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A CHARACTERIZATION OF THE SPATIAL POISSON PROCESS AND CHANGING TIME

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Watanabe proved that if X_t is a point process such that $X_t - t$ is a martingale, then X_t is a Poisson process and this result was generalized by Brémaud for doubly stochastic Poisson processes. Here we define two-parameter point processes and extend this property without needing the strong martingale condition. Using this characterization, we study the problem of transforming a two-parameter point process into a two-parameter Poisson process by means of a family of stopping lines as a time change. Nualart and Sanz gave conditions in order to transform a square integrable strong martingale into a Wiener process. Here, we do the same for the Poisson process by a similar method but under more general conditions.

0. Introduction and notation. It is well known that the classical Poisson process is closely related to the theory of discontinuous martingales. A beautiful characterization was given by Watanabe: A point process X_t is a Poisson process if and only if the process $X_t - t$ is a martingale [12]. Several generalizations of the Poisson process were introduced in order to represent models in different applications. The most important seems to be the doubly stochastic Poisson process (also called the Cox process). Intuitively it is a generalized Poisson process such that the intensity of the process is a random measure. Also for such processes, a characterization in terms of martingales was obtained by Brémaud [1]. In this paper, we deal with stochastic processes indexed by points of the positive quadrant of the plane R_+^2, or of a rectangle $R_{z_0} = [(0,0), z_0]$. In this context, spatial point processes have a geometric interpretation with applications in different branches. In the first section we define two-parameter point processes and study their first properties. The spatial Poisson process and the spatial doubly stochastic Poisson process were defined and studied by several authors ([3], [8], [5]). At the same time, the concept of martingale can be generalized in the two-parameter case to different, nonequivalent kinds of martingales [15]. A first approach to extend the result of Watanabe in the spatial case was given by Papangelou [11] and by Mecke. In the second section we extend the result of Brémaud in terms of martingales. A similar result is the celebrated theorem of Lévy: A continuous martingale is a Wiener process if and only if its associated increasing process is the deterministic function $f(t) = t$. In the two-parameter

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case, this result was extended first by Wong and afterwards by Zakai (see [15]) in terms of strong martingales. The surprising fact is that in our characterization, the condition on the process to be a strong martingale is not essential.

The purpose of the last section is to study the problem of transforming a two-parameter point process into a two-parameter Poisson process by means of a two-dimensional time change. In the one-parameter case, the construction of a time change transforming every point process into a standard Poisson process was done first by Meyer [7], and was studied extensively by Papangelou [10]. The time change used is a family of stopping times. The stopping time has two kinds of generalizations for two-parameter processes: the stopping lines and the stopping points with respect to different filtrations corresponding to different kinds of past. Cairoli and Walsh [2] proved that a given two-parameter strong martingale cannot be time changed into a Brownian sheet by using a family of stopping points (in spite of the fact that this can be done in the one-parameter case via the Dubins–Schwarz theorem). Their construction shows that it is the same in our case: A given two-parameter strong martingale point process cannot be time changed into a Poisson process by using a family of stopping points. Starting from this fact, Nualart and Sanz [9] gave sufficient conditions for the existence of a family of stopping lines (or stopping sets) such that a strong martingale can be transformed into a Brownian sheet (i.e., a Wiener process). Also, they treat the particular case of changing time for the Wiener process obtaining some characterizations of all families of deterministic stopping sets which transform a Brownian sheet into another one.

Here, we deal only with point processes and the method is close to that of [9], but simpler, using the characterization theorem of Poisson processes in terms of martingales.

