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A CHARACTERIZATION OF THE MEAN CURVATURE FUNCTIONS OF CODIMENSION-ONE FOLIATIONS

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Abstract. Walczak posed a problem on the characterization of the mean curvature functions of codimension-one foliations. An affirmative answer to this problem is given here. As an application, we get a simpler proof of the topological characterization, due to the author, of codimension-one foliations consisting of constant mean curvature hypersurfaces.

1. Introduction. Let F be a transversely oriented codimension-one foliation of a closed connected manifold M. If we choose a Riemannian metric g on M, then we have a smooth function H on M, where $H(x)$ is the mean curvature at x of the leaf L_x of F through x with respect to the unit vector field N which is orthogonal to F and whose direction coincides with the given transverse orientation. We call H the mean curvature function of F with respect to g. In the previous papers [5], [6], the author studied the following question posed by Walczak [8]: Which smooth function on M can be written as the mean curvature function with respect to some Riemannian metric on M? Such a smooth function on M is said to be admissible. Some characterizations of such functions are given in [5], [6]. However, it is not so easy to check whether the given function is admissible or not by the characterizations given there.

On the other hand, Walczak also posed the following problem on the characterization of admissible functions (see Langevin [3]):

PROBLEM. *Show that f is admissible if and only if $f(x)>0$ somewhere in any $N_{\max}$ and $f(y)<0$ somewhere in any $N_{\min}$, where $N_{\max}$ means a maximal Novikov component and $N_{\min}$ means a minimal Novikov component.*

In this paper we study this problem. After reformulating the problem, we give an affirmative answer to this problem in §3. As an application, we give in §4 a simpler proof of the topological characterization in Oshikiri [6] of codimension-one foliations consisting of constant mean curvature hypersurfaces.

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2. Preliminaries and the result. In this paper, we work in the C^{∞}-category. In what follows, we always assume that foliations are of codimension-one and transversely oriented, and that the ambient manifolds are closed, connected, oriented, and of dimension $n+1 \geq 3$, unless otherwise stated.

First we fix a transversely oriented codimension-one foliation F on M. Let g be a Riemannian metric on M. Then there is a unique unit vector field orthogonal to F whose direction coincides with the given transverse orientation. We denote this vector field by N. We give an orientation to F as follows: Let $\{E_1, \ldots, E_n\}$ be an oriented local orthonormal frame for TF. The orientation of M given by $\{N, E_1, \ldots, E_n\}$ coincides with the given one of M.

We denote by $H(x)$ the mean curvature of the leaf L at x with respect to N, that is,

$$H = \sum_{i=1}^{n} \langle \nabla_{E_i} E_i, N \rangle$$

where $\langle\ ,\ \rangle$ means $g(\ ,\ )$, ∇ is the Riemannian connection of (M, g), $\{E_i\}$ is a local orthonormal frame for TF and $\dim F = n$. We call H the mean curvature function of F with respect to g. We also define an n-form χ_F on M by

$$\chi_F(V_1, \ldots, V_n) = \det(\langle E_i, V_j \rangle)_{i,j=1,\ldots,n} \qquad \text{for} \quad V_j \in TM,$$

where $\{E_1, \ldots, E_n\}$ is an oriented local orthonormal frame for TF. The restriction $\chi_F|_L$ is the volume element of $(L, g|_L)$ for $L \in F$.

PROPOSITION R (Rummler [7]). *$d\chi_F = -H dV(M, g) = \mathrm{div}_g(N) dV(M, g)$, where $dV(M, g)$ is the volume element of (M, g) and $\mathrm{div}_g(N)$ is the divergence of N with respect to g, i.e.,*

$$\mathrm{div}_g(N) = \sum_{i=1}^{n} \langle \nabla_{E_i} N, E_i \rangle .$$

Let f be a smooth function on M. We say f to be *admissible* if there is a Riemannian metric g on M so that $-f$ coincides with the mean curvature function of F with respect to g. We set

$$C_{\mathrm{ad}}(F) = \{f \in C^{\infty}(M) : f \text{ is admissible}\} .$$

Let D be a compact saturated domain of M. We call D a *$(+)$-foliated compact domain* ($(+)$-fcd, for short) if the transverse orientation of F is outward everywhere on ∂D, and we call D a *$(-)$-foliated compact domain* ($(-)$-fcd, for short) if the transverse orientation of F is inward everywhere on ∂D. Note that for a foliated compact domain D, $\mathrm{Int}\, D$ is a maximal Novikov component (resp. minimal Novikov component) if and only if D is a minimal $(+)$-fcd (resp. a minimal $(-)$-fcd). Here *minimal* is in the usual set-theoretical sense. For the notion of the Novikov component, see the original paper of Novikov [4]. We do not give the definition of the Novikov component, because we use only the notions of $(+)$-fcd's and $(-)$-fcd's.

