A CHARACTERIZATION OF THE MULTIVARIATE NORMAL DISTRIBUTION[1]

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1. Introduction: independence of linear forms. Let $X_1, \cdots, X_n$ be independent p-dimensional random row vectors, and let there exist non-zero constants $a_1, \cdots, a_n, b_1, \cdots, b_n$, such that $\sum X_i a_i$ is independent of $\sum X_i b_i$. By considering all linear combinations $\theta X_i'$, where $\theta = (\theta_1, \cdots, \theta_p)$, it follows from the well-known univariate result, first proved completely by Skitovič [7], that the X_i are normally distributed. (For a history of the subject, see Lukacs [4, Section 5].) However, when the scalars a_i, b_i are replaced by $p \times p$ matrices A_i, B_i, this reduction to the univariate case no longer holds. The matrix case for $n = 2$ was treated in [2]. In this paper we treat the general multivariate case.

Another peculiarity of the matrix case stems from the distinction between singularity and vanishing of a matrix. In the one-dimensional problem, if one of the coefficients a_i or b_i is zero, the distribution of the corresponding random variable can be completely arbitrary. The same is true in the matrix case if one of the matrices A_i or B_i is zero. However, if a matrix A_i, say, is singular but not zero, then some linear combinations of elements of the corresponding random vector X_i are normally distributed, but the distribution of X_i is partly arbitrary. An example of a possible consequence is the following:

Let X_1, X_2 be independent random row vectors, and let A be a singular matrix of rank r such that $X_1 + X_2$ and $X_1 + X_2 A$ are independent. There exist non-singular matrices M and N such that $A = M\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}N$, where I_r is the identity matrix of order r. Writing

$$X_i M = Y_i = (Y_{i1}, Y_{i2}), \quad \text{and} \quad NM = B = (B_{ij}), i, j = 1, 2,$$

we have that $(Y_{11}, Y_{12}) + (Y_{21}, Y_{22})$ is independent of $(Y_{11}, Y_{12}) + (Y_{21}B_{11}, Y_{21}B_{12})$. Consequently, the hypothesis does not restrain Y_{22} sufficiently to determine its distribution, and in fact, if Y_{22} is independent of Y_{21}, it can have any distribution without affecting the hypothesis.

We now state the principal result and outline its proof. The main details, which have an intrinsic interest, are given in the next section.

Theorem. *Let $X_1, \cdots, X_n$ be n mutually independent p-dimensional random*

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row-vectors, and let $A_1, \cdots, A_n, B_1, \cdots, B_n$ *be non-singular* $p \times p$ *matrices. If* $\sum A_iX_i'$ *is independent of* $\sum B_iX_i'$, *then the* X_i *are normally distributed.*

Remarks on the Proof. Let t and u denote real-valued p-dimensional row vectors. In terms of characteristic functions, the hypothesis states that $E \exp i(\sum tA_jX_j' + \sum uB_jX_j') = E \exp (i \sum tA_jX_j')E \exp (i \sum uB_jX_j')$, or equivalently,

$$\prod_1^n \varphi_j(tA_j + uB_j) = \prod_1^n \varphi_j(tA_j)\varphi_j(uB_j) \equiv F(t)G(u),$$

where $\varphi_j(t) \equiv E \exp (itX_j')$.

The proof involves a series of steps. We first show that the φ_j have no zeros (Lemma 1), and then show that the above functional equation, which is a generalization of the equation of Skitovič [7], implies that $\sum \log \varphi_j(t)$ is a polynomial in the vector t. By letting $t = uv$, where u is a real variable and v a fixed vector, and using the univariate theorem of Marcinkiewicz [5], it follows that $\sum \log \varphi_j(uv)$ is a quadratic polynomial in u for each fixed vector v. This implies that $\sum \log \varphi_j(t)$ is a quadratic polynomial in the vector t. Finally, as a consequence of the multivariate theorem of Cramér [1, p. 112], each $\log \varphi_j(t)$ is a quadratic polynomial.