Let us introduce some notation. The usual partial order on R_+^2 will be denoted by $<$: $z < z'$ iff $s \leq s'$ and $t \leq t'$ where $z = (s, t)$ and $z' = (s', t')$. We write $z \ll z'$ if $s < s'$ and $t < t'$. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and $\{\mathscr{F}_z, z \in R_+^2\}$ be a filtration (a family of sub-σ-fields of $\mathscr{F}$) which is increasing ($z < z'$ implies $\mathscr{F}_z \subset \mathscr{F}_{z'}$), right continuous ($\mathscr{F}_z = \bigcap_{z' \gg z} \mathscr{F}_{z'}$) and complete (each $\mathscr{F}_z$ contains all null sets of $\mathscr{F}$). We define $\mathscr{F}_z^1 = \bigvee_{u \geq 0} \mathscr{F}_{(s,u)}$, $\mathscr{F}_z^2 = \bigvee_{v \geq 0} \mathscr{F}_{(v,t)}$, and $\mathscr{F}_z^* = \mathscr{F}_z^1 \vee \mathscr{F}_z^2$. In addition, we assume the hypothesis (F4) of conditional independence: For each $z \in R_+^2$, $\mathscr{F}_z^1$ and $\mathscr{F}_z^2$ are conditionally independent given $\mathscr{F}_z$. (This hypothesis is verified in most of the examples and permits the development of the stochastic calculus; see [13].)

Recall now some results about two-parameter martingales. For the sake of simplicity, we suppose that all the processes are vanishing on the axes. An adapted process $M = \{M_z, z \in R_+^2\}$ is called a martingale if it is integrable and for each pair $z < z'$, $E[M_{z'} | \mathscr{F}_z] = M_z$. It is a strong martingale if it is a martingale such that for each pair $z \ll z'$, $E[M(z, z'] | \mathscr{F}_z^*] = 0$, where $M(z, z'] = M_{z'} - M_{s,t'} - M_{s',t} + M_z$ is the increment of M over the interval $(z, z']$. M is called a weak martingale if it is an adapted process such that for each pair $z \ll z'$, $E[M(z, z'] | \mathscr{F}_z] = 0$. Generally the class of strong martingales is strictly smaller than the class of martingales, which is strictly smaller than the class of weak martingales. A process M is a martingale if and only if it is an

i-martingale for $i = 1$ and $i = 2$; that is, for each pair $z \ll z'$, $E[M(z, z']|\mathscr{F}_z^i] = 0$ and M is $\mathscr{F}_z^i$-adapted. In the product space $\Omega \times R_+^2$ we can define different σ-fields. We are interested here in the σ-field $\mathscr{P}$ of predictable sets: It is generated by simple sets of the form $F \times (z, z']$, $F \in \mathscr{F}_z$. It is also generated by the continuous and adapted processes. In the same manner, we define the σ-field $\mathscr{P}^i$ of i-predictable sets ($i = 1, 2$) and the σ-field $\mathscr{P}^*$ of $*$-predictable sets. We have: $\mathscr{P} = \mathscr{P}^1 \cap \mathscr{P}^2$ and $\mathscr{P}^* = \mathscr{P}^1 \vee \mathscr{P}^2$.

To each square integrable martingale M, we can associate a unique predictable increasing (right-continuous and with a nonnegative increment over any interval) process $\langle M \rangle$ such that $M^2 - \langle M \rangle$ is a weak martingale, and if M is a strong martingale then there exists a unique $\mathscr{P}^i$ increasing process $\langle M \rangle^i$ such that $M^2 - \langle M \rangle^i$ is an i-martingale [6].

To every increasing and integrable (or integrable variation) process A, we can associate its dual predictable projection A^{π}. It is the unique predictable increasing (integrable variation) process such that $A - A^{\pi}$ is a weak martingale [6]. Until now we do not know how to define a general dual projection such that the difference is a strong martingale.

A stopping line λ is a random connected curve such that for every pair $z \ll z'$, we have $z \notin \lambda(\omega)$ or $z' \notin \lambda(\omega)$ and $\{\omega: z < \lambda(\omega)\} \in \mathscr{F}_z$. The relation $z < \lambda$ means that there exists a random point z' such that $z < z'$ and $z' \in \lambda$, and in the same manner, we define a partial order between the stopping lines.

