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A CHARACTERIZATION OF THE VON MISES DISTRIBUTION

BY LOUIS GORDON AND MALCOLM HUDSON[1]

ALZA *Corporation and Stanford University*

The von Mises, or circular normal density on the unit circle is the hitting density of a two-dimensional Brownian motion starting at the origin, with constant drift velocity and direction. The concentration and location parameters of the density have a natural relation to the drift parameters.

1. Introduction and summary. The von Mises, or circular normal distribution is discussed in Mardia (1972). Several characterizations are there available, often analogous to characterizations for the normal distribution on the line. Extensive use of random walk models and the von Mises distribution are made in Kendall (1974). Kendall, however, was apparently unaware of the characterization given below.

The circular normal density on the circle is proportional to $\exp(\delta \cos(\theta - \alpha))$, $\theta \in [0, 2\pi]$ where θ is interpreted as an angle with the positive x-axis, and the circle is centered at (0, 0). The parameters δ and α are respectively concentration and location parameters. We use a likelihood ratio argument to establish that the density is also the hitting density on the unit circle of a standard two-dimensional Brownian motion starting at (0, 0) with drift velocity δ and drift direction α. The use of likelihood ratio martingales in the theory of stochastic processes has been exploited by a number of authors. Robbins and Siegmund (1970) used the martingale exploited below in the evaluation of boundray corssing probabilities for a univariate Wiener process. Kailath (1971) discusses likelihood ratio martingales in an hypothesis testing framework.

We here evaluate the hitting distribution of a standard 2-dimensional Brownian motion starting from (0, 0) with drift velocity δ in direction α, to the unit circle centered at (0, 0). We first reduce the problem to a one-dimensional problem by means of symmetry considerations. We then complete the proof by using the appropriate likelihood ratio martingale.

2. The characterization. Since the von Mises density is proportional to $\exp(\delta \cos(\theta - \alpha))$, for $\theta, \alpha \in [0, 2\pi]$, we need only prove the following theorem.

THEOREM. *The hitting density to a unit circle centered at* (0, 0) *for two-dimensional standard Brownian motion with drift velocity* $\delta > 0$ *and direction angle* α *is proportional to* $\exp(\delta \cos(\theta - \alpha))$, *where* $\theta, \alpha \in [0, 2\pi]$ *and* θ *is the angle of the hit vector with the positive x axis.*

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[1] Currently at MacQuarie University, New South Wales.

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PROOF. Let X, Y be independent standard one-dimensional Brownian motions. Since we may choose as axes any two orthogonal axes, we reduce the problem without generality loss to the case $\alpha = 0$. Again from symmetry considerations, it suffices to evaluate the hitting density only in the upper half of the circle.

Define $X_\delta(t) = X(t) + \delta t$. Let $\mathscr{F}_t = \sigma\{X(s) \mid s \leqq t\}$ be the history σ-field of the X, Y process to time t. Let $L_\delta(t) = \exp(\delta X(t) - \delta^2 t/2)$ and note that $(L_\delta(t), \mathscr{F}_t, t < \infty)$ is a likelihood ratio martingale for the process X_δ with respect to the process X, but with associated history σ-fields $\mathscr{F}_t$ somewhat larger than customarily used.

Let T be the first time (X, Y) hits the unit circle. Again from symmetry, since there is no drift in any direction, $(X(T), Y(T))$ is uniformly distributed on the circle and is stochastically independent of T.

Since $|X(T\Lambda t)| < 1$, the martingale $(L_\delta(T\Lambda t), \mathscr{F}_{T\Lambda t}, t < \infty)$ is uniformly integrable and so $L_\delta(T)$ is the likelihood ratio for the process $X_\delta(T\Lambda t)$ with respect to $X(T\Lambda t)$ on sets in $\mathscr{F}_T$. Because $X_\delta(T)$ is the cosine of the hitting angle, we evaluate the hitting density by computing $Eg(X_\delta(T))$ for g any bounded function. However,

$$\begin{aligned} Eg(X_\delta(T)) &= Eg(X(T)) \exp(\delta X(T) - \delta^2 T/2) \\ &= Eg(X(T)) \exp(\delta X(T)) E \exp(-\delta^2 T/2) \\ &= \textstyle\int_0^{2\pi} g(\cos(\theta)) \exp(\delta \cos(\theta))\, d\theta E \exp(-\delta^2 T/2) \end{aligned}$$

where the right-hand expectations are all taken under the drift 0 measure. The theorem follows immediately.

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ALZA
950 PAGE MILL ROAD
PALO ALTO, CALIFORNIA 94304

MACQUARIE UNIVERSITY
NEW SOUTH WALES
AUSTRALIA