2. An extension of the functional equation of Skitovič. We first show that the φ_j have no zeros.

Lemma 1. *Let* $\varphi_j(t)$ *be characteristic functions on* R_p. *If there exist non-singular matrices* $A_j, B_j, j = 1, 2, \cdots, n$, *such that*

$$\prod_1^n \varphi_j(tA_j + uB_j) = \prod_1^n \varphi_j(tA_j)\varphi_j(uB_j), \tag{1}$$

for all $t, u \in R_p$, *then the* φ_j *have no zeros.*

Proof. The general outline of the proof follows that of Skitovič for the case $p = 1$. Denote the right-hand side of (1) by $F(t)G(u)$, and suppose that one of the functions, say φ_j, has a zero. Then there exists a vector v such that $\varphi_j(vA_j) = 0$, and consequently that $F(v)G(v) = 0$. Let λ^2 be the largest characteristic root of $(A_j B_j^{-1})(A_j B_j^{-1})'$. Choose an r such that $0 < r < \min(1, |\lambda|^{-1})$, and define

$$v_1 = (1 - r)v, \qquad v_2 = r\, vA_jB_j^{-1}.$$

Then $v_1v_1' < vv'$ and $v_2v_2' < vv'$. On putting $t = v_1$ and $u = v_2$ in (1), we obtain

$$\prod_1^n \varphi_i(v_1A_i + v_2B_i) = F(v_1)G(v_2) = 0,$$

since $v_1A_j + v_2B_j = vA_j$. Hence, either v_1 or v_2 is a zero of $F(t)G(t)$. Thus, corresponding to every zero of $F(t)G(t)$, there exists another which is nearer the origin. But this contradicts the fact that the zeros of $F(t)G(t)$, which is a characteristic function, form a closed set, bounded away from the origin.‖[3]

Hence, $f_j(t) = \log \varphi_j(t), j = 1, \cdots, n$, is defined for all t. From (1) we obtain

$$(2) \qquad \sum_1^n f_j(tA_j + uB_j) = \sum_1^n f_j(tA_j) + \sum_1^n f_j(uB_j), \qquad \text{for all } t, u \in R_p .$$

This equation can be simplified somewhat by writing $C_j \equiv B_j A_j^{-1}$ and $g_j(t) \equiv f_j(tA_j)$, namely,

$$(3) \qquad \sum_1^n g_j(t + uC_j) = \sum_1^n g_j(t) + \sum_1^n g_j(uC_j).$$

Equation (2) is a generalization of the equation of Skitovič. A further generalization is given in Lemma 3. The proof of Skitovič, when A_j, B_j are scalars, uses a complicated method of exhaustion. Zinger and Linnik [8] give a very elegant solution. Kemperman [3] discusses this equation under weak conditions on the f_j. We treat an extension of (2), which incidentally yields a stronger result for the scalar case.

Lemma 2. *Let $P(u \mid t)$ be a complex-valued function of the real, p-dimensional vectors $t = (t_1, \cdots, t_p)$, $u = (u_1, \cdots, u_p)$, which for each fixed value of t, is a polynomial in u of degree $\leqq r$. If to each vector v, there corresponds a vector $w \equiv w(v)$, depending only on v, such that*

$$Q(t, u) = P(u + w \mid t + v) - P(u \mid t)$$

is a polynomial in (t, u) of degree $\leqq s$, then $P(u \mid t)$ is a polynomial in (t, u) of degree $\leqq s + 1$.

Proof. By the hypothesis, $P(u \mid t)$ can be written in the form $\sum_0^r P_j(u \mid t)$, where

$$P_j(u \mid t) = \sum_j p_j(\alpha_1, \cdots, \alpha_p; t) u_1^{\alpha_1} \cdots u_p^{\alpha_p},$$

and $\sum_j$ denotes summation over all $\alpha_i \in \{0, 1, \cdots, r\}$ with $\alpha_1 + \cdots + \alpha_p = j$. Hence, $Q(t, u) = \sum_0^r [P_j(u + w \mid t + v) - P_j(u \mid t)]$.

The proof is by induction on r. The lemma is true for $r = 0$, since, if $P_0(t + v) - P_0(t)$ is a polynomial in t of degree $\leqq s$ for every v, then $P_0(t)$ is a polynomial of degree $\leqq s + 1$. Suppose the lemma to hold for $r = 0, 1, \cdots, m - 1$ and let $P(u \mid t) = \sum_0^m P_j(u \mid t)$ be a function satisfying the hypothesis of the lemma. Then

$$\begin{aligned} Q(t, u) = {} & \sum_0^{m-1} [P_j(u + w \mid t + v) - P_j(u \mid t)] \\ (4) \qquad & + \sum_m [p_m(\alpha_1, \cdots, \alpha_p; t + v)(u_1 + w_1)^{\alpha_1} \cdots (u_p + w_p)^{\alpha_p} \\ & - p_m(\alpha_1, \cdots, \alpha_p; t) u_1^{\alpha_1} \cdots u_p^{\alpha_p}]. \end{aligned}$$

[3] The symbol ‖ denotes end of proof.