Let Z be a random point and denote by $\overline{Z}$ (resp. $\underline{Z}$) the set of all the points greater (resp. smaller) than or equal to Z such that one of the coordinates is the same as Z. Z will be called a stopping point if $\overline{Z}$ is a stopping line. Note that this condition is equivalent to the following: For each $z \in R_+^2$, $\{\omega: Z(\omega) < z\} \in \mathscr{F}_z$. Note also that the random line $\underline{Z}$ is not necessarily a stopping line. Every stopping line λ determines a stopping set $D(\lambda) = \{(\omega, z): z < \lambda(\omega)\}$ and conversely.

Let A be an increasing process. Then the random variable A_λ is well defined, being the measure of the set $D(\lambda)$ induced by A: $\int I_{D(\lambda)}(z)\, dA_z$. If λ is a stopping line such that $D(\lambda) = \bigcup_i R_{Z_i}$ where $Z_i = (S_i, T_i)$ are random points, R_{Z_i} are the stochastic intervals $[(0,0), Z_i]$, and the union is finite for a.e. ω, then λ is called a stepped stopping line and A_λ is also equal to $\sum_i (A_{Z_i} - A_{(S_i, T_{i-1})})$, provided that the points Z_i are ordered in a suitable manner.

1. Point processes.

DEFINITION. A point process over R_+^2 is a right-continuous increasing adapted step process $N = \{N_z\}$ taking its values in $N \cup \{0\} \cup \{\infty\}$ and which vanishes on the axes. The jump ΔN_z of the process at some point z is defined to be $N_z - N_{s^-, t} - N_{s, t^-} + N_{z^-}$, where $z = (s, t)$, and we suppose that the nonzero jumps of N are equal to one. We have $N_z = \sum_{z' \leq z} \Delta N_{z'}$, and all the jumps are on an increasing sequence of stopping lines $L_k = \text{Debut}\{N_z \geq k\}$ (where the Debut of a set is defined to be its minimal points).

Another increasing sequence of stopping lines associated with the point process N is the following:

$$L'_1 = L_1, \qquad L'_n = \text{Debut}\{z\colon \Delta N_z = 1, L'_{n-1} \ll z\}, \qquad n > 1,$$

and Debut $\phi = \infty$.

To every point process we can associate a sequence of random points Z_n which can take the value ∞ such that:

(i) $Z_0 = (0,0)$ and if $n \neq 0$ then Z_n is not on the axes.
(ii) $Z_n \ll \infty \Rightarrow Z_{n+1} \not< Z_n$.
(iii) $Z_n \ll \infty \Rightarrow \Delta N_{Z_n} = 1$.
(iv) For all Z such that $[Z] \cap (\bigcup_n [Z_n])$ is an evanescent set, we have $\Delta N_Z = 0$ ($[Z]$ is the graph of Z).
(v) For all z, $\{\omega\colon Z_n(\omega) < z\} \in \mathscr{F}_z^*$.

Note that in general these random points are not stopping points, but

$$\bigcup_n [Z_n] \subset \bigcup_n [L'_n] \subset \bigcup_n [L_n].$$

A point process is, by definition, nonexplosive iff for z: $N_z < \infty$ a.s. If $Z_\infty = \lim_{n\to\infty} Z_n = \infty$ then N is nonexplosive, but the converse does not hold. In this paper we suppose that all the processes are nonexplosive.

The points $\{Z_n\}$ are obtained considering for any $\omega \in \Omega$ the denumerable set of points where the process N has a jump and ordering this set in a suitable manner. For example, the sequence can be ordered as follows. We first consider the intersections of the jump set with the sets $R_{(n+1,\,n+1)} \setminus R_{(n,\,n)}$ and then in every set, we order the points by the lexicographical order induced by the first coordinate.

Conversely, given a sequence of random points $\{Z_n\}_{n=0}^{\infty}$ such that $Z_0 = (0,0)$, for all $n \neq 0$, Z_n does not belong to the axes, and for all z, Card$\{n\colon Z_n < z\} < \infty$, we can define its associated counting process N_z which is a point process equal to $n - 1$ if n is the number of sets $[Z_i, \infty)$, $i = 0, 1, \ldots,$ which contain the point z. However, we have to require that the random variables N_z are $\mathscr{F}_z$-measurable for all z.