In this setting, we reformulate Walczak's problem as follows:

PROBLEM. *Assume that F contains at least one* $(+)$-fcd. *f is admissible if and only if $f(x)>0$ somewhere in any minimal* $(+)$-fcd *and $f(y)<0$ somewhere in any minimal* $(-)$-fcd.

In the next section we give an affirmative answer to this problem, that is, we prove the following theorem:

THEOREM. *Let F be a transversely oriented codimension-one foliation of a closed connected oriented manifold M. Assume that F contains at least one* $(+)$-fcd. *f is admissible if and only if $f(x)>0$ somewhere in any minimal* $(+)$-fcd *and $f(y)<0$ somewhere in any minimal* $(-)$-fcd.

3. Proof of the theorem. To prove the theorem, we need the following result in Oshikiri [6]:

THEOREM O1. *For any $f \in C^\infty(M)$, the following conditions are equivalent.*

(1) $f \in C_{\mathrm{ad}}(F)$.

(2) *There is an oriented volume form dV on M so that*

 (i) $\int_M f dV=0$, *and*

 (ii) $\int_D f dV>0$ *for any* $(+)$-fcd D.

If f is admissible, then by the condition (2), (ii) in Theorem O1, we have $f(x)>0$ somewhere on each $(+)$-fcd. If D is a $(-)$-fcd, then $M-\operatorname{Int} D$ is a $(+)$-fcd and, by Theorem O1, (2), we have $\int_D f dV<0$. Thus $f(y)<0$ somewhere on D.

In order to show the converse, we need some preparation. Let I be the unit closed interval $[0, 1]$ and L a compact leaf of F. A foliated trivial I-bundle over L is the manifold $L \times I$ with a codimension-one foliation whose leaves are transverse everywhere to the fibers $\{p\} \times I$, for any $p \in L$. It is known that if F has an infinite number of compact leaves, then all but a finite number of compact leaves are contained in a foliated trivial I-bundle over some compact leaves (cf. Hector-Hirsch [2]). With this fact in mind, define a finite number of foliated compact domains $\{D_i\}_{i=0}^k$ satisfying the following:

 (i) $\bigcup_{i=0}^k D_i=M$,

 (ii) $(\operatorname{Int} D_i) \cap (\operatorname{Int} D_j)=\varnothing$ if $i \neq j$,

 (iii) D_i is either a minimal $(+)$-fcd, a minimal $(-)$-fcd, a maximal foliated trivial I-bundle over a compact leaf of F, or other ones that do not contain any compact leaves which divide the foliated compact domain D_i into two components.

For later use, we fix a minimal $(+)$-fcd and denote it by D_0. First choose all minimal $(+)$-fcd's and minimal $(-)$-fcd's; $D_0, D_1, \ldots, D_s$. Next choose all maximal foliated trivial I-bundles; $D_{s+1}, \dots, D_t$. Finally set M_1 to be the closure of $M-D_0-\cdots-D_t$ in M. Note that ∂M_1 consists of a finite number of compact leaves, and $\operatorname{Int} M_1$ contains only a finite number of compact leaves. Take $C_{t+1}, \dots, C_k$ to be the connected components of $M_1-\{\text{compact leaves in } \operatorname{Int} M_1\}$ and set $D_i=\bar{C}_i$ for $i=t+1, \dots, k$, where the closure $\bar{C}_i$ of C_i is taken in M_1. Then it is easy to see that the resulting set $\{D_0, D_1, \dots, D_k\}$ is the desired one.

Let $P=\{G_1, G_2, \dots, G_l\}$ be a mutually distinct ordered set with $G_j \in \{D_i\}$ ($j=1, \dots, l$). We say P to be a positive path from G_1 to G_l if $G_i \cap G_{i+1}$ contains a compact leaf L with the unit vector field N being outward on $L \subset G_i$ and inward on $L \subset G_{i+1}$ for any $i=1, \dots, l-1$. In this case, we say that P is a positive path from G_1 to G_l, and write $S(P)=G_1$ and $T(P)=G_l$.

For each minimal $(+)$-fcd D_u, set

$$A_u=\#\{\text{minimal } (-)\text{-fcd's connected by a positive path } P \text{ with } S(P)=D_u\}$$

and for each minimal $(-)$-fcd D_v, set

$$B_v=\#\{\text{minimal } (+)\text{-fcd's connected by a positive path } P \text{ with } T(P)=D_v\}$$

where $\#\{W\}$ denotes the cardinality of the set W.

