\} \in E(G) \text{ for each } i \geqslant 0.$$

If $\sigma = (v_0, v_1, v_2, \ldots)$ is a $*$–y slide and $z = v_0$, then we say that σ is a z–y slide. By virtue of the definition of a geometric function, if $(v_0, v_1, v_2, \ldots)$ is a $*$–y slide in J and $m \geqslant 0$, then $(v_0, \ldots, v_m)$ is a path in J.

It follows from Lemma in [5] that if J is a geometric function associated with a finite connected graph G and $y \in V(G)$, then there exists no $*$–y slide in J.

Example. Let G be the graph defined as follows:

$$V(G) = \{\ldots, u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots\}$$

and

$$E(G) = \{\ldots, \{u_{-2}, u_{-1}\}, \{u_{-1}, u_0\}, \{u_0, u_1\}, \{u_1, u_2\}, \ldots\},$$

where the vertices $\ldots, u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots$ are mutually distinct. Then G is infinite and connected. For all f and g, $f \leqslant g$, we put

$$J(u_f, u_g) = \{u_f\} \text{ if } f = g,$$
$$J(u_f, u_g) = \{u_f, u_{f+1}, \ldots, u_g\} \text{ if } 0 \leqslant f < g \text{ or } f < g \leqslant 0,$$
$$J(u_f, u_g) = \{u_f, u_{f-1}, u_{f-2}, \ldots\} \cup \{u_g, u_{g+1}, u_{g+2}, \ldots\} \text{ if } f < 0 < g$$

and

$$J(u_g, u_f) = J(u_f, u_g).$$

It is easy to see that J is a geometric function associated with G. Moreover, we see that J satisfies Axioms (X) and (Z) but it does not satisfy Axiom (Y). Finally, we see that if $i < 0 < j$ or $j < 0 < i$, then
(a) there exists no path from u_i to u_j in J and
(b) there exists a u_i–u_j slide in J.

Lemma 2. *Let J be a geometric function associated with a graph G, let J satisfy Axiom* (Y)*, let u_0 and x be distinct vertices of G, and let $(u_0, u_1, u_2, \ldots)$ be a $*$–x slide in J. Assume that y is a vertex of G adjacent to x. Then there exist at most two distinct $j \geqslant 0$ such that $u_{j+1} \notin J(u_j, y)$.*

Proof. Suppose, to the contrary, that there exist f, g and h, $0 \leqslant f < g < h$, such that $u_{f+1} \notin J(u_f, y)$, $u_{g+1} \notin J(u_g, y)$ and $u_{h+1} \notin J(u_h, y)$.

First, let $y \in J(u_g, x)$. Recall that J is a geometric function. Since

$$u_{f+1} \in J(u_f, x), \ldots,\ u_g \in J(u_{g-1}, x),$$

we get

$$u_{f+1} \in J(u_f, y), \ldots,\ u_g \in J(u_{g-1}, y);$$

a contradiction.

Now, let $y \notin J(u_g, x)$. By virtue of (Y), $x \in J(u_{g+1}, y)$. Since

$$u_{g+2} \in J(u_{g+1}, x), \ldots,\ u_{h+1} \in J(u_h, x),$$

we get

$$u_{g+2} \in J(u_{g+1}, y), \ldots,\ u_{h+1} \in J(u_h, y);$$

a contradiction. $\square$

Corollary 1. *Let J be a geometric function associated with a graph G, let J satisfy Axiom* (Y)*, let x and y be adjacent vertices of G, and let $(u_0, u_1, u_2, \ldots)$ be a $(*\text{–}x)$ slide in J. Then there exists $k \geqslant 0$ such that $(u_k, u_{k+1}, u_{k+2}, \ldots)$ is a $(*\text{–}y)$ slide in J.*

Theorem 1. *Let J be a geometric function associated with a graph G, let J satisfy Axiom* (Y)*, and let u and x be vertices of G. If there exists a path from u to x in G, then there exists no u–x slide in J.*

Proof. Let there exist a path from u to x in G. Suppose, to the contrary, that there exists a u–x slide in J, say a u–x slide $(u_0, u_1, u_2, \ldots)$. By virtue of Corollary 1, there exists $m \geqslant 0$ such that $(u_m, u_{m+1}, u_{m+2}, \ldots)$ is a $*$–u slide in J. Hence $u_{m+1} \in J(u_m, u)$. Since $(u_0, u_1, u_2, \ldots)$ is a $*$–x slide, we see that $(u_0, \ldots, u_m, u_{m+1})$ is a path in J. Since $u = u_0$, we get $u_m \in J(u, u_{m+1})$. This implies that $u_m = u_{m+1}$. The vertices u_m and u_{m+1} are not adjacent in G, which contradicts the definition of a slide. $\square$