Remarks. 1. The notion of point process is well-adapted to the study of two-servers queueing processes. A two-servers queueing process Q_z is an integer-valued process of the form $Q_z = Q_{(0,0)} + A_z - D_z$, where A_z and D_z are (nonexplosive) point (or counting) processes without common jumps. $Q_{(0,0)}$ is called the initial state and Q_z the state process. For each $z = (s,t) > (0,0)$ the random variable Q_z can be interpreted as the number of customers waiting in the first line at time s and waiting in the second line at time t. This kind of problem occurs where the two lines (or servers) are not in proximity to one another, and we do not obtain information from the lines at the same time. The process A_z (D_z) is the number of arrivals (departures) in the rectangle R_z, and is called the arrival (departure) process.

2. If N_z is a point process and θ an increasing path, then the trace of N on θ is a one-parameter point process. The converse is false.

3. Let L be a stopping line, then the process stopped at L: $N_z^L = N_{L \wedge z}$ is always well defined (since N is increasing) and is still a point process.

DEFINITION. Let $N = \{N_z, z \in R_+^2\}$ be a point process. The process $\overline{N} = \{\overline{N}_z, z \in R_+^2\}$ is called the $(i, *)$-intensity of N if it is a nonnegative $(\mathscr{F}^i, \mathscr{F}^*)$-predictable process such that for all z: $\int_{R_z} \overline{N}_\xi \, d\xi < \infty$ a.s., and for all nonnegative processes $\varphi = \{\varphi_z, z \in R_+^2\}$ bounded and $(\mathscr{F}^i, \mathscr{F}^*)$-predictable, we have

$$E\left[\int_{R_+^2} \varphi_z \, dN_z\right] = E\left[\int_{R_+^2} \varphi_z \overline{N}_z \, dz\right].$$

The notion of intensity was defined and extensively studied in the one-parameter case by Brémaud [1]. The following remarks about intensity are simple to prove and a few of them will be used in this paper.

1. Let $\overline{N}$ be the $(i, *)$-intensity of N and suppose that N is integrable. Then, the process $\{N_z - \int_{R_z} \overline{N}_\xi \, d\xi, z \in R_+^2\}$, which is called the compensator of N, is a weak (resp. i, strong) martingale. Since $\overline{N}$ is predictable and following the uniqueness of the dual predictable projection of N, it follows that $N_z^\pi = \int_{R_z} \overline{N}_\xi \, d\xi$, and the intensity is uniquely determined.

2. Under a condition of continuity, for all $z \in R_+^2$, we have $\overline{N}_z = \lim_{z' \downarrow z} (1/|(z, z']|) E[N(z, z'] | \mathscr{F}_z]$, where $|(z, z']|$ denotes the area of the rectangle $(z, z']$.

3. The definition of the intensity depends on the filtration. In particular, we have the following result: If $\overline{N}$ is the intensity of N with respect to the filtration $\mathscr{F}$ and if there exists another filtration $\mathscr{G} = \{\mathscr{G}_z, z \in R_+^2\}$ such that for all $z \in R_+^2$: $\mathscr{G}_z \subseteq \mathscr{F}_z$ and $\overline{N}$ is $\mathscr{G}$-predictable, then $\overline{N}$ is also the intensity of N with respect to the filtration $\mathscr{G}$.

A very important class of point processes is the following:

DEFINITION ([1]). Let $N = \{N_z, z \in R_+^2\}$ be an adapted point process and let $\lambda = \{\lambda_z, z \in R_+^2\}$ be a nonnegative, $\mathscr{F}^*_{(0,0)}$-measurable and integrable process.

If, for all $z \ll z'$, $u \in R$,

$$E\left[e^{iuN_{(z,z']}} | \mathscr{F}_z^*\right] = \exp\left\{(e^{iu} - 1) \int_{(z, z']} \lambda_\xi \, d\xi\right\},$$

then N is called a double stochastic $\mathscr{F}^*$-Poisson process with the stochastic intensity λ_z. If the filtration $\mathscr{F}^*$ is replaced in the definition by the filtration $\mathscr{F}^i$, then N is called a doubly stochastic $\mathscr{F}^i$-Poisson process.