The monomials in u of degree m occur only in the last expression. It follows from the hypothesis that $[p_m(\alpha_1, \cdots, \alpha_p; t + v) - p_m(\alpha_1, \cdots, \alpha_p; t)]$, which is the coefficient of $u_1^{\alpha_1} \cdots u_p^{\alpha_p}$ with $\alpha_1 + \cdots + \alpha_p = m$, is a polynomial in t of degree $\leqq s - m$. Thus the $p_m(\alpha_1, \cdots, \alpha_p; t)$, for $\alpha_1 + \cdots + \alpha_p = m$, are polynomials of degree $\leqq s - m + 1$, from which we obtain that $P_m(u \mid t)$ is a polynomial in (t, u) of degree $\leqq s + 1$. Consequently, the last expression in (4) is a polynomial in (t, u) of degree $\leqq s + 1$, as is the second term, which is the difference $Q - \sum_m$. By the induction hypothesis, this implies that $\sum_0^{m-1} P_j(u \mid t)$ is a polynomial. Hence $\sum_0^m P_j(u \mid t)$ is a polynomial in (t, u) of degree $\leqq s + 1$, which completes the proof.||

LEMMA 3. *Let $f_1(t), \cdots, f_n(t)$ be complex-valued functions on R_p which are bounded on every finite set. Let $A(x \mid y)$ and $B(x \mid y)$ be defined on $R_p \times R_p$, and for each fixed $y \varepsilon R_p$, be polynomials in x of degree $\leqq a$ and b, respectively. If there exist real, non-singular $p \times p$ matrices, $C_1, \cdots, C_n$ such that*

$$\sum_1^n f_i(t + uC_i) = A(t \mid u) + B(u \mid t), \quad \textit{for all } t, u \,\varepsilon\, R_p, \tag{5}$$

then

(i) *$\sum_1^n f_i(t + uC_i)$ is a polynomial in (t, u) of degree $\leqq a + b + n$.*

(ii) *If, in addition, there exists a non-empty set $N = \{i_1, \cdots, i_r\} \subset \{1, \cdots, n\}$, such that $C_i = C_j$ whenever $i, j \varepsilon N$ and $|C_i - C_j| \neq 0$ whenever $i \varepsilon N, j \not\varepsilon N$, then $\sum_{i\varepsilon N} f_i(t)$ and $\sum_{i \not\varepsilon N} f_i(t + uC_i)$ are polynomials of degree $\leqq a + b + n - r + 1$.*

REMARKS. The lemma is similar to Theorem 6.1 of Kemperman [3], in which $B(u \mid t)$ is assumed to be of degree zero in u, for each given t. However, his concern is somewhat different, and his assumptions regarding f_i are weaker than ours. Presumably, the lemma could be proven with weaker conditions on the f_i, but at the cost of simplicity of presentation. To a certain degree, our proof is an elaboration on that of Kemperman.

PROOF. We first prove (i) by induction on n. We use the notation $\Delta_t^r(h_1, \cdots, h_r)$ to denote $\prod_1^r \Delta_t(h_i)$, where $\Delta_t(h)$ is the difference operator defined by $\Delta_t(h)F(t, u) = F(t + h, u) - F(t, u)$. Here $h_1, \cdots, h_r$ are p-dimensional vectors.

Choose $h_0, h_1, \cdots, h_a \,\varepsilon\, R_p$ and difference (5) $a + 1$ times with these increments in t. This yields

$$\sum_1^n g_i(t + uC_i) = \Delta_t^{a+1}(h_0, h_1, \cdots, h_a)B(u \mid t) \equiv P(u \mid t), \tag{6}$$

where

$$g_i(t) = \Delta_t^{a+1}(h_0, h_1, \cdots, h_a)f_i(t). \tag{7}$$

For the present, the h_i are held constant. We show by induction on n that (6) implies that $P(u \mid t)$ is a polynomial in (t, u) of degree $\leqq b + n - 1$.

