A characterization of trans-separable spaces*

J.C. Ferrando   Jerzy Kąkol   M. López Pellicer

Dedicated to Professor J. Schmets on the occasion of his 65th birthday

Abstract

The paper shows that a uniform space X is trans-separable if and only if every pointwise bounded uniformly equicontinuous subset of the space of continuous real-valued functions $C_c(X)$ equipped with the compact-open topology is metrizable. This extends earlier results of Pfister and Robertson and also applies to show that if $C_c(X)$ is angelic then X is trans-separable. The precise relation among DCCC spaces and trans-separable spaces has been also determined.

1 Preliminaries

A uniform space $(X, \mathcal{N})$ is called *trans-separable* if for every vicinity U of $\mathcal{N}$ there is a countable subset C of X such that $U[C] = X$, [8, 9]. Separable uniform spaces and Lindelöf uniform spaces are trans-separable but the converse statements are not true in general, although every uniform pseudometrizable trans-separable space is separable. Clearly a uniform space is trans-separable iff it is uniformly isomorphic to a subspace of a uniform product of separable pseudometric spaces. For topological vector spaces (tvs) X trans-separability means exactly that X is isomorphic to a subspace of a product of metrizable and separable tvs. In particular, if X is a locally convex space (lcs) with topological dual X', then X is trans-separable provided with

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(the translation-invariant uniformity of) the weak topology $\sigma(X, X')$. Among other permanence properties, the class of trans-separable uniform spaces is hereditary, productive and closed under uniform continuous images. Hence each uniformly continuous image of a trans-separable space onto a uniform pseudometrizable space is separable. Trans-separable lcs are closed under transition to linear subspaces, topological products and continuous linear images.

Trans-separability has been successfully used to study several problems both in analysis and topology, for example while studying the metrizability of precompact sets in general uniform spaces and in the class of lcs, [3, 6, 15, 16] (see also [5, 11]). Pfister [14] showed that a lcs X is trans-separable iff the equicontinuous subsets in the weak* dual $(X', \sigma(X', X))$ are metrizable, and then he applied this fact to show that precompact sets in (DF)-spaces are metrizable.

In the Theorem given in [15], Robertson demonstrated that a uniform space $(X, \mathcal{N})$ which is covered by a family $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of precompact sets such that $K_\alpha \subseteq K_\beta$ if $\alpha \leq \beta$ is trans-separable. In [6] we proved that in a completely regular Hausdorff space X every compact set is metrizable iff the space $C_c(X)$ of all realvalued continuous functions on X equipped with the compact-open topology τ_c is trans-separable. As a nice application of this result, we showed in [6] that the compact (precompact) sets of a lcs X are metrizable iff the dual X' equipped with the locally convex topology of uniform convergence on the compact (precompact) subsets of X is trans-separable. In this paper we complete the picture by showing the following theorem, where $\tau_\mathcal{N}$ denotes the uniform topology on X, that is, the topology defined by the uniformity $\mathcal{N}$ for X.

Theorem 1. *For a uniform space $(X, \mathcal{N})$ the following statements are equivalent:*

1. *$(X, \mathcal{N})$ is trans-separable.*
2. *Every pointwise bounded uniformly equicontinuous set of functions from $(X, \mathcal{N})$ to $\mathbb{R}$ is metrizable in $C_c(X, \tau_\mathcal{N})$.*
3. *Every pointwise bounded uniformly equicontinuous set of functions from $(X, \mathcal{N})$ to $\mathbb{R}$ has countable tightness in $C_c(X, \tau_\mathcal{N})$.*

Theorem 1 applies to show that a uniform space $(X, \mathcal{N})$ is trans-separable whenever the space $C_c(X, \tau_\mathcal{N})$ is angelic (Corollary 4), but the converse statement fails (Example 3). Since for a large class of topological spaces X (introduced and studied by Orihuela [13] under the name of *web-compact* spaces) the space $C_p(X)$ of the realvalued continuous functions defined on X endowed with the pointwise convergence topology τ_p is angelic, Corollary 4 applies to show that every web-compact uniform space is trans-separable, which extends Robertson's Theorem of [15]. The relation among Wheeler's DCCC spaces and trans-separable spaces has been also determined by showing that each uniform space $(X, \mathcal{N})$ such that $(X, \tau_\mathcal{N})$ has the DCCC is trans-separable, but there exist trans-separable spaces $(X, \mathcal{N})$ such that $(X, \tau_\mathcal{N})$ does not have the DCCC.

