

A CHARACTERIZATION OF THE RIEMANN EXTENSION  
IN TERMS OF HARMONICITY

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*We dedicate our paper to Professor Ivan Kolář, with our warm birthday wishes*

*Abstract.* If  $(M, \nabla)$  is a manifold with a symmetric linear connection, then  $T^*M$  can be endowed with the natural Riemann extension  $\bar{g}$  (O. Kowalski and M. Sekizawa (2011), M. Sekizawa (1987)). Here we continue to study the harmonicity with respect to  $\bar{g}$  initiated by C. L. Bejan and O. Kowalski (2015). More precisely, we first construct a canonical almost para-complex structure  $\mathcal{P}$  on  $(T^*M, \bar{g})$  and prove that  $\mathcal{P}$  is harmonic (in the sense of E. Garcíá-Río, L. Vanhecke and M. E. Vázquez-Abal (1997)) if and only if  $\bar{g}$  reduces to the classical Riemann extension introduced by E. M. Patterson and A. G. Walker (1952).

*Keywords:* semi-Riemannian manifold; cotangent bundle; natural Riemann extension; harmonic tensor field

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1. INTRODUCTION

Harmonicity represents a very interesting topic, not only in differential geometry, but also in analysis, partial differential equations, theoretical physics and so on.

We recall that a function  $f: (N, h) \rightarrow \mathbb{R}$  on a (semi-)Riemannian manifold is harmonic if  $f$  is in the kernel of the Laplacian, i.e.  $f$  satisfies the Laplace equation. More general, a  $C^2$ -map  $\varphi: (N, h) \rightarrow (\bar{N}, \bar{h})$  between (semi-)Riemannian manifolds is harmonic if its tension field  $\tau(\varphi)$  vanishes identically, that is  $\varphi$  satisfies the Euler-Lagrange equations. Later, Garcíá-Río, Vanhecke and Vázquez-Abal defined the harmonicity of a  $(1, 1)$ -tensor field  $\mathcal{T}$  on a manifold  $N$ .

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More precisely, from [8], a  $(1, 1)$ -tensor field  $\mathcal{T}$  on a (semi-)Riemannian manifold  $(N, h)$  is called harmonic if it is a harmonic map when it is viewed as a map  $\mathcal{T}: (TN, h^c) \rightarrow (TN, h^c)$  between (semi-)Riemannian manifolds, where  $c$  denotes the complete lift (see Definition 4.3). Moreover, the harmonicity of a  $(1, 1)$ -tensor field is characterized in [8] as being divergence-free, which means that a  $(1, 1)$ -tensor field  $\mathcal{T}$  is harmonic if and only if  $\delta\mathcal{T} = 0$  (see Proposition 4.2).

Now, we establish the framework of our paper.

Let  $(M^n, \nabla)$  be a manifold endowed with a symmetric linear connection. Then, on its phase space (which is the total space of the cotangent bundle  $T^*M$ ), Patterson and Walker defined in [13] the (classical) Riemann extension. This notion was generalized by Sekizawa in [16] (see also Kowalski-Sekizawa [11], [12]) to natural Riemann extension  $\bar{g}$ , which is a semi-Riemannian metric on  $T^*M$  of signature  $(n, n)$ . The technique used there is that of lifting structures, which is well known in mathematical literature (see [19], [18], [15] and [10]). Natural Riemann extension is a special class of both the modified Riemann extension (see [6] and [9]) and the general Riemann extension. Bejan and Kowalski characterized in [5] some harmonic functions on  $(T^*M, \bar{g})$ .

The present paper goes further and characterizes the harmonicity of a canonical endomorphism field on  $T^*M$ . More precisely, we construct a canonical almost para-complex structure  $\mathcal{P}$  on  $(T^*M, \bar{g})$  and prove that  $\bar{g}$  is the (classical) Riemann extension introduced in [13] if and only if  $\mathcal{P}$  is harmonic.

## 2. PRELIMINARIES

Let  $M$  be a connected smooth  $n$ -dimensional manifold,  $n \geq 2$ , and let  $p: T^*M \rightarrow M$  be the natural projection from its cotangent bundle  $T^*M$  to  $M$ .