Corollary 2. *Let J be a geometric interval function associated with a connected graph G, and let J satisfy Axioms* (Y) *and* (Z)*. Then there exists a path from u to v in J for each ordered pair of vertices u and v of G.*

Lemma 3. *Let G be a connected graph, let J_1 and J_2 be geometric functions associated with G, and let $u, x \in V(G)$. Assume that J_1 satisfies Axioms* (Y) *and* (Z) *and that $J_1(u,x) - J_2(u,x) \neq \emptyset$. Then there exists a path α from u to x in J_1 such that $\alpha \notin P(J_2)$.*

Proof. Obviously, there exists $v \in J_1(u,x) - J_2(u,x)$. Since $v \notin J_2(u,x)$, we get $u \neq v \neq x$. Corollary 2 implies that there exist $u_0, \ldots, u_k, v_0, \ldots, v_m \in V(G)$ such that $k \geqslant 1$, $m \geqslant 1$, $u_0 = u$, $u_k = v = v_0$, $v_m = x$, and

$$(u_0, \ldots, u_k), (v_0, \ldots, v_m) \in P(J_1).$$

Since $v \in J_1(u_0, v_m)$, it follows from Observation 2 that

$$(u_0, \ldots, u_k = v_0, \ldots, v_m) \in P(J_1).$$

Since $v \notin J_2(u,x)$, we get

$$(u_0, \ldots, u_k = v_0, \ldots, v_m) \notin P(J_2).$$

$\square$

Let G be the graph defined in Example. Then $G - u_0$ is not connected. It is easy to find a geometric function associated with $G - u_0$ which satisfies Axioms (X), (Y) and (Z).

Clearly, if G is a connected graph, then $P(I_G)$ is the set of all geodesics (i.e. shortest paths) in G.

The next theorem gives a characterization of the interval function of a (finite or infinite) connected graph.

Theorem 2. *Let J be a geometric function associated with a connected graph G. Then J is the interval function of G if and only if J satisfies Axioms* (X), (Y) *and* (Z).

Proof. Put $V = V(G)$, $I = I_G$ and $d = d_G$. Let $J = I$. It is obvious that J satisfies (Z). Moreover, it is easy to show that J satisfies (X) and (Y); cf. [5].

Conversely, let J satisfy (X), (Y) and (Z). We will prove that $J = I$. Suppose, to the contrary, that $J \neq I$. Then there exist $n \geqslant 0$ and $u, x \in V(G)$ such that $d(u, x) = n$, $J(u, x) \neq I(u, x)$,

$$(1) \qquad J(v, y) = I(v, y) \text{ for all } v, y \in V \text{ such that } d(v, y) < n$$

and

$$(2) \qquad \text{if } I(u, x) \subseteq J(u, x), \text{ then } I(w, z) \subseteq J(w, z) \text{ for all } w, z \in V \text{ such that } d(w, z) = n.$$

Since J is a geometric function associated with G, we have $n \geqslant 2$. We distinguish two cases.

Case 1. Assume that $I(u, x) \subseteq J(u, x)$. Then $J(u, x) - I(u, x) \neq \emptyset$. Recall that $P(I)$ is the set of all geodesics in G. By virtue of Lemma 3, there exist $x_0, \ldots, x_{m+n} \in V$ (where $m > n$) such that $x_0 = u = x_{m+n}$, $x_m = x$,

$$(x_0, x_1, \ldots, x_m) \in P(J) - P(I) \text{ and } (x_m, x_{m+1}, \ldots, x_{m+n}) \in P(I).$$

Put

$$(3) \qquad \alpha_f = (x_f, x_{f+1}, \ldots, x_{f+m}), \ \ \beta_f = (x_{f+m}, x_{f+m+1}, \ldots, x_{f+m+n}) \text{ for each } f, \ 0 \leqslant f \leqslant n, \ \text{ where } x_{n+m+1} = x_1, x_{n+n+2} = x_2, \ldots, x_{m+2n} = x_n.$$

Let $\alpha_n \in P(J)$. Then $x_{n+1} \in J(x_n, x_{n+m})$. Since $m > n$, $x_{m+n} = x_0$ and $\alpha_0 \in P(J)$, we have $x_n \in J(x_{n+1}, x_{n+m})$. This implies that $x_{n+1} \in J(x_n, x_n)$ and thus $x_n = x_{n+1}$, which is a contradiction. We get $\alpha_n \notin P(J)$.