Note that $P(u \mid t)$ is, for each t, a polynomial in u of degree $\leqq b$. Hence, if

$n = 1$ in (6), we see by putting $t = 0$ that g is a polynomial of degree $\leqq b$. Consequently, if $n = 1$, $P(u \mid t)$ is a polynomial in (t, u) of degree $\leqq b$.

Let H_n denote the induction hypothesis that (6) implies that $P(u \mid t)$ is a polynomial in (t, u) of degree $\leqq b + n - 1$. Since H_1 is true, we need to show that H_{n-1} implies H_n.

Suppose we have functions $g_1, \cdots, g_n$ and $P(u \mid t)$ satisfying (6), where $P(u \mid t)$ is a polynomial in u, for each fixed t, of degree $\leqq b$. Choose a vector $v \varepsilon R_p$, and let $v_i = v(I - C_n^{-1}C_i)$. Then from (6), we have

$$\sum_1^n g_i(t + v_i + uC_i) = \sum_1^n g_i(t + v + (u - vC_n^{-1})C_i) = P(u - vC_n^{-1} \mid t + v). \tag{8}$$

Subtracting (6) from (8), writing $g_i^*(t) = g_i(t + v_i) - g_i(t)$, and noting that $v_n = 0$, we obtain

$$\sum_1^{n-1} g_i^*(t + uC_i) = P(u - vC_n^{-1} \mid t + v) - P(u \mid t), \tag{9}$$

which is of the same form as (6) with $n - 1$ terms, instead of n, on the left-hand side. The function on the right-hand side of (9) is, for each t, a polynomial in u of degree $\leqq b$.

By the induction hypothesis H_{n-1}, the right-hand side of (9) is a polynomial in (t, u) of degree $\leqq b + n - 2$. Consequently, by Lemma 2, $P(u \mid t)$ is of degree $\leqq b + n - 1$, thus establishing that H_{n-1} implies H_n.

As a result, if we consider any rth degree monomial in u in $P(u \mid t)$, its coefficient is a polynomial in t of degree $\leqq b + n - 1 - r$. By (6) and (7), this means that the coefficient of the corresponding monomial in u in $B(u \mid t)$ is a polynomial in t of degree $\leqq a + b + n - r$.

Write $F_i(t) = f_i(tC_i)$, so that (5) becomes

$$\sum_1^n F_i(u + tC_i^{-1}) = B(u \mid t) + A(t \mid u), \tag{10}$$

which is of the same type as (5) with the roles of A and B interchanged. Hence, $A(t \mid u)$ is also a polynomial in (t, u) of degree $\leqq a + b + n$, thus verifying part (i).

To prove (ii), let C denote the common value of C_i for $i \varepsilon N$ and $g(t) = \sum_{i \varepsilon N} f_i(t)$. Then (5) becomes

$$g(t + uC) + \sum_{i \notin N} f_i(t + uC_i) = A(t \mid u) + B(u \mid t), \tag{11}$$

which is again of the same form as discussed above, but with n reduced to $n - r + 1$. Hence, each side of (11) is a polynomial of degree $\leqq a + b + n - r + 1$.

Choose row vectors $k_0, \cdots, k_b \varepsilon R_p$, and take the $(b + 1)$st difference of (11) with these increments in u. Denoting the resulting functions with asterisks, we obtain

$$g^*(t + uC) + \sum_{i \notin N} f_i^*(t + uC_i) = A^*(t \mid u). \tag{12}$$

Rewrite (12) with $v = t + uC$ in the form

$$\sum_{i \notin N} f_i^*(v + uC_i^*) = A^*(v - uC \mid u) - g^*(v), \tag{13}$$

where $C_i^* = C_i - C$.

Since $A^*(t \mid u)$ is a polynomial in t of degree $\leqq a$, we see that this equation has the form of (5) with $b = 0$ and n replaced by $n - r$. Hence the right-hand side of (13) is a polynomial of degree $\leqq a + n - r$. In particular, $A^*(v \mid 0) - g^*(v)$ is of degree $\leqq a + n - r$. Since the first term is of degree $\leqq a$, the degree of g^* itself cannot exceed $a + n - r$. This holds for each set $\{k_0, \cdots, k_b\}$, and since $g^*(t) = \Delta_t^{b+1}(k_0C, \cdots, k_bC)g(t)$, it follows that $g(t)$ is a polynomial of degree at most $a + b + n - r + 1$.||