Any local chart  $(U; x^1, \dots, x^n)$  around  $x \in M$  induces a local chart  $(p^{-1}(U); x^1, \dots, x^n, x^{1*}, \dots, x^{n*})$  around  $(x, w) \in T^*M$ , where for any  $i = \overline{1, n}$ , we identify the function  $x^i \circ p$  on  $p^{-1}(U)$  with  $x^i$  on  $U$ . We denote  $x^{i*} = w_i = w((\partial/\partial x^i)_x)$  at any point  $(x, w) \in p^{-1}(U)$ . We obtain a basis for the tangent space  $(T^*M)_{(x, w)}$  at each point  $(x, w) \in T^*M$ :

$$\{(\partial_1)_{(x, w)}, \dots, (\partial_n)_{(x, w)}, (\partial_{1*})_{(x, w)}, \dots, (\partial_{n*})_{(x, w)}\},$$

where we put  $\partial_i = \partial/\partial x^i$  and  $\partial_{i*} = \partial/\partial w_i$ ,  $i = \overline{1, n}$ .

Let  $W \in \chi(T^*M)$  denote the canonical vertical vector field on  $T^*M$  which is a global vector field defined in local coordinate systems, by

$$(2.1) \quad W = \sum_{i=1}^n w_i \partial_{i*}.$$

We recall now the construction of the vertical and the complete lifts for which we refer to ([19], [18], [15] and [10]). If  $\alpha \in \Omega^1(M)$  is a differential one-form on  $M$ , then its *vertical lift*  $\alpha^v$  is the vector field which is tangent to  $T^*M$  and defined by:

$$(2.2) \quad \alpha^v(Z^v) = (\alpha(Z))^v, \quad Z \in \chi(M).$$

In local coordinates one can write:

$$(2.3) \quad \alpha^v = \sum_{i=1}^n \alpha_i \partial_{i*},$$

where  $\alpha = \sum_{i=1}^n \alpha_i dx^i$  and we identified  $f^v = f \circ p \in \mathcal{F}(T^*M)$  with  $f \in \mathcal{F}(M)$ . We note that  $\alpha^v(f^v) = 0$  for all  $f \in \mathcal{F}(M)$ .

The *complete lift of a vector field*  $X \in \chi(M)$  is defined as the vector field  $X^c \in \chi(T^*M)$  such that

$$(2.4) \quad X^c(Z^v) = [X, Z]^v, \quad Z \in \chi(M).$$

In local coordinates, it can be written at each point  $(x, w) \in T^*M$  as

$$X^c_{(x,w)} = \sum_{i=1}^n \xi^i(x) (\partial_i)_{(x,w)} - \sum_{h,i=1}^n w_h (\partial_i \xi^h)(x) (\partial_{i*})_{(x,w)},$$

where  $X = \xi^i \partial_i$ . We note that  $X^c(f^v) = (Xf)^v$  for all  $f \in \mathcal{F}(M)$ .

In general context, this technique is used for lifting structures to bundles, for which we refer to [10].

### 3. THE NATURAL RIEMANN EXTENSION

This section deals with the main notion used in our paper, which was introduced in [16] as a generalization of the (classical) Riemann extension defined in [13]:

**Definition 3.1.** Let  $(M, \nabla)$  be a manifold endowed with a torsion free linear connection. Then the natural Riemann extension  $\bar{g}$  is defined at each point  $(x, w) \in T^*M$  so that

$$(3.1) \quad \begin{aligned} \bar{g}_{(x,w)}(X^c, Y^c) &= -aw(\nabla_{X_x} Y + \nabla_{Y_x} X) + bw(X_x)w(Y_x), \\ \bar{g}_{(x,w)}(X^c, \alpha^v) &= a\alpha_x(X_x), \\ \bar{g}_{(x,w)}(\alpha^v, \beta^v) &= 0 \end{aligned}$$

for all vector fields  $X, Y$  and all differential one-forms  $\alpha, \beta$  on  $M$ , where  $a, b$  are arbitrary constants. We may assume  $a > 0$  without loss of generality.

We shall see later that the signature of  $\bar{g}$  is  $(n, n)$ . In particular, when  $a = 1$  and  $b = 0$ , it follows that  $(T^*M, \bar{g})$  is the classical Riemann extension of  $(M, \nabla)$ , for which we cite [13], [17].