LEMMA 1. *For each minimal* $(+)$-fcd D_u, *there exists a minimal* $(-)$-fcd D_v *connected by a positive path P with* $S(P)=D_u$ *and* $T(P)=D_v$. *For each minimal* $(-)$-fcd D_v, *there exists a minimal* $(+)$-fcd D_u *connected by a positive path P with* $S(P)=D_u$ *and* $T(P)=D_v$. *Thus, in particular,* $A_u \geq 1$ *and* $B_v \geq 1$ *for each u and v.*

PROOF. Let D be a minimal $(+)$-fcd. Set Y to be the union of all D_i's which appear in some positive path from D. It follows that $Y=M$ or that Y is a $(-)$-fcd; Indeed, if Y is not a $(-)$-fcd and different from M, then we can find a new positive path from D which contains some D_j, which is not contained in Y. By a standard set-theoretical argument, every $(-)$-fcd contains a minimal $(-)$-fcd. Thus D is connected to a minimal $(-)$-fcd by a positive path.

By the same argument, we can show that for a minimal $(-)$-fcd there is a positive path from a minimal $(+)$-fcd to the minimal $(-)$-fcd. This completes the proof.

LEMMA 2. *Let D be a* $(+)$-fcd *and L a compact leaf of F with $L \subset D$. Then there is a positive path P such that $S(P)$ is a minimal* $(+)$-fcd *contained in D, and $T(P)$ is a minimal* $(-)$-fcd *contained in* $M-(\operatorname{Int} D)$.

PROOF. Assume that $D_j \in \{D_i\}$ contains L, and that D_j is contained in D. Set Y_1 to be the union of all D_i's which appear in some positive path P with $T(P)=D_j$, and Y_2 the union of all D_i's which appear in some positive path P with $S(P)=D_j$. Since D is a $(+)$-fcd, we have $Y_1 \subset D$ and $Y_2 \subset M-(\operatorname{Int} D)$. By the same argument as in the proof of Lemma 1, Y_1 is a $(+)$-fcd and Y_2 is a $(-)$-fcd. Thus Y_1 contains a minimal $(+)$-fcd, and Y_2 contains a minimal $(-)$-fcd. This completes the proof.

We now return to the proof of the Theorem. To show the converse, assume that f satisfies the condition $f(x)>0$ somewhere in any minimal $(+)$-fcd and $f(y)<0$ somewhere in any minimal $(-)$-fcd. We show that there is a volume form dV on M satisfying the condition (2) in Theorem O1. Choose a Riemannian metric g on M such that

$$\int_{D_i} |f|dV(M,g)<\frac{1}{2(k+1)} \qquad \text{for} \quad i=0,1,\dots,k\,.$$

Set $C=\sum_{i\neq u,v}\int_{D_i} fdV(M,g)$, where the summation is taken over all D_i's except minimal $(+)$-fcd's D_u and minimal $(-)$-fcd's D_v. We denote this summation by $\sum_{i\neq u,v}$.

By assumption, we have

$$|C|<(k+1)\times\frac{1}{2(k+1)}=\frac{1}{2}\,.$$

Now deform the volume element $dV(M,g)$ into dV so that

$$\int_{D_0} fdV=A_0-C\geq 1-\frac{1}{2}>0\,,$$

$$\int_{D_u} fdV=A_u \text{ for each minimal } (+)\text{-fcd } D_u\,,$$

$$\int_{D_v} fdV=-B_v \text{ for each minimal } (-)\text{-fcd } D_v\,.$$

Note that the D_0 is the previously chosen minimal $(+)$-fcd.

We can easily find such a dV with $dV=dV(M,g)$ near the boundaries of D_u and D_v, because f is positive somewhere on each D_u and negative somewhere on each D_v. Here D_u is one of the minimal $(+)$-fcd's and D_v is one of the minimal $(-)$-fcd's. On other D_i's, we set $dV=dV(M,g)$. We show that this dV satisfies the condition (2) in Theorem O1.

First note that $\sum_u A_u-\sum_v B_v=0$, because if a minimal $(+)$-fcd is connected to a minimal $(-)$-fcd by a positive path, then the minimal $(-)$-fcd is connected to the minimal $(+)$-fcd by the same positive path. Here the summation $\sum_u$ is taken over all minimal $(+)$-fcd's, and the summation $\sum_v$ is taken over all minimal $(-)$-fcd's.

From this observation, we have

$$\int_M fdV=\sum_{i=0}^{k}\int_{D_i} fdV=A_0-C+\sum_{u\neq 0} A_u-\sum_v B_v+C=0\,.$$

The summation convection is the same as above. This shows that the condition (i) in (2) is satisfied.