Corollary 3.1. (Skitovič [7], Zinger and Linnik [8], Kemperman [3]). *In the special case where all the C_i are scalars, let $c_1, \cdots, c_k$, $(c_j \neq 0)$ denote the distinct points of $\{C_1, \cdots, C_n\}$, and let $\sum_i^*$ denote summation over all j such that $C_j = c_i$. If the hypotheses of Lemma 3 are satisfied, then $\sum_i^* f_j(t)$ is a polynomial of degree $\leqq a + b + k$.*

Remark. We note that this inequality for the degree is sharp in the sense that equality can be achieved. For, suppose we are given k distinct scalars $c_1, \cdots, c_k$, we can always find k real numbers α_i, such that the terms

$$t^{a+k-1}u^{b+1}, \cdots, t^{a+1}u^{b+k-1} \quad \text{in} \quad \sum_1^k \alpha_i(t + c_i u)^{a+b+k}$$

have zero coefficients. Consequently, this sum can be written in the form of the right-hand side of (5). This follows from the non-singularity of the Vandermonde matrix with distinct arguments c_i.

3. Related results. As indicated in the previous remark, Corollary 3.1 gives a complete solution to the functional equation of Skitovič, in which the C_i are all scalars. This was possible because of the fact that, given two scalars, either they are equal or their difference is non-singular. This, of course, is no longer true in the matrix case. In order to see what may happen when two of the matrices are not equal, but their difference is singular, we consider formula (5) in detail for $n = 2$. In particular, we find that there is a partial resolution of the problem corresponding to the partial distinctness between C_1 and C_2.

Corollary 3.2. *If in (5), $n = 2$, and $B = C_1 - C_2 : p \times p$, has rank r, then $f_i(tB)$, $i = 1, 2$, are polynomials in t of degree $\leqq a + b + 2$. Furthermore, there exists a non-singular matrix N such that $f_i((x_1, 0)N)$, $i = 1, 2$, are polynomials of degree $\leqq a + b + 2$, where x_1 and x_2 have dimensionality r and $(p - r)$, respectively.*

Proof. From Lemma 3, $\sum f_i(t + uC_i)$ is a polynomial in (t, u) of degree

$\leqq a + b + 2$. Writing $v = t + uC_2$, we have that $f_1(v + uB) + f_2(v)$ is a polynomial in (v, u) of degree $\leqq a + b + 2$. Putting $v = 0, f_1(uB)$ is a polynomial in u of degree $\leqq a + b + 2$. Putting $u = 0$, $v = tB$, $f_1(tB) + f_2(tB)$ is a polynomial in t of degree $\leqq a + b + 2$, so that $f_2(tB)$ is a polynomial in t of degree $\leqq a + b + 2$.

Note that $f_i(uB)$ is a function on an r-dimensional space. This is emphasized in the second assertion of the corollary, which follows from the fact that there exist non-singular matrices M and N such that $B = M\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}N$. Writing $uM = x$, we then see that $f_i((x_1, 0)N)$, $i = 1, 2$, is a polynomial.||

COROLLARY 3.3. *Let $f_1(t), \cdots, f_n(t)$ be complex-valued functions on R_p, which are bounded on every finite set. If there exist non-singular matrices A_{ij}, $i = 1, \cdots, n$, $j = 1, \cdots, r$, such that $\sum_{i=1}^n \Delta_t^r(u_1A_{i1}, \cdots, u_rA_{ir})f_i(t)$ is independent of t for all vectors $u_1, \cdots, u_r \varepsilon R_p$, then $\sum_1^n f_i(t)$ is a polynomial of degree $\leqq nr$.*

PROOF. Using Lemma 3, the result is easily proved by induction on r. First note that if $r = 1$, the hypothesis states that $\sum_1^n f_i(t + u_1A_{i1}) = -\sum_1^n f_i(0) + \sum_1^n f_i(t) + \sum_1^n f_i(u_1A_{i1})$, which is the hypothesis of Lemma 3 with $a = b = 0$. Hence the statement is true for $r = 1$. Suppose that it is true for $r = 1, 2, \cdots, s - 1$, and that we are given the data of the corollary with $r = s$. Let $f_i^*(t) = \Delta_t(u_sA_{is})f_i(t)$. Then $\sum_1^n \Delta_t^{s-1}(u_1A_{i1}, \cdots, u_{s-1}A_{i,s-1})f_i^*(t)$ is independent of t. Hence, by the induction hypothesis, $\sum_1^n f_i^*(t)$ is a polynomial in t of degree $\leqq n(s - 1)$, whose coefficients might depend on u_s; i.e., in the notation of (5),

$$\sum_1^n f_i(t + u_sA_{is}) - \sum_1^n f_i(t) = A(t \mid u_s),$$

with $a = n(s - 1), b = 0$. Consequently $\sum f_i(t)$ is a polynomial of degree $\leqq n(s - 1) + n = ns$, which completes the induction.||