Let us mention here a very useful fact given in [19]:

**Proposition 3.1.** *Let  $X$  and  $Y$  be two vector fields on  $T^*M$ . If  $X(Z^v) = Y(Z^v)$  holds for all  $Z \in \chi(M)$ , then  $X = Y$ .*

Later on we use the following conventions and formulas:

**Notation 3.1.** If  $\mathcal{T}$  is a  $(1, 1)$ -tensor field on a manifold  $M$ , then the *contracted vector field*  $\mathcal{C}(\mathcal{T}) \in \chi(T^*M)$  is defined at any point  $(x, w) \in T^*M$  by its value on any vertical lift as follows:

$$\mathcal{C}(\mathcal{T})(X^v)_{(x,w)} = (\mathcal{T}X)_{(x,w)}^v = w((\mathcal{T}X)_x), \quad X \in \chi(M).$$

For the Levi-Civita connection  $\bar{\nabla}$  of the Riemann extension  $\bar{g}$ , we get the following formulas (see e.g. [11]):

$$\begin{aligned} (3.2) \quad (\bar{\nabla}_{X^c} Y^c)_{(x,w)} &= (\nabla_X Y)_{(x,w)}^c + C_w((\nabla X)(\nabla Y) + (\nabla Y)(\nabla X))_{(x,w)} \\ &\quad + C_w(R_x(\cdot, X)Y + R_x(\cdot, Y)X)_{(x,w)} \\ &\quad - \frac{c}{2}\{w(Y)X^c + w(X)Y^c + 2w(Y)C_w(\nabla X) + 2w(X)C_w(\nabla Y) \\ &\quad + w(\nabla_X Y + \nabla_Y X)W\}_{(x,w)} + c^2 w(X)w(Y)W_{(x,w)}, \\ (\bar{\nabla}_{X^c} \beta^v)_{(x,w)} &= (\nabla_X \beta)_{(x,w)}^v + \frac{c}{2}\{w(X)\beta^v + \beta(X)W\}_{(x,w)}, \\ (\bar{\nabla}_{\alpha^v} Y^c)_{(x,w)} &= - (i_\alpha(\nabla Y))_{(x,w)}^v + \frac{c}{2}\{w(Y)\alpha^v + \alpha(Y)W\}_{(x,w)}, \\ (\bar{\nabla}_{\alpha^v} \beta^v)_{(x,w)} &= 0, \quad X, Y \in \chi(M), \quad \alpha, \beta \in \Omega^1(M), \end{aligned}$$

where  $c$  denotes the fraction  $b/a$ . For any  $(1, 1)$ -tensor field  $\mathcal{T}$  and a one-form  $\alpha$  on  $M$ , we denote by  $i_\alpha(\mathcal{T})$  the one-form of  $M$  defined by

$$(i_\alpha(\mathcal{T}))(X) = \alpha(\mathcal{T}X), \quad X \in \chi(M).$$

Let  $(x, w)$  be an arbitrary fixed point of  $T^*M$ , where  $w \neq 0$ . We take  $\{\alpha_1, \dots, \alpha_n\}$  to be a basis of covectors on  $T_x^*M$  such that

$$(3.3) \quad \alpha_1 = w$$

and let  $\{e_1, \dots, e_n\}$  be its dual basis on  $T_x M$ . We denote by the same letter  $e_i$  the parallel extension of each  $e_i$  (along geodesics starting at  $x$ ) to a normal neighborhood of  $x$  in  $M$  for  $i = \overline{1, n}$ , (see [11]). We obtain a local frame  $\{e_1, \dots, e_n\}$  defined around  $x$  in  $M$  such that

$$(3.4) \quad (\nabla_{e_i} e_j)_x = 0, \quad i, j = \overline{1, n}.$$

We note that:

$$\bar{g}_{(x,w)}(e_i^c, e_j^c) = bw(e_{i,x})w(e_j, x), \quad i, j = \overline{1, n}.$$

Next, we denote by the same letter  $\{\alpha_1, \dots, \alpha_n\}$  the local coframe defined around  $x$  on  $M$  which is dual to the local frame  $\{e_1, \dots, e_n\}$ , i.e.,  $\alpha_i(e_j) = \delta_{ij}$ ,  $i, j = \overline{1, n}$ , and we have automatically  $\alpha_{1,x} = w$ .

