AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **30** (2004), Pages 31–39

A characterization of trees with equal total domination and paired-domination numbers

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Abstract

Let $G = (V, E)$ be a graph without isolated vertices. A set $S \subseteq V$ is a total dominating set if every vertex in V is adjacent to at least one vertex in S. A total dominating set $S \subseteq V$ is a paired-dominating set if the induced subgraph $G[S]$ has at least one perfect matching. The paired-domination number $\gamma_{\rm pr}(G)$ is the minimum cardinality of a paired-domination set of G. In this paper, we provide a constructive characterization of those trees with equal total domination and paired-domination numbers, and of those trees for which the paired domination number is twice the matching number.

* Research of the first and second authors supported by the National Nature Science Foundation of China under grant 10101010 and the Young Science Foundation of Shanghai Education Committee under grant 01QN6262.

† Research supported in part by the South African National Research Foundation and the University of KwaZulu-Natal.

1 Introduction

In this paper, we continue the study of total domination and paired-domination in graphs. For a graph $G = (V, E)$, a set $S \subseteq V$ is a *dominating set* if every vertex in $V - S$ is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. Domination and its many variations have been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [5, 6]. We are interested in two variations of domination called total domination and paired-domination.

A set $S \subseteq V$ is a *total dominating set* (TDS) if every vertex in V is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS. A minimum TDS of a graph G is called a $\gamma_t(G)$-set.

A set $S \subseteq V$ is a *paired-dominating set* (PDS) if S dominates V and the induced subgraph $G[S]$ has a perfect matching. Obviously, every PDS is a TDS. Paired-domination was introduced by Haynes and Slater [7] with the following application in mind. If we think of a vertex in S as the location of a guard capable of protecting each vertex in its closed neighborhood, then domination requires every vertex to be protected, and for total domination, each guard must be protected by another guard. For paired-domination, the guards' locations must be selected as adjacent pairs of vertices so that each guard is assigned one other and they are designated as backups for each other. The *paired-domination number* $\gamma_{\rm pr}(G)$ is defined to be the minimum cardinality of a PDS of G. A minimum PDS of a graph G is called a $\gamma_{\rm pr}(G)$-set. Paired-domination is also studied in [2, 8, 9, 11]. Both total domination and paired-domination require that there be no isolated vertices in the graph.

The *edge-independence number* (also called the *matching number*) $\beta_1(G)$ of G is the maximum cardinality of an edge-independent set of G. A graph G is said to have a *perfect matching* if $\beta_1(G) = |V|/2$.

As a direct consequence of the definitions of the above four parameters, we have the following observation due to Haynes and Slater [7].

Observation 1 *For any graph G without isolated vertices,*

$$\gamma(G) \leq \gamma_t(G) \leq \gamma_{\rm pr}(G) \leq 2\beta_1(G).$$

An area of research that has received considerable attention is the study of classes of graphs for which some of these parameters are equal (or not equal). For any graph theoretical parameters λ and μ, we define G to be a (λ, μ)-graph if $\lambda(G) = \mu(G)$ and a $(\lambda, 2\mu)$-graph if $\lambda(G) = 2\mu(G)$. The class of (γ, i)-trees, that is, trees T for which $\gamma(T) = i(T)$ where $i(T)$ denotes the independent domination number of T, was characterized in [1, 3]. Several classes of (γ, i)-graphs have been found in [10]. Furthermore, a characterization of (i, γ_r)-trees and (i, γ_w)-trees, where γ_r and γ_w are respectively the restrained domination and weak domination numbers, is given in [4]. In this article, we have two aims: First to provide a constructive characterization of

$(\gamma_t, \gamma_{\rm pr})$-trees and, secondly, to present a constructive characterization of $(\gamma_{\rm pr}, 2\beta_1)$-trees.