Another very general definition was given by Grandell [3], where the parameter set is a locally compact Hausdorff space with countable basis (which includes the R_+^2 case). An existence theorem is also given. If λ_z is deterministic then N_z is called a $\mathscr{F}^*$- (or $\mathscr{F}^i$-) Poisson process; and if, moreover, λ_z is constant then N_z is called the $\mathscr{F}^*$- (or $\mathscr{F}^i$-) standard Poisson process.

In the deterministic case, the Poisson process was defined and constructed by Kingman [4], and by Neveu ([8], page 260) and studied by Mazziotto and Szpirglas [5]. It is a point process N_z such that for every rectangle A, the random variable $N(A)$ has the Poisson distribution with parameter $\Lambda(A) = \int_A \lambda_z \, dz$, and for any finite sequence of disjoint rectangles $A_1, \dots, A_n$, the random variables $N(A_1), \dots, N(A_n)$ are mutually independent.

The Poisson process can always be chosen right-continuous with left limits (since it is an increasing process), and the filtration generated by the Poisson process is right-continuous and satisfies the property (F4) of conditional independence. Therefore, the Poisson process is a $\mathscr{F}^*$-Poisson process with respect to its filtration.

2. A characterization of generalized Poisson processes. In this section, we give a characterization for the different kinds of (doubly stochastic) Poisson processes defined in the last section. For the special case of the Poisson process and without using the concept of martingale, the result was essentially proved by Mecke and by Papangelou [11], by another method. As we shall see, the characterization obtained is better than in the Wiener case, and we do not use the property (F4) of conditional independence.

THEOREM 1. *Let* $N = \{N_z, z \in R_+^2\}$ *be a point process,* $\overline{N} = \{\overline{N}_z, z \in R_+^2\}$ *its intensity and* $\tilde{N} = \{N_z - \int_{R_z} \overline{N}_\xi \, d\xi\}$ *its compensator.*

(i) *N is a doubly stochastic $\mathscr{F}^*$-Poisson process if and only if $\overline{N}$ is $\mathscr{F}^*_{(0,0)}$-measurable and $\tilde{N}$ is a strong martingale.*
(ii) *N is a doubly stochastic $\mathscr{F}^i$-Poisson process if and only if $\overline{N}$ is $\mathscr{F}^i_{(0,0)}$-measurable and $\tilde{N}$ is an i-martingale ($i = 1, 2$).*
(iii) *N is an $\mathscr{F}^*$-Poisson process if and only if $\overline{N}$ is deterministic and $\tilde{N}$ is a strong martingale.*
(iv) *N is an $\mathscr{F}^i$-Poisson process if and only if $\overline{N}$ is deterministic and $\tilde{N}$ is an i-martingale ($i = 1, 2$).*

PROOF. The first parts of these assertions follow easily from the definitions of the different Poisson processes.

Conversely, assume that $\tilde{N}$ is a strong martingale. Then, if we fix $t < t'$, the process

$$\left\{ N_{(s,t')} - N_{(s,t)} - \int_{(0,s]\times(t,t']} \overline{N}_\xi \, d\xi, \mathscr{F}^*_{(s,t)}, s \geq 0 \right\}$$

is a one-parameter martingale, and following the characterization of the doubly stochastic one-parameter Poisson process proved by Brémaud [1], the process $\{N_{(s,t')} - N_{(s,t)}, s \geq 0\}$ is a doubly stochastic $\{\mathscr{F}^*_{(s,t)}, s \geq 0\}$-Poisson process. That means, for any $s < s'$, $N_{(s',t')} - N_{(s',t)} - N_{(s,t')} + N_{(s,t)}$ is independent of $\mathscr{F}^*_{(s,t)}$ given $\mathscr{F}^*_{(0,0)}$ and we obtain $E[e^{iuN_{(z,z']}} | \mathscr{F}_z^*] = \exp\{(e^{iu} - 1)\int_{(z,z']} \overline{N}_\xi \, d\xi\}$. □