We construct as in [5] an orthonormal basis  $\{E_i, E_{i*}\}_{i=\overline{1, n}}$  with respect to  $\bar{g}$  in  $T_{(x,w)}(T^*M)$ , which is defined at any point  $(x, w) \in T^*M$  by

$$(3.5) \quad \begin{aligned} E_1 &= e_1^c + \frac{1-b}{2a}\alpha_1^v; & E_{1*} &= e_1^c - \frac{1+b}{2a}\alpha_1^v; \\ E_k &= \frac{1}{\sqrt{2a}}(e_k^c + \alpha_k^v); & E_{k*} &= \frac{1}{\sqrt{2a}}(e_k^c - \alpha_k^v). \end{aligned}$$

Therefore we have  $\bar{g}(E_i, E_i) = 1$  and  $\bar{g}(E_{i*}, E_{i*}) = -1$ ,  $i = \overline{1, n}$ , from which we can see that  $\bar{g}$  is of signature  $(n, n)$ .

#### 4. HARMONICITY OF AN ALMOST PARA-COMPLEX STRUCTURE

This section deals with para-complex geometry which has interesting features and the mathematical literature contains several papers on this subject. To mention only the ones published by the first author, we recall the classification of the para-Hermitian manifolds given in [1] (and cited in [7]), the existence problem studied in [2], and some examples of manifolds with almost para-Hermitian structures given in [3].

Our paper provides an almost para-complex structure  $\mathcal{P}$  on  $T^*M$ . We recall

**Definition 4.1.** An almost product structure  $\mathcal{P}$  (i.e.,  $\mathcal{P}^2 = \text{Id}$  and  $\mathcal{P} \neq \pm \text{Id}$ ) on a manifold  $N$  whose eigenvalues  $\pm 1$  have the same multiplicity, is called a para-complex structure.

Inspired by [4], in which the authors studied the harmonicity of an almost complex structure, here we shall construct a canonical almost para-complex structure for which we characterize its harmonicity.

**Definition 4.2.** We define the endomorphism

$$(4.1) \quad \mathcal{P}: \chi(T^*M) \rightarrow \chi(T^*M), \quad \text{such that} \quad \mathcal{P}X^c = X^c \quad \text{and} \quad \mathcal{P}\alpha^v = -\alpha^v,$$

where  $X^c$  and  $\alpha^v$  are the complete lift of a vector field  $X \in \chi(M)$  and the vertical lift of a differential one-form  $\alpha$  on  $M$ , respectively. We say that  $\mathcal{P}$  is canonical, since its eigen distributions are spanned by the complete lift (of all vector fields on  $M$ ) and respectively the vertical lift (of all one-forms on  $M$ ).

**Remark 4.1.** A similar endomorphism is constructed in [14], by using the horizontal lift.

From Definition 4.1, one can easily see that the structure  $\mathcal{P}$  defined by (4.1) satisfies  $\mathcal{P}^2 = \text{Id}$ ,  $\mathcal{P} \neq \pm \text{Id}$  and the eigen distributions of  $\mathcal{P}$  corresponding to the eigenvalues  $\pm 1$  of  $\mathcal{P}$  have the same rank. Therefore we obtain

**Proposition 4.1.** *Let  $M$  be an  $n$ -dimensional manifold. Then the total space of its cotangent bundle  $T^*M$  endowed with the structure  $\mathcal{P}$  is an almost para-complex manifold  $(T^*M, \mathcal{P})$ .*

We need the following notion introduced in [8]:

**Definition 4.3.** Any  $(1,1)$ -tensor field  $\mathcal{T}$  on a (semi-)Riemannian manifold  $(N, h)$  is called harmonic if  $\mathcal{T}$  viewed as an endomorphism field

$$\mathcal{T}: (TN, h^c) \rightarrow (TN, h^c)$$

is a harmonic map, where  $h^c$  denotes the complete lift (see [18]) of the semi-Riemannian metric  $h$ .