1.1 Notation

Let $G = (V, E)$ be a graph without isolated vertices, and let v be a vertex in V. The *open neighborhood* of v is $N(v, G) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v, G] = N(v, G) \cup \{v\}$. Furthermore, for a set S of vertices, the open neighborhood of S is defined by $N(S, G) = \bigcup_{v \in S} N(v, G)$ and set $N[S, G] = \bigcup_{v \in S} N[v, G] = N(S, G) \cup S$. The subgraph of G induced by the vertices in S is defined by $G[S]$. For $X \subseteq V$, the *private neighborhood* $\mathrm{pn}(v, X)$ of $v \in X$ is defined by $\mathrm{pn}(v, X) = N[v, G] - N[X - \{v\}, G]$. When there is no risk of confusion, we shall write briefly the notation as $N(v)$, $N[v]$, $N(S)$, $N[S]$ and $\mathrm{pn}(v)$, respectively. For $n \geq 1$, the complete bipartite graph $K_{1,n}$ is called a *star*. A *subdivided star* $K^*_{1,n}$ is the graph obtained from a star $K_{1,n}$ by subdividing each edge of the star. A *double star* is a tree that contains exactly two vertices that are not leaves.

2 A characterization of $(\gamma_t, \gamma_{\rm pr})$-trees

Before presenting a characterization of $(\gamma_t, \gamma_{\rm pr})$-trees, we shall need some additional notation.

A vertex v is said to be *totally dominated* by a set $S \subseteq V$ if it is adjacent to a vertex of S (other than itself). We define an *almost total dominating set* (ATDS) of G *relative* to v as a set of vertices of G that totally dominates all vertices of G, except possibly for v. The *almost total domination number* of G *relative* to v, denoted $\gamma_t(G; v)$, is the minimum cardinality of an ATDS of G relative to v. An ATDS of G relative to v of cardinality $\gamma_t(G; v)$ we call a $\gamma_t(G; v)$-set. Note that it is possible for v to belong to a $\gamma_t(G; v)$-set although v itself may not be totally dominated. For ease of presentation, we sometimes consider a tree as rooted tree. The concept of rooted tree can be found in [10]. A vertex of a tree T is said to be *remote* if it is adjacent to a leaf of T.

We shall need the following two observations.

Observation 2 *Let T be a tree that is not a star. Then,*
(a) *there exists a $\gamma_t(T)$-set that contains no leaf, and*
(b) *if T is a $(\gamma_t, \gamma_{\rm pr})$-tree, there exists a $\gamma_{\rm pr}(T)$-set that contains no leaf.*

Proof. Replacing each leaf, if any, in a $\gamma_t(T)$-set by its neighboring remote vertex yields (a). To prove (b), let S be a $\gamma_{\rm pr}(T)$-set that contains as few leaves as possible. By definition of a PDS, the set S contains all remote vertices of T. Suppose that S contains a leaf u. Let v be the remote vertex adjacent to u. Then, $\{u, v\} \subseteq S$. If S contains a neighbor of v different from u, then $S - \{u\}$ is a TDS of T, and so

$\gamma_t(T) \le |S| - 1 = \gamma_{\rm pr}(T) - 1$, contradicting the fact that T is a $(\gamma_t, \gamma_{\rm pr})$-tree. Hence, u is the only neighbor of v in S. Since T is not a star, there is a non-leaf neighbor w of v. Since $w \notin S$, the set $(S - \{u\}) \cup \{w\}$ is a PDS of T that contains fewer leaves than does S, contradicting our choice of the set S. Hence, S contains no leaf. □

Let $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ be the following three operations on a tree T. (By *attaching a path* P to a vertex v of T we mean adding the path P and joining v to a leaf of P.)

Operation $\mathcal{T}_1$. Attach a path P_1 to a vertex of T, which is in some $\gamma_{\rm pr}(T)$-set.

Operation $\mathcal{T}_2$. Attach a path P_3 to a vertex v of T, where $\gamma_t(T;v) = \gamma_t(T)$.

Operation $\mathcal{T}_3$. Attach a path P_4 to any vertex of T.

Let $\mathcal{T}$ be the family defined by $\mathcal{T} = \{T \mid T$ is obtained from P_2 by a finite sequence of operations $\mathcal{T}_1, \mathcal{T}_2$ or $\mathcal{T}_3\}$. We show first that every tree in the family $\mathcal{T}$ has equal total domination and paired-domination numbers.