The same proof can be applied to the i-martingale case considering the one-parameter martingale (in the case $i = 1$) $\{N_{(s',t)} - N_{(s,t)} - \int_{(s,s']\times(0,t]}\overline{N}_\xi \, d\xi\}$. □

THEOREM 2. *Each of the following conditions implies that N is the Poisson process.*

(i) *N is an $\mathscr{F}^*$-Poisson process.*
(ii) *N is an $\mathscr{F}^i$-Poisson process ($i = 1$ or 2).*
(iii) *The hypothesis* (F4) *is satisfied; N is a point process with a deterministic intensity and its compensator is a martingale.*

PROOF. The condition (i) clearly implies that N is the Poisson process; then suppose that N is an $\mathscr{F}^1$-Poisson process. If we fix a finite number of disjoint intervals $I_i = (t_i, t_{i+1}]$, $i = 1, \dots, k$, then the one-parameter k-vector process $\{\{N((0,s] \times I_i), s \geq 0\}, \; i = 1, \dots, k\}$ is a k-variate point process which is adapted to the filtration $\{\mathscr{F}_s^1, s \geq 0\}$ and such that for every $i = 1, \dots, k$, the processes $\{N((0,s] \times I_i) - \int_{(0,s]\times I_i} \overline{N}_\xi \, d\xi, s \geq 0\}$ are $\mathscr{F}_s^1$-martingales. Therefore, by the multichannel Watanabe theorem (proved by Brémaud in [1]), the $\{N((0,s] \times I_i), s \geq 0\}$, $i = 1, \dots, k$, are independent $\{\mathscr{F}_s^1, s \geq 0\}$-Poisson processes with the intensities $\int_{I_i} \overline{N}_{(s,t)} \, dt$, $i = 1, \dots, k$, respectively. That means, for any $s < s'$, we have:

$$E\left[\exp\left\{i \sum_{j=1}^{k} u_j N\big((s,s'] \times I_j\big)\right\} \middle| \mathscr{F}_s^1\right] = \prod_{j=1}^{k} \exp\left\{(e^{iu_j} - 1) \int_{(s,s'] \times I_j} \overline{N}_\xi \, d\xi\right\}$$

for any finite sequence $\{u_j\}_{j=1}^{k}$ of real numbers. It follows from this equality that N is the Poisson process.

Suppose now that condition (iii) is fulfilled. Following hypothesis (F4), the compensator is an i-martingale. By (iv) of Theorem 1, N is an $\mathscr{F}^i$-Poisson process. Therefore it is the Poisson process. □

REMARKS. 1. In the Wiener case, this result does not hold, since the compensator needs to be a strong martingale or at least a martingale with orthogonal increments [15].

2. The proof of the multichannel Watanabe theorem uses the assumption that the intensities are deterministic. Therefore the proof of Theorem 2 cannot be applied to the doubly stochastic Poisson process.

3. Changing time. Let $N = \{N_z, z \in R_+^2\}$ be a Poisson process and $\tilde{N} = \{\tilde{N}_z, z \in R_+^2\}$ its compensator. Let

$$\phi_z = \begin{cases} 1 & \text{if } |R_z| \leq 1, \\ 2 & \text{if } |R_z| > 1, \end{cases}$$

and define

$$M_z = \int_{R_z} \phi_\xi \, dN_\xi.$$

M is a point process with deterministic intensity, and its compensator $\tilde{M}_z = \int_{R_z}\phi_\xi \, d\tilde{N}_\xi$ is a strong martingale [5]. Nevertheless, it is exactly the same proof that in [2] shows that M cannot be transformed into a Poisson process by any family of stopping points. Here we deal with stopping lines and show in this case M can be transformed into a Poisson process.

In general the (F4) hypothesis is not preserved when we stop a filtration by a family of stopping lines. Therefore we need the following result. In the case of the Wiener process, it was proved by Wong and Zakai [13].

From now on, we suppose that N is a Poisson process and $\{\mathscr{F}_z\}$ is the filtration generated by N. Let L be a stopping line and define $\mathscr{F}_L$ to be the σ-algebra generated by the process $N_{z\wedge L}$, that is the σ-algebra generated by the random variable $\{N_z, z < L\}$.