We recall from [8] the following characterization:

**Proposition 4.2.** *Let  $(N, h)$  be a (semi-)Riemannian manifold and let  $\nabla$  be the Levi-Civita connection of  $h$ . Then any  $(1,1)$ -tensor field  $\mathcal{T}$  on  $(N, h)$  is harmonic if and only if  $\delta\mathcal{T} = 0$ , where*

$$\delta\mathcal{T} = \text{trace}_h(\nabla\mathcal{T}) = \text{trace}_h\{(X, Y) \rightarrow (\nabla_X\mathcal{T})Y\}.$$

To obtain our main characterization result, we need the following

**Lemma 4.1.** *Let  $M$  be a connected smooth  $n$ -dimensional manifold with the total space of its cotangent bundle  $T^*M$ , endowed with the natural Riemann extension  $\bar{g}$ . Then the almost para-complex structure  $\mathcal{P}$  is harmonic on  $(T^*M, \bar{g})$  if and only if*

$$(4.2) \quad \frac{c(n+1)}{2}W_{(x,w)} = \sum_{s=1}^n (i_{\alpha_s}(\nabla e_s))_{(x,w)}^v,$$

where the frame  $\{e_1, \dots, e_n\}$  and its dual  $\{\alpha_1, \dots, \alpha_n\}$  are defined above in Section 3 and  $(x, w)$  is an arbitrary fixed point in  $T^*M$  such that  $w \neq 0$ .

*Proof.* Any relation written here will be calculated at each point  $(x, w) \in T^*M$ .

Let  $\bar{\nabla}$  be the Levi-Civita connection of  $\bar{g}$  given by (3.2). From Proposition 4.2 we have the following equivalences:

The almost product structure  $\mathcal{P}$  on  $(T^*M, \bar{g})$  is harmonic

$$(4.3) \quad \begin{aligned} &\Leftrightarrow 0 = \delta\mathcal{P} = \text{trace}_{\bar{g}} \bar{\nabla}\mathcal{P} \\ &\Leftrightarrow \sum_{i,j=1}^{2n} \bar{g}^{ij}(\bar{\nabla}_{H_i}\mathcal{P})H_j = 0, \end{aligned}$$

where  $\{H_i\}_{i=\overline{1,2n}}$  is a local basis on  $T^*M$  and  $\bar{g}^{ij}$  is the inverse matrix of the matrix  $(\bar{g}(H_i, H_j))_{i,j=\overline{1,2n}}$ . Then

$$(4.4) \quad (4.3) \Leftrightarrow \sum_{i=1}^{2n} \varepsilon_i(\bar{\nabla}_{F_i}\mathcal{P})F_i = 0,$$

where  $\{F_i\}_{i=\overline{1,2n}}$  is a local orthonormal frame on  $(T^*M, \bar{g})$  and  $\varepsilon_i = \bar{g}(F_i, F_i)$ ,  $i = \overline{1, 2n}$ . From (3.5) we have the equivalences

$$(4.4) \Leftrightarrow \sum_{s=1}^n \{(\bar{\nabla}_{E_s}\mathcal{P})E_s - (\bar{\nabla}_{E_{s*}}\mathcal{P})E_{s*}\} = 0 \Leftrightarrow (4.5),$$

where

$$(4.5) \quad \begin{aligned} &\bar{\nabla}_{E_1}\mathcal{P}E_1 - \mathcal{P}\bar{\nabla}_{E_1}E_1 - \bar{\nabla}_{E_{1*}}\mathcal{P}E_{1*} + \mathcal{P}\bar{\nabla}_{E_{1*}}E_{1*} \\ &= \sum_{k=2}^n \{\bar{\nabla}_{E_{k*}}\mathcal{P}E_{k*} - \mathcal{P}\bar{\nabla}_{E_{k*}}E_{k*} - \bar{\nabla}_{E_k}\mathcal{P}E_k + \mathcal{P}\bar{\nabla}_{E_k}E_k\}. \end{aligned}$$

From ([5], (4.6)) we obtain

$$(\bar{\nabla}_{E_{1*}}E_{1*} - \bar{\nabla}_{E_1}E_1)_{(x,w)} = -\frac{1}{a}\{(\nabla_{e_1}\alpha_1)^v + c\alpha_1^v + cW - (i_{\alpha_1}\nabla e_1)^v\}_{(x,w)}.$$