Lemma 3 *If $T \in \mathcal{T}$, then T is a $(\gamma_t, \gamma_{\rm pr})$-tree.*

Proof. We proceed by induction on the number $s(T)$ of operations required to construct the tree T. If $s(T) = 0$, then $T = P_2$ and T is a $(\gamma_t, \gamma_{\rm pr})$-tree. This establishes the base case. Assume, then, that $k \ge 1$ is an integer and that each tree $T' \in \mathcal{T}$ with $s(T') < k$ is a $(\gamma_t, \gamma_{\rm pr})$-tree. Let $T \in \mathcal{T}$ be a tree with $s(T) = k$. Then, T can be obtained from a tree $T' \in \mathcal{T}$ with $s(T') < k$ by one of the operations $\mathcal{T}_1$, $\mathcal{T}_2$ or $\mathcal{T}_3$. Applying the inductive hypothesis to the tree T', T' is a $(\gamma_t, \gamma_{\rm pr})$-tree. If T is a star or a double-star, then $\gamma_t(T) = \gamma_{\rm pr}(T) = 2$ and so T is a $(\gamma_t, \gamma_{\rm pr})$-tree. Hence we may assume that $\mathrm{diam}(T) \ge 4$. We now consider three possibilities depending on whether T is obtained from T' by operation $\mathcal{T}_1$, $\mathcal{T}_2$ or $\mathcal{T}_3$.

Case 1. T is obtained from T' by operation $\mathcal{T}_1$. Suppose T is obtained from T' by adding a vertex u and the edge uv where $v \in V(T')$. Then, v is in some $\gamma_{\rm pr}(T')$-set. Any such set is a PDS of T, whence $\gamma_{\rm pr}(T) \le \gamma_{\rm pr}(T')$. By Observation 2, there is $\gamma_t(T)$-set that contains no leaves of T. Such a set is a TDS of T', and so $\gamma_t(T') \le \gamma_t(T)$. Thus, $\gamma_t(T) \le \gamma_{\rm pr}(T) \le \gamma_{\rm pr}(T') = \gamma_t(T') \le \gamma_t(T)$. Consequently we must have equality throughout this inequality chain. In particular, $\gamma_t(T) = \gamma_{\rm pr}(T)$.

Case 2. T is obtained from T' by operation $\mathcal{T}_2$. Suppose T is obtained from T' by adding the path u_1, u_2, u_3 and the edge u_1v where $v \in V(T')$. Then, $\gamma_t(T';v) = \gamma_t(T')$. Any $\gamma_{\rm pr}(T')$-set can be extended to a PDS of T by adding to it the vertices u_1 and u_2. Hence, $\gamma_{\rm pr}(T) \le \gamma_{\rm pr}(T') + 2$. We show next that $\gamma_t(T') \le \gamma_t(T) - 2$. Let S be a $\gamma_t(T)$-set that contains no leaf. Then, $\{u_1, u_2\} \subset S$. Let $S' = S - \{u_1, u_2\}$. If S' contains a neighbor of v, then S' is a TDS of T', whence $\gamma_t(T') \le \gamma_t(T) - 2$. If S' contains no neighbor of v, i.e., if $S \cap N[v] \subseteq \{u_1, v\}$, then S' is an ATDS of T' relative to v, and so $\gamma_t(T') = \gamma_t(T';v) \le |S'| = \gamma_t(T) - 2$. In both cases, $\gamma_t(T') \le \gamma_t(T) - 2$. Thus,

$$\gamma_t(T) \le \gamma_{\rm pr}(T) \le \gamma_{\rm pr}(T') + 2 = \gamma_t(T') + 2 \le \gamma_t(T). \tag{1}$$

Consequently we must have equality throughout the inequality chain (1). In particular, $\gamma_t(T) = \gamma_{\rm pr}(T)$.