Recall that a topological space X is said to have *countable tightness* [1] if for every set A in X and every $x \in \overline{A}$ there is a countable subset of A whose closure contains x. A Hausdorff space X is *angelic* [7] if relatively countably compact sets in X are relatively compact and for every relatively compact subset A of X each point in the closure of A is the limit of a sequence of A. In angelic spaces (relatively) compact sets, (relatively) countably compact sets and (relatively) sequentially compact sets are the same. A family $\mathcal{F}$ of functions from a uniform space $(X,\mathcal{N})$ into a uniform space $(Y,\mathcal{M})$ is called *uniformly equicontinuous* [2, X.2.1 Definition 2] if for each $V\in\mathcal{M}$ there is $U\in\mathcal{N}$ such that $(f(x),f(y))\in V$ whenever $f\in\mathcal{F}$ and $(x,y)\in U$.

2 Proof of Theorem 1 and Applications

Henceforth we assume $\mathbb{R}$ equipped with the uniformity generated by the usual metric on $\mathbb{R}$. If (X,d) is a pseudometric space, we denote by $\mathcal{N}_d$ the pseudometric uniformity for X generated by the pseudometric d on X. If $(X,\mathcal{N})$ is a uniform space, we represent by $\tau_{\mathcal{N}}$ the topology on X defined by the uniformity $\mathcal{N}$.

We are ready to prove Theorem 1.

Proof. $1\Rightarrow 2$. Assume that $(X,\mathcal{N})$ is trans-separable and let $\mathcal{C}$ be a pointwise bounded uniformly equicontinuous set of realvalued functions on $(X,\mathcal{N})$ closed in $C_c(X,\tau_{\mathcal{N}})$. Given that $\mathcal{C}$ is pointwise bounded, the map

$$(x,y)\mapsto \sup\nolimits_{f\in\mathcal{C}}|f(x)-f(y)|$$

defines a pseudometric m on X. Since $\mathcal{C}$ is uniformly equicontinuous, given $\epsilon>0$ there is a vicinity U on $X\times X$ such that $m(x,y)<\epsilon$ for every $(x,y)\in U$. The latter means that the identity map φ from $(X,\mathcal{N})$ onto $(X,\mathcal{N}_m)$ is uniformly continuous. Moreover, since obviously $\sup_{f\in\mathcal{C}}|f(x)-f(y)|<\epsilon$ iff $m(x,y)<\epsilon$, then $\mathcal{C}$ is uniformly equicontinuous on $(X,\mathcal{N}_m)$ and in particular $\mathcal{C}\subseteq C_c(X,m)$. As in addition $\mathcal{C}$ is clearly closed in $C_c(X,m)$, Ascoli's theorem [2, X.2.5 Corollary 2] guarantees that the set $\mathcal{C}$ is compact in $C_c(X,m)$.

Given that $(X,\mathcal{N})$ is trans-separable, the uniform continuity of φ implies that (X,m) is separable. Let D be a countable and dense subset of (X,m). If for each $x\in X$ we set $\delta_x(f):=f(x)$ for $f\in C(X)$, then $\{\delta_x|_{\mathcal{C}}:x\in D\}$ is a countable family of continuous functions that separates the points of $\mathcal{C}$. Hence $\mathcal{C}$, as a compact subset of $C_c(X,m)$, is metrizable. Since the map $T:C_c(X,m)\to C_c(X,\tau_{\mathcal{N}})$ defined by $Tf=f\circ\varphi$ is one to one and continuous, it follows that $\mathcal{C}$ is a (compact and) metrizable subset of $C_c(X,\tau_{\mathcal{N}})$.

$2\Rightarrow 3$ is obvious. Let us prove $3\Rightarrow 1$. Let $\mathcal{P}$ be the family of all pseudometrics for X generating $\mathcal{N}$ and let $d\in\mathcal{P}$. First note that if $\mathcal{C}$ is a set of pointwise bounded uniformly equicontinuous functions from $(X,\mathcal{N}_d)$ into $\mathbb{R}$ then, by Ascoli's theorem, $\mathcal{C}$ is a relatively compact subset of $C_c(X,d)$. So the compactness of $\overline{\mathcal{C}}^{C_c(X,d)}$ in $C_c(X,d)$ and the continuity of the immersion from $C_c(X,d)$ into $C_c(X,\tau_{\mathcal{N}})$ imply that $\overline{\mathcal{C}}^{C_c(X,d)}=\overline{\mathcal{C}}^{C_c(X,\tau_{\mathcal{N}})}$. As in addition $\mathcal{N}_d$ is smaller than $\mathcal{N}$ and therefore $\overline{\mathcal{C}}^{C_c(X,\tau_{\mathcal{N}})}$ is uniformly equicontinuous on $(X,\mathcal{N})$, it follows from the hypothesis that $\overline{\mathcal{C}}^{C_c(X,d)}$ has countable tightness in $C_c(X,d)$.