By applying  $\mathcal{P}$  defined by (4.1) we get:

$$(4.6) \quad (\mathcal{P}(\overline{\nabla}_{E_{1*}} E_{1*} - \overline{\nabla}_{E_1} E_1))_{(x,w)} = \frac{1}{a} ((\nabla_{e_1} \alpha_1)^v + c\alpha_1^v + cW - (i_{\alpha_1}(\nabla e_1))^v)_{(x,w)}.$$

A direct calculation using (3.2) yields

$$(4.7) \quad (\overline{\nabla}_{E_1} \mathcal{P} E_1 - \overline{\nabla}_{E_{1*}} \mathcal{P} E_{1*})_{(x,w)} = \frac{1}{a} (-i_{\alpha_1}(\nabla e_1)^v - (\nabla_{e_1} \alpha_1)^v)_{(x,w)}.$$

By virtue of (4.6) and (4.7), the left hand side of (4.5) becomes

$$(4.8) \quad \begin{aligned} & (\overline{\nabla}_{E_1} \mathcal{P} E_1 - \mathcal{P} \overline{\nabla}_{E_1} E_1 - \overline{\nabla}_{E_{1*}} \mathcal{P} E_{1*} + \mathcal{P} \overline{\nabla}_{E_{1*}} E_{1*})_{(x,w)} \\ &= \frac{1}{a} (c\alpha_1^v + cW - 2(i_{\alpha_1}(\nabla e_1))^v)_{(x,w)} \\ &= \frac{2}{a} (cW - (i_{\alpha_1}(\nabla e_1))^v)_{(x,w)}, \end{aligned}$$

where we used (2.1), (3.3) and (2.3).

Using ([5], (4.8)), we obtain:

$$(4.9) \quad \sum_{k=2}^n (\mathcal{P}(\overline{\nabla}_{E_{k*}} E_{k*} - \overline{\nabla}_{E_k} E_k))_{(x,w)} = \frac{1}{a} \sum_{k=2}^n ((\nabla_{e_k} \alpha_k)^v - (i_{\alpha_k}(\nabla e_k))^v + cW)_{(x,w)}.$$

A direct calculation using (3.2) leads

$$(4.10) \quad \sum_{k=2}^n (\overline{\nabla}_{E_k} \mathcal{P} E_k - \overline{\nabla}_{E_{k*}} \mathcal{P} E_{k*})_{(x,w)} = \frac{1}{a} \sum_{k=2}^n (-(i_{\alpha_k}(\nabla e_k))^v - (\nabla_{e_k} \alpha_k)^v)_{(x,w)}.$$

By virtue of (4.9) and (4.10), the right hand side of (4.5) becomes

$$(4.11) \quad \begin{aligned} & \sum_{k=2}^n (\overline{\nabla}_{E_{k*}} \mathcal{P} E_{k*} - \mathcal{P} \overline{\nabla}_{E_{k*}} E_{k*} - \overline{\nabla}_{E_k} \mathcal{P} E_k + \mathcal{P} \overline{\nabla}_{E_k} E_k)_{(x,w)} \\ &= \frac{1}{a} \sum_{k=2}^n (2(i_{\alpha_k}(\nabla e_k))^v - cW)_{(x,w)}. \end{aligned}$$

From (4.8), (4.11) and  $\alpha_1 = w$ , we get (4.2), which completes the proof.  $\square$

The above lemma yields the main result of the paper. To obtain it, we first remark that the classical Riemann extension has  $b = 0$  and therefore  $c = 0$ . Then, the left hand side of (4.2) is zero. Moreover, each term of the right hand side of (4.2) is zero, since applying the relation (2.3) and taking into account (3.4) gives  $(i_{\alpha_s}(\nabla e_s))_{(x,w)}^v = 0$ ,  $s = \overline{1, n}$ . Hence we proved the following:



**Theorem 4.1.** *Let  $M$  be a connected smooth  $n$ -dimensional manifold with the total space of its cotangent bundle  $T^*M$  endowed with the natural Riemann extension  $\bar{g}$ . The almost para-complex structure  $\mathcal{P}$  is harmonic if and only if  $\bar{g}$  is the classical Riemann extension.*

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