Theorem 3. *Let L_1 and L_2 be stopping lines and $L_3 = L_1 \wedge L_2$. Then $\mathscr{F}_{L_1}$ and $\mathscr{F}_{L_2}$ are conditionally independent given $\mathscr{F}_{L_3}$.*

Proof. Every stopping line L can be approximated by a decreasing sequence of stepped stopping lines $\{L^n\}_{n=1}^{\infty}$ (for example, the dyadic approximation). Since $N(A)$ and $N(B)$ are independent where A and B are Borel disjoint sets in R_+^2, it follows that $\mathscr{F}_{L_1^n}$ and $\mathscr{F}_{L_2^n}$ are conditionally independent given $\mathscr{F}_{L_3^n}$. As was done in [12], it follows by the smoothing property of conditional expectations that $\mathscr{F}_{L_1^+}$ and $\mathscr{F}_{L_2^+}$ are conditionally independent given $\mathscr{F}_{L_3^+}$, where $\mathscr{F}_{L^+} = \bigcap_{n=1}^{\infty}\mathscr{F}_{L^n}$. In order to complete the proof, we show that $\mathscr{F}_{L^+} = \mathscr{F}_L$. Here, too, the result is obtained using the same idea of Proposition 5.3 of [13] and the fact (proved in [14]) that the family of random variables $\exp\{\int_A f\, dN - \int_A(e^f - 1)\, dz\}$ with f Borel and bounded function with compact support, is dense in the space of square integrable functionals of N. □

The following changing-time result holds under the (F4) hypothesis.

Theorem 4. *Let M be a point process with intensity $\overline{M}$ such that its compensator is a martingale. Suppose that the function $s \to \int_0^t \overline{M}_{s,u}\, du$ is nondecreasing for all $t > 0$ and tends to infinity with t (or the same after exchanging s and t). Then there exists a family of stopping lines $\{L_z, z \in R_+^2\}$ such that M_{L_z} is a standard Poisson process.*

Proof. Suppose that the condition of the theorem is satisfied; that is, $\int_0^\infty \overline{M}_{s,u}\, du = \infty$, and therefore define:

$$t(s) = \inf\left\{t' \colon \int_0^{t'} \overline{M}_{s,u}\, du \geq t\right\},$$

$$\tau_z = \{(x, y) \colon x \leq s,\ y \leq t(x)\} \quad \text{where } z = (s, t).$$

Therefore, $\int_{\tau_z} \overline{M}_\xi \, d\xi = |z|$, where $|z|$ denotes the area of R_z, and the family of random sets $\{\tau_z, z \in R_+^2\}$ is increasing and satisfies: $\tau_z \cap \tau_{z'} \equiv \tau_{z \wedge z'}$ for every pair of two points z and z' (since $\int_0^t \overline{M}_{s,u}\, du$ is nondecreasing in s).

Let L_z be the boundary (without the axes) of τ_z. Since $\overline{M}$ is predictable, then L_z is a stopping line.

Define now $M_z^L = M_{L_z}$ and $\mathscr{F}_z^L = \mathscr{F}_{L_z}$. The σ-algebras $\mathscr{F}_z^{L_i}$, $i = 1, 2$ are defined similarly and following Theorem 3 satisfy the hypothesis (F4) of conditional independence.

Following its construction, the process M_z^L is $\mathscr{F}_z^L$-adapted (since M is $\mathscr{F}$-adapted) and is also a point process.