Finally, we wish to draw attention to the rather ingenious method of Zinger and Linnik [8] for solving the equation of Skitovič when $p = 1$. We elaborate on their method, and obtain a stronger result—in particular, the conclusion of Corollary 3.1 for $p = 1$—while avoiding the use of the Zinger-Linnik extension of Cramér's Theorem.

LEMMA 4[4]. *Let $f_1(t), \cdots, f_n(t)$ be continuous, complex-valued functions of a real variable. If there exist distinct, non-zero numbers $c_1, \cdots, c_n$ such that*

$$\sum_1^n f_i(t + c_iu) = A(t \mid u) + B(u \mid t), \tag{14}$$

where $A(x \mid y)$ and $B(x \mid y)$ are, for fixed $y \varepsilon R_1$, polynomials in x of degree $\leqq a$ and b, respectively, then the $f_i(t)$ are polynomials of degree $\leqq a + b + n$.

PROOF. Choose $h_1, \cdots, h_b$, and difference (14) b times with these increments

[4] While this paper was in proof, the Editor received a communication from B. Ramachandran, Catholic University, in which a result similar to Lemma 4 is independently obtained by essentially the same method used here.

in u. Denoting the resulting functions with asterisks, we obtain

$$\sum_1^n f_i^*(t + c_i u) = b^*(t) + \sum_0^a a_j^*(u) t^j. \tag{15}$$

Continuity of the f_i implies continuity of the functions on the right-hand side of (15). Multiply (15) by $(x - t)^r$, and integrate with respect to t over $(0, x)$, namely,

$$\begin{aligned} &\sum_1^n \int_0^x f_i^*(t + c_i u)(x - t)^r \, dt \\ &\quad = \int_0^x b^*(t)(x - t)^r \, dt + \sum_{j=0}^a a_j^*(u) \int_0^x t^j (x - t)^r \, dt \\ &\quad = \int_0^x b^*(t)(x - t)^r \, dt + \sum_{j=0}^a B(j + 1, r + 1) a_j^*(u) x^{j+r+1}, \end{aligned} \tag{16}$$

where $B(p, q)$ is the Beta function. In each term of the sum in the left-hand side of (16), make a change of variable from t to v by $t = v - c_i u$, $i = 1, \cdots, n$, respectively. This yields

$$\begin{aligned} \sum_1^n \int_0^{x+c_i u} f_i^*(v)(x + c_i u - v)^r \, dv &= \sum_1^n \int_0^{c_i u} f_i^*(v)(x + c_i u - v)^r \, dv \\ &+ \int_0^x b^*(t)(x - t)^r \, dt + \sum_0^a B(j + 1, r + 1) a_j^*(u) x^{j+r+1}. \end{aligned} \tag{17}$$

The left-hand side is differentiable $(r + 1)$ times with respect to u, and on the right-hand side the terms containing u appear as coefficients of powers of x. Hence, the coefficient of each power of x is differentiable $(r + 1)$ times with respect to u. Performing this operation and setting $u = 0$, we obtain

$$\sum_1^n r\,! \, c_i^{r+1} f_i^*(x) = \sum_0^{a+r+1} A_j^* x^j. \tag{18}$$

Letting $r = 1, 2, \cdots, n$, we have a system of n linear equations in the n unknowns f_i^*, the coefficients of which form a Vandermonde matrix. Since the c_i are distinct, the matrix is non-singular, so that each $f_i^*(x)$ is obtained as a polynomial of degree $\leqq a + n + 1$. But $f_i^*(x)$ is the bth difference of f_i, and hence f_i is a polynomial of degree $\leqq a + b + n + 1$. By substituting such polynomials for the f_i, we find that (14) cannot be satisfied if any of the polynomials is of degree $a + b + n + 1$. Consequently the degree $a + b + n$ of the lemma cannot be improved upon.||

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