Case 3. T is obtained from T' by operation $\mathcal{T}_3$. Suppose T is obtained from T' by adding the path u_1, u_2, u_3, u_4 and the edge $u_1 v$ where $v \in V(T')$. By Observation 2, there is a $\gamma_t(T)$-set S that contains no leaf. Thus, $\{u_2, u_3\} \subset S$ and $u_4 \notin S$. If $u_1 \in S$, then we can simply replace u_1 in S by some other neighbor of v. Hence we may assume that $S \cap \{u_1, u_2, u_3, u_4\} = \{u_2, u_3\}$. Thus, $S - \{u_2, u_3\}$ is a TDS of T', and so $\gamma_t(T') \leq \gamma_t(T) - 2$. Further, any $\gamma_{\rm pr}(T')$-set can be extended to a PDS of T by adding to it the vertices u_2 and u_3. Hence, $\gamma_{\rm pr}(T) \leq \gamma_{\rm pr}(T') + 2$. Thus the inequality chain (1) holds, whence $\gamma_t(T) = \gamma_{\rm pr}(T)$. $\square$

We show next that every $(\gamma_t, \gamma_{\rm pr})$-tree belongs to the family $\mathcal{T}$.

Lemma 4 *If T is a $(\gamma_t, \gamma_{\rm pr})$-tree, then $T \in \mathcal{T}$.*

Proof. We proceed by induction on the order $n \geq 2$ of a $(\gamma_t, \gamma_{\rm pr})$-tree. If T is a star or a double-star, then T can be obtained from P_2 by repeated applications of operation $\mathcal{T}_1$. Hence we may assume that $\mathrm{diam}(T) \geq 4$. Let T be rooted at a leaf r of a longest path P. Let P be a r-u path, and let v be the neighbor of u. Further, let w denote the parent of v on this path, x the parent of w and y the parent of x. Then, u is a leaf of T.

By Observation 2(b) since T is a $(\gamma_t, \gamma_{\rm pr})$-tree, it contains a $\gamma_{\rm pr}(T)$-set S that contains no leaf of T. In particular, $u \notin S$, $\{v, w\} \subset S$, and the vertices v and w are matched in $T[S]$.

Suppose $d_T(v) \geq 3$. Then v is adjacent to at least two leaves. Let $T' = T - u$. Then, $\gamma_t(T') = \gamma_t(T)$ and $\gamma_{\rm pr}(T') = \gamma_{\rm pr}(T)$. Hence, T' is a $(\gamma_t, \gamma_{\rm pr})$-tree. By the inductive hypothesis, $T' \in \mathcal{T}$. Since v is a remote vertex in T', the vertex v is in every $\gamma_{\rm pr}(T')$-set. Hence, T can be obtained from T' by operation $\mathcal{T}_1$. Thus we may assume $d_T(v) = 2$, for otherwise $T \in \mathcal{T}$, as desired.

Since the PDS S contains no leaf of T, it follows that the vertex w is adjacent to no remote vertex other than v. Suppose w is adjacent to a leaf v'. Let $T' = T - v'$. Then, S is a PDS of T', and so $\gamma_{\rm pr}(T') \leq |S| = \gamma_{\rm pr}(T)$. By Observation 2(a), there is $\gamma_t(T')$-set that contains no leaf of T'. Such a TDS of T' contains the vertex w, and is therefore also a TDS of T, whence $\gamma_t(T) \leq \gamma_t(T')$. Therefore, $\gamma_t(T') \leq \gamma_{\rm pr}(T') \leq \gamma_{\rm pr}(T) = \gamma_t(T) \leq \gamma_t(T')$. Consequently, we must have equality throughout this inequality chain. In particular, $\gamma_t(T') = \gamma_{\rm pr}(T')$ and S is a $\gamma_{\rm pr}(T')$-set. By the inductive hypothesis, $T' \in \mathcal{T}$. Since the vertex w is in some $\gamma_{\rm pr}(T')$-set, namely S, the tree T can be obtained from T' by operation $\mathcal{T}_1$. Thus we may assume $d_T(w) = 2$, for otherwise $T \in \mathcal{T}$, as desired.