Therefore, following Theorem 2, it suffices to prove that the process $M_z^L - |z|$ is a martingale with respect to the new filtration $\{\mathscr{F}_z^L\}$. Let $(s, s'] \times (0, t]$ be a generalized rectangle such that $F \in \mathscr{F}_z^{L_i}$ and $z = (s, t)$, $z' = (s', t)$. Then

$$E\int I_{F\times(s,s']\times(0,t]}\,dM^L = E\left[I_F M^L((s, s'] \times (0, t])\right] = E\left[I_F\left(M_{z'}^L - M_z^L\right)\right]$$

$$= E\left[I_F\left(M_{L_{z'}} - M_{L_z}\right)\right] = E\int I_{(F\times R_+^2)\cap(L_z, L_{z'}]}\,dM,$$

where $(L_z, L_{z'}]$ is the stochastic interval open on the left $\{(\omega, \xi): L_z(\omega) \ll \xi < L_{z'}(\omega)\}$. This stochastic interval is a predictable set; therefore, since $F \in \mathscr{F}_{L_z}^i$, the set $(F \times R_+^2) \cap (L_z, L_{z'}]$ is $\mathscr{F}^i$-predictable. The process $M - \int\overline{M}$ is an i-martingale and it induces a measure which vanishes on the $\mathscr{F}^i$-predictable sets and the last relation equals

$$E\int I_{(F\times R_+^2)\cap(L_z, L_{z'}]}\overline{M}_z\,dz = E\left[I_F\overline{M}(L_z, L_{z'}]\right]$$

$$= E\left[I_F|(z, z']|\right] = E\int I_{F\times(z,z']}(\xi)\,d\xi.$$

This implies that $M_z^L - |z|$ is an i-martingale. Therefore M_{L_z} is a standard Poisson process. □

REMARK. Without using the (F4) hypothesis, the result remains valid if we require that the compensator of M is a strong martingale.

EXAMPLE . Consider the example of the beginning of this section. The family of stopping lines $\{L_z\}$ transforming M into a Poisson process can be constructed as in [9]. The intensity of M is ϕ and satisfies the conditions of the theorem. We have $t(s) = t \wedge [\frac{1}{2}(t + 1/s)]$, the corresponding stopping lines are $L_z = \{(x, \frac{1}{2}(t + 1/x)), x > 0\} \wedge \underline{z}$, where $z = (s, t)$, and $\{M_{L_z}\}$ is the standard Poisson process.

The increasing condition in Theorem 4 can be cancelled in special cases:

THEOREM 5. *Let M be a point process with intensity $\overline{M}$ such that its compensator is a martingale and $\int_0^\infty \overline{M}_{(s,u)}\,du = \infty$ for all s. Suppose that there exists a positive decreasing function β such that $\overline{M}_{(s,u)}\beta(s)^{-1}$ is a nondecreasing function of s, for all $u \geq 0$.*

Then there exists a family of stopping lines $\{L_z\}$ such that M_{L_z} is a standard Poisson process.

PROOF. Define

$$f_t(s) = \inf\left\{t': \int_0^{t'} \overline{M}_{(s,u)}\, du = t\beta(s)\right\},$$

$$\alpha(s) = \inf\left\{s': \int_0^{s'} \beta(u)\, du = s\right\},$$

and

$$\tau_z = \{(x, y): 0 \le x \le \alpha(s), 0 \le y \le f_t(x)\}.$$

The proof now follows as in Theorem 4. □

As an example of the application of Theorem 5, suppose that $\overline{M}$ is a smooth function and assume that $\overline{M}^1_{(s,t)}(s)$ is continuous. Assume also that $\overline{M} \ge a > 0$ and $\overline{M}^1_{(s,t)}(s) \ge -K > -\infty$. Then the conditions of the theorem hold with $\beta(x) = e^{-(K/a)x}$.

REMARK. Let N be a standard Poisson process and $\{L_z\}$ a family of deterministic stopping lines (that is, L_z is a nonincreasing connected line). Which conditions must this family satisfy in order that $\{N_{L_z}\}$ remain a standard Poisson process? The answer to this question is exactly the same as in the Wiener case as it was done in [9], since the arguments of the proofs are only geometric deterministic and not probabilistic.

The main result is the following: Let $\{L_z, z \in R_{z_0}\}$ a family of deterministic stopping lines contained in R_{z_0} such that $\{N_{L_z}, z \in R_{z_0}\}$ remains a standard Poisson process. Then for all $z = (s, t)$, we have $L_z = \underline{z}$ or $L_z = \underline{(ts_0t_0^{-1}, st_0s_0^{-1})}$ where $z_0 = (s_0, t_0)$.

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