Let D be an arbitrary $(+)$-fcd. To show that the condition (ii) in (2) is satisfied, we have only to show that $\int_D f dV > 0$. Set U to be the subset of indices $\{0, 1, \ldots, k\}$ such that $u \in U$ implies D_u is a minimal $(+)$-fcd and is contained in D. Set also V to be the subset of indices $\{0, 1, \ldots, k\}$ such that $v \in V$ implies D_v is a minimal $(-)$-fcd and is contained in D. Note that $\sum_{u \in U} A_u - \sum_{v \in V} B_v \geq 1$; Indeed, if a minimal $(-)$-fcd in D is connected by a positive path to a minimal $(+)$-fcd, then the minimal $(+)$-fcd must be in D since D is a $(+)$-fcd. By Lemma 2, at least one $(+)$-fcd in D is connected by a positive path to a minimal $(-)$-fcd outside of D. This means $\sum_{u \in U} A_u - \sum_{v \in V} B_v \geq 1$. By this observation, if D_0 is contained in D, then we have

$$\int_D f dV \geq \sum_{u \in U} A_u - \sum_{v \in V} B_v - 2C > 1 - 2 \times \frac{1}{2} = 0 .$$

If D_0 is not contained in D, then we have

$$\int_D f dV \geq \sum_{u \in U} A_u - \sum_{v \in V} B_v - C > 1 - \frac{1}{2} > 0 .$$

This completes the proof of the Theorem.

4. Foliations of constant mean curvature. Let (M, F, g) be a codimension-one foliation of a Riemannian manifold. We call F a foliation of constant mean curvature if the mean curvature function H of the foliation is constant on each leaf of F. We say a codimension-one foliation *tense*, if we can find a Riemannian metric so that the foliation is of constant mean curvature with respect to this metric. In this section, we give a simpler proof of the topological characterization in Oshikiri [6] of tense foliations.

We say that a compact leaf L_0 is *contained in a continuous family* if there is a compact saturated set D which contains L_0 and is diffeomorphic to $L_0 \times [0, 1]$ so that the foliation induced on D by F corresponds to $L_0 \times \{t\}$, $t \in [0, 1]$. Denote by $C(F)$ the union of all compact leaves which are contained in continuous families.

The characterization of tense foliations given in Oshikiri [6] is the following:

THEOREM O2. *Let (M, F) be a transversely oriented codimension-one foliation of a connected, closed, and oriented manifold M with* $\dim M \geq 3$. *Then F is tense if and only if each connected component of $M - C(F)$ does not contain a* $(+)$-fcd *and a* $(-)$-fcd *simultaneously.*

PROOF. The first half of the proof is the same as the one in [6]. However, for the sake of convenience, we repeat it here.

If F is tense, then there is a Riemannian metric g on M so that each leaf L of F is a hypersurface of constant mean curvature. We denote the mean curvature of F by H. Since the set $\{x \in M \mid dH_x \neq 0\}$ consists of compact leaves of F (cf. Barbosa-Kenmotsu-Oshikiri [1]), the mean curvature function H is constant on each connected component of $M - C(F)$. If a connected component X of $M - C(F)$ contains both $(+)$-fcd C_+ and

$(-)$-fcd C_-, then by assumption and Proposition R,

$$-\int_{C_+} HdV = \int_{C_+} d\chi_F = \int_{\partial C_+} \chi_F = \mathrm{Vol}(\partial C_+) > 0 .$$

On the other hand, by the same argument, we have

$$-\int_{C_-} HdV = -\mathrm{Vol}(\partial C_-) < 0 .$$

Since H has the same constant value on both C_+ and C_-, this is impossible.

Now we show the converse. This part is fairly simplified. We have only to construct an admissible function f which is constant on each leaf L of F. To do so, first define f on each connected component X on $M-C(F)$ by $f\equiv 1$ if X contains a $(+)$-fcd, $f\equiv -1$ if X contains a $(-)$-fcd, and $f\equiv 0$ if X contains neither $(+)$-fcd's nor $(-)$-fcd's. On the closed unti interval $[0, 1]$, there is a smooth function h with $h\equiv a$ on $[0, 1/4)$ and $h\equiv b$ on $(3/4, 1]$. Here a and b are arbitrary real numbers. On each $L\times[0, 1]$, where L is a compact leaf of $\partial(C(F))$, define $f(x, t)=h(t)$ for $(x, t)\in L\times[0, 1]$ with suitable a and b in order to get a smooth function f on M. By construction, it is clear that f is constant on each leaf of F, and that f satisfies the condition in the Theorem. Thus f is admissible. This completes the proof.

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