Suppose $x \notin \mathrm{pn}(w, S)$. Let $T' = T - \{u, v, w\}$. Then, $S - \{v, w\}$ is a PDS of T', and so $\gamma_{\rm pr}(T') \leq \gamma_{\rm pr}(T) - 2$. Any $\gamma_t(T')$-set can be extended to a TDS of T by adding to it the vertices v and w, and so $\gamma_t(T) \leq \gamma_t(T') + 2$. Hence, $\gamma_t(T') \leq \gamma_{\rm pr}(T') \leq \gamma_{\rm pr}(T) - 2 = \gamma_t(T) - 2 \leq \gamma_t(T')$. Consequently, we must have equality throughout this inequality chain. In particular, $\gamma_t(T') = \gamma_{\rm pr}(T')$ and $\gamma_t(T) = \gamma_t(T') + 2$. By the inductive hypothesis, $T' \in \mathcal{T}$. Now, $\gamma_t(T'; x) \le \gamma_t(T')$. Any $\gamma_t(T'; x)$-set can be extended to a TDS of T by adding to it the vertices v and w, whence $\gamma_t(T) \le \gamma_t(T'; x) + 2 \le \gamma_t(T') + 2 = \gamma_t(T)$. Consequently, $\gamma_t(T'; x) = \gamma_t(T')$. Thus, T can be obtained from T' by operation $\mathcal{T}_2$. Hence we may assume that $x \in {\rm pn}(w, S)$, for otherwise $T \in \mathcal{T}$, as desired.

Since $x \in {\rm pn}(w, S)$, it follows from our choice of the PDS S (which contains no leaf of T) that $d_T(x) = 2$. Let $T' = T - \{u, v, w, x\}$. Then, $S - \{v, w\}$ is a PDS of T', and so $\gamma_{\rm pr}(T') \le \gamma_{\rm pr}(T) - 2$. Further, $\gamma_t(T) \le \gamma_t(T') + |\{v, w\}| = \gamma_t(T') + 2$. Therefore, $\gamma_t(T') \le \gamma_{\rm pr}(T') \le \gamma_{\rm pr}(T) - 2 = \gamma_t(T) - 2 \le \gamma_t(T')$, whence $\gamma_t(T') = \gamma_{\rm pr}(T')$. By the inductive hypothesis, $T' \in \mathcal{T}$. Thus, T can be obtained from T' by operation $\mathcal{T}_3$. □

As an immediate consequence of Lemmas 3 and 4 we have the following characterization of $(\gamma_t, \gamma_{\rm pr})$-trees.

Theorem 5 *A tree T is a $(\gamma_t, \gamma_{\rm pr})$-tree if and only if $T \in \mathcal{T}$.*

3 A characterization of $(\gamma_{\rm pr}, 2\beta_1)$-trees

We now turn our attention to a characterization of $(\gamma_{\rm pr}, 2\beta_1)$-trees. First we introduce two types of operations that are used to construct $(\gamma_{\rm pr}, 2\beta_1)$-trees. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the following two operations on a tree T. (By *attaching a subdivided star* to a vertex v of T we mean adding a subdivided star $K^*_{1,k}$ to T and joining v to either a leaf of the subdivided star if $k = 1$ or to the vertex of degree k in the subdivided star if $k \ge 2$.)

Operation $\mathcal{F}_1$. Attach a path P_1 to a vertex of T, which is in every $\gamma_{\rm pr}(T)$-set.

Operation $\mathcal{F}_2$. Attach the subdivided star to a vertex of T, which is in every $\gamma_{\rm pr}(T)$-set.

Lemma 6 *If $\gamma_{\rm pr}(T') = 2\beta_1(T')$ and T is obtained from T' by operation $\mathcal{F}_1$, then $\gamma_{\rm pr}(T) = 2\beta_1(T)$.*

Proof. Suppose that T is obtained from T' by adding a new vertex u and the edge uv where $v \in V(T')$. Then, v is in every $\gamma_{\rm pr}(T')$-set. It is easily seen that $\gamma_{\rm pr}(T) = \gamma_{\rm pr}(T')$. We show next that $\beta_1(T) = \beta_1(T')$. For otherwise, $\beta_1(T) = \beta_1(T') + 1$. If M is a maximum matching of T, then $uv \in M$ and $M' = M - \{uv\}$ is a maximum matching of T'. Clearly, $V(M')$ is a PDS of T'. Note that

$$\gamma_{\rm pr}(T') \le |V(M')| = |V(M)| - 2 = 2(\beta_1(T) - 1) = 2\beta_1(T').$$

Hence, $V(M')$ is a $\gamma_{\rm pr}(T')$-set, but $v \notin V(M')$. This contradiction implies that $\beta_1(T) = \beta(T')$. So $\gamma_{\rm pr}(T) = 2\beta_1(T)$. □