Now we show that (X,d) is separable, for which we may assume that d is bounded. Let $\mathcal{K}(X,d)$ be the family of all compact subsets of (X,d) and define $f_A(x):=d(x,A)$ for $x\in X$ and $A\in\mathcal{K}(X,d)$. Then the family $\mathcal{H}:=$

$\{f_A : A \in \mathcal{K}(X,d)\}$ of uniformly bounded realvalued functions is uniformly equicontinuous on $(X, \mathcal{N}_d)$ and consequently $\overline{\mathcal{H}}^{C_c(X,d)}$ has countable tightness in $C_c(X,d)$. Since $c_0 \in \overline{\mathcal{H}}^{C_c(X,d)}$, where c_0 is the null function on X, it follows that there exists a sequence $(A_n)_n$ in $\mathcal{K}(X,d)$ such that $c_0 \in \overline{\{f_{A_n} : n \in \mathbb{N}\}}^{C_c(X,d)}$. If $x \in X$ and $\epsilon > 0$, there is $k \in \mathbb{N}$ such that $f_{A_k}(x) < \epsilon$, so $\bigcup_{n=1}^{\infty} A_n$ is a dense subspace of (X,d). But since every compact pseudometric space A_n is separable, it follows that (X,d) is separable. Given that $(X,\mathcal{N})$ is uniformly isomorphic to a subspace of the uniform product $\prod_{d\in\mathcal{P}}(X,\mathcal{N}_d)$ of pseudometric separable spaces, the space $(X,\mathcal{N})$ is trans-separable. ∎

It is well known [1, III.3.7] that every compact subset of $C_p(X)$ is Fréchet-Urysohn (hence has countable tightness) provided X is compact. On the other hand, since every pointwise bounded uniformly equicontinuous set of realvalued functions on a uniform space $(X,\mathcal{N})$ which is closed in $C_c(X,\tau_{\mathcal{N}})$ is compact (by Ascoli's theorem), Theorem 1 applies to get the following

Corollary 2. *Let $(X,\mathcal{N})$ be a uniform space. If every compact subset of $C_c(X,\tau_{\mathcal{N}})$ has countable tightness, then $(X,\mathcal{N})$ is trans-separable.*

Surely the converse of Corollary 2 holds for separable uniform spaces, but fails in general, as the following example shows.

Example 3. *A trans-separable space $(X,\mathcal{N})$ such that $C_c(X,\tau_{\mathcal{N}})$ contains a compact set K which does not have countable tightness.* Let ω_1 be the first ordinal of uncountable cardinal and let E denote the locally convex direct sum of $\aleph_1$ copies of $\mathbb{R}$. Then the dual of E is isomorphic to $F := \mathbb{R}^{[0,\omega_1)}$. If $(X,\mathcal{N})$ denotes the linear space E equipped with the translation invariant uniformity $\mathcal{N}$ associated to the weak topology $\sigma(E,F)$ on E, then $(X,\mathcal{N})$ is trans-separable and, since the product topology on F coincides with the strong topology $\beta(F,E)$, then $\mathbb{R}^{[0,\omega_1)}$ is embedded in $C_c(X,\tau_{\mathcal{N}})$, where $\tau_{\mathcal{N}} = \sigma(E,F)$. Let $\mathcal{A}$ denote the family of all finite subsets of $[0,\omega_1)$ and, if A is a finite subset of $[0,\omega_1)$, define $g_A : [0,\omega_1) \to \mathbb{R}$ so that $g_A(\gamma) = 1$ if $\gamma \in A$ and $g_F(\gamma) = 0$ if $\gamma \in [0,\omega_1) \setminus A$. In the compact set $K := [0,1]^{[0,\omega_1)}$, the constant function $f : [0,\omega_1) \to \mathbb{R}$ such that $f(\gamma) = 1$ for each $\gamma \in [0,\omega_1)$ belongs to the closure of the set $\{g_A : A \in \mathcal{A}\}$. But no countable family $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{A}$ verifies that $f \in \overline{\{g_{A_n} : n \in \mathbb{N}\}}$. Indeed, if $\xi \in [0,\omega_1) \setminus \bigcup_{n=1}^{\infty} A_n$ then $U(f,\xi) := \left\{h \in \mathbb{R}^{[0,\omega_1)} : |h(\xi) - 1| < \frac{1}{2}\right\}$ is a neighborhood of f in $\mathbb{R}^{[0,\omega_1)}$ such that $g_{A_n} \notin U(f,\xi)$ for every $n \in \mathbb{N}$.