Recall that $\beta_0 \in P(I)$ and $I(u,x) \subseteq J(u,x)$. Combining these facts with (1), we get $\beta_0 \in P(J)$. There exists h, $0 \leqslant h < n$, such that α_h, $\beta_h \in P(J)$ and

$$\alpha_{h+1} \notin P(J) \ \textit{or}\ \ \beta_{h+1} \notin P(J). \tag{4}$$

Put

$$r = x_h, \ \ s = x_{h+1}, \ \ y = x_{h+m} \quad \text{and} \quad z = x_{h+m+1}. \tag{5}$$

Let $\beta_{h+1} \in P(J)$. Then $r \in J(s,z)$. Since $\alpha_h, \beta_h \in P(J)$, we have $s, z \in J(r,y)$. By (X), $y \in J(s,z)$. Since $\alpha_h \in P(J)$, Observation 1 implies that $\alpha_{h+1} \in P(J)$, which contradicts (4). Thus $\beta_{h+1} \notin P(J)$.

Clearly, $d(s,z) \leqslant n$. Assume that $d(s,z) = n$. Then $\beta_{h+1} \in P(I)$. By (2), $I(s,z) \subseteq J(s,z)$. Combining this fact with (1), we get $\beta_{h+1} \in P(J)$; a contradiction. Thus $d(s,z) < n$.

Since $\alpha_h \in P(J)$, we have $(x_{h+1}, \dots, x_{h+m}) \in P(J)$. If $d(s,y) < n$, then (1) implies that $(x_{h+1}, \dots, x_{h+m}) \in P(I)$ and thus $m - 1 < n$; a contradiction. Thus $d(s,y) \geqslant n$.

We get $d(s,y) = n$ and $d(s,z) = n - 1$. Hence $z \in I(s,y)$. By (2), $z \in J(s,y)$. Since $s \in J(r,y)$, we have $s \in J(r,z)$. Obviously, $d(r,z) < n$. By (1), $s \in I(r,z)$. Therefore, $d(s,z) < n - 1$; a contradiction.

Case 2. Assume that $I(u,x) - J(u,x) \neq \emptyset$. There exist $x_0, \dots, x_{m+n} \in V$ (where $m \geqslant n$) such that $x_0 = u = x_{m+n}$, $x_m = x$,

$$(x_0, x_1, \dots, x_m) \in P(J) \quad \text{and} \quad (x_m, x_{m+1}, \dots, x_{m+n}) \in P(I) - P(J).$$

Let us use notation (3). Since $\beta_0 \notin P(J)$, we get $\alpha_n \notin P(J)$. There exists h, $0 \leqslant h < n$, such that $\alpha_h \in P(J)$, $\beta_h \notin P(J)$ and

$$\alpha_{h+1} \notin P(J) \quad \text{or} \quad \beta_{h+1} \in P(J). \tag{6}$$

Now let us use also notation (5).

Recall that $\alpha_h \in P(J)$. Let $d(r,y) < n$; by virtue of (1), $\alpha_h \in P(I)$; this means that $m \leqslant n - 1$; a contradiction. Thus $d(r,y) = n$. We get $\beta_h \in P(I)$. Since $d(r,z) = n - 1$, (1) implies that $(x_{h+m+1}, \dots, x_{h+m+n}) \in P(J)$. Since $\beta_h \notin P(J)$, it follows from Observation 1 that $z \notin J(r,y)$.

Let $\alpha_{h+1} \in P(J)$. By (6), $\beta_{h+1} \in P(J)$. Thus $r, y \in J(s,z)$. Since $\alpha_h \in P(J)$, we get $s \in J(r,y)$. By (X), $z \in J(r,y)$; a contradiction. Hence $\alpha_{h+1} \notin P(J)$. Since $\alpha_h \in P(J)$, we get $y \notin J(s,z)$.

Recall that $s \in J(r, y)$, $y \notin J(s, z)$ and $z \notin J(r, y)$. According to (Y), $s \in J(r, z)$. By (1), $s \in I(r, z)$. Hence $d(s, z) = n - 2$. Since $d(r, y) = n$, we have $d(s, y) = n - 1$. Therefore, $z \in I(s, y)$. By (1), $z \in J(s, y)$. Since $s \in J(r, y)$, we get $z \in J(r, y)$; a contradiction.

Hence $J = I$. □

Combining Theorem 2 with Lemma 1, we get Theorem 0. This new proof of Theorem 0 is simpler than the proof of Theorem 0 given in [5] and than the proof of a modification of Theorem 0 given in [6].

References

[1] *H.-J. Bandelt and V. Chepoi*: A Helly theorem in weakly modular space. Discrete Math. *160* (1996), 25–39.
[2] *H.-J. Bandelt, M. van de Vel and E. Verheul*: Modular interval spaces. Math. Nachr. *163* (1993), 177–201.
[3] *H. M. Mulder*: The Interval Function of a Graph. Mathematish Centrum, Amsterdam, 1980.
[4] *H. M. Mulder*: Transit functions on graphs. In preparation.
[5] *L. Nebeský*: A characterization of the interval function of a connected graph. Czechoslovak Math. J. *44(119)* (1994), 173–178.
[6] *L. Nebeský*: Characterizing the interval function of a connected graph. Math. Bohem. *123* (1998), 137–144.

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