Recall that in [13] Orihuela introduced and studied a large class of topological spaces X (web-compact spaces) for which the space $C_p(X)$ is angelic. In particular, separable spaces, countably K-determined spaces and K-analytic spaces are web-compact. The first part of the next corollary implies that every web-compact uniform space is trans-separable.

Corollary 4. *Let $(X,\mathcal{N})$ be a uniform space. If the space $C_c(X,\tau_{\mathcal{N}})$ is angelic, then $(X,\mathcal{N})$ is trans-separable. In particular, a uniform space $(X,\mathcal{N})$ covered by a family $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of precompact sets such that $K_\alpha \subseteq K_\beta$ if $\alpha \leq \beta$ is trans-separable.*

Proof. If $C_c(X, \tau_{\mathcal{N}})$ is angelic, then each compact subset of $C_c(X, \tau_{\mathcal{N}})$ is Fréchet-Urysohn and Corollary 2 applies. For the second statement note that if $(\widehat{X}, \widehat{\mathcal{N}})$ denotes the completion of $(X, \mathcal{N})$, then the uniform subspace Y of $(\widehat{X}, \widehat{\mathcal{N}})$ covered by the closures of the sets K_α in $\left(\widehat{X}, \tau_{\widehat{\mathcal{N}}}\right)$ is web-compact [13], so the first part shows that Y provided with the relative uniformity is trans-separable. Since X is a uniform subspace of Y, we are done. ∎

Corollary 5. *Let X be a metric space. Then $C_p(X)$ is angelic if and only if $C_c(X)$ is angelic if and only if X is separable.*

According to [17] a topological space X is said to have the *Discrete Countable Chain Condition* (DCCC) if every discrete family of open sets is countable, which is equivalent to require that each continuous metrizable image of X is separable. It is shown in [17, Theorem 3.5] that if $C_p(X)$ is angelic, then X has the DCCC. Hence each web-compact space has the DCCC. Since a topological space X has the DCCC iff every pointwise bounded equicontinuous subset of $C(X)$ is τ_p-metrizable [4, Theorem 4] and τ_p and τ_c coincide on the equicontinuous sets, it follows from Theorem 1 that if $(X, \mathcal{N})$ is a uniform space such that $(X, \tau_{\mathcal{N}})$ has the DCCC then $(X, \mathcal{N})$ is trans-separable. However, as the following example shows there exists a trans-separable space $(X, \mathcal{N})$ such that $(X, \tau_{\mathcal{N}})$ does not have the DCCC.

Example 6. Let $X = [0, \omega_1)$ be the ordinal interval, ω_1 being the first ordinal of uncountable cardinality. For each $\gamma \in X$ set $U_\gamma := \{(\alpha, \beta) : \alpha = \beta \vee (\alpha \geq \gamma \wedge \beta \geq \gamma)\}$. Then the family $\{U_\gamma : 0 \leq \gamma < \omega_1\}$ is a base of a uniformity $\mathcal{N}$ for X. For a given vicinity U_γ with $0 \leq \gamma < \omega_1$ choose $C_\gamma := \{\alpha \in X : \alpha \leq \gamma\}$. Given $\beta \in X$ then $(\alpha, \beta) \in U_\gamma$ for $\alpha = \beta$ if $\beta < \gamma$ and $(\alpha, \beta) \in U_\gamma$ for $\alpha = \gamma$ if $\beta \geq \gamma$, which means that $U_\gamma[C_\gamma] = X$. Since the set C_γ is countable, it follows that the uniform space $(X, \mathcal{N})$ is trans-separable. On the other hand, since $\tau_{\mathcal{N}}$ is the discrete topology on X (see for instance [10, Chapter 6, Exercise C]) and X is uncountable, then $(X, \tau_{\mathcal{N}})$ does not have the DCCC.

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Centro de Investigación Operativa. Universidad Miguel Hernández, E-03202 Elche (Alicante), Spain
email:jc.ferrando@umh.es

Faculty of Mathematics and Informatics. A. Mickiewicz University,
61-614 Poznań, Poland
email:kakol@amu.edu.pl

Depto. de Matemática Aplicada and IMPA.
Universidad Politécnica de Valencia, E-46022 Valencia, Spain
email:mlopezpe@mat.upv.es