Lemma 7 *If $\gamma_{\rm pr}(T') = 2\beta_1(T')$ and T is obtained from T' by operation $\mathcal{F}_2$, then $\gamma_{\rm pr}(T) = 2\beta_1(T)$.*

Proof. Suppose that T is obtained from T' by attaching a subdivided star $K^*_{1,k}$, $k \geq 1$, to a vertex v in T', where v is in every $\gamma_{\rm pr}(T')$-set. Let u be the vertex of the subdivided star that is adjacent to v. Let $N(u) - \{v\} = \{u_1, \ldots, u_k\}$ and for $i = 1, \ldots, k$, let $N(u_i) - \{u\} = \{u'_i\}$. Let M be a maximum matching of T. Without loss of generality, we may assume that $u_j u'_j \in M$, $1 \leq j \leq k$. If $uv \notin M$, then $M - \{u_j u'_j \mid j = 1, \ldots, k\}$ is a matching of T', and so $\beta_1(T') \geq \beta_1(T) - k$. If $uv \in M$, then $M' = M - \{uv, u_j u'_j \mid j = 1, \ldots, k\}$ is a matching of T'. Since every $\gamma_{\rm pr}(T')$-set contains the vertex v and $\gamma_{\rm pr}(T') = 2\beta_1(T')$, it follows that every maximum matching of T' must contain the vertex v. Hence, M' is not maximum matching of T', that is, $\beta_1(T') \geq \beta(T) - k$. Furthermore, we note that any maximum matching of T' can be extended to a matching of T by adding all edges $u_j u'_j$, $j = 1, \ldots, k$, whence $\beta_1(T) \geq \beta_1(T') + k$. Consequently, $\beta_1(T) = \beta_1(T') + k$.

Now let S be a $\gamma_{\rm pr}(T)$-set. Then, $\{u_1, \ldots, u_k\} \subset S$. If $u \notin S$, then we must have that $\{u'_1, \ldots, u'_k\} \subset S$. This means that $S' = S - \{u_j, u'_j \mid j = 1, 2, \cdots, k\}$ is a PDS of T' (with u_j and u'_j paired in $T[S]$). Thus, we have

$$\gamma_{\rm pr}(T') \leq |S'| = |S| - 2k = \gamma_{\rm pr}(T) - 2k \leq 2(\beta_1(T) - k) = 2\beta_1(T').$$

Since $\gamma_{\rm pr}(T') = 2\beta_1(T')$, we must have equality throughout the above inequality chain. In particular, $\gamma_{\rm pr}(T) = 2\beta_1(T)$. Suppose $u \in S$ and there exists a vertex u_i paired with u in $T[S]$. As above, it is easily verified that $\gamma_{\rm pr}(T) = 2\beta_1(T)$. Finally, suppose $u \in S$ with u and v paired in $T[S]$. Then, $\{u'_1, \ldots, u'_k\} \subset S$. If $N(v) \subset S$, then $S - \{u, v\}$ is a PDS of T, contradicting the minimality of S. Hence there exists a vertex $w \in N(v) \cap V(T')$ such that $w \notin S$. Thus, $S^* = (S \cup \{w\}) - \{u\}$ is a minimum PDS of T and $u \notin S^*$. As before, the desired result follows. □

We now define the family of trees $\mathcal{F}$ as:

$\mathcal{F} = \{T \mid T$ is obtained from P_2 by a finite sequence of operations $\mathcal{F}_1$ and $\mathcal{F}_2\}$.

As an immediate consequence of Lemmas 6 and 7, we have the following result.

Lemma 8 *If $T \in \mathcal{F}$, then T is a $(\gamma_{\rm pr}, 2\beta_1)$-tree.*

We show next that every $(\gamma_{\rm pr}, 2\beta_1)$-tree belongs to the family $\mathcal{F}$.

Lemma 9 *If T is a $(\gamma_{\rm pr}, 2\beta_1)$-tree, then $T \in \mathcal{F}$.*

Proof. We proceed by induction on the order $n \geq 2$ of a $(\gamma_{\rm pr}, 2\beta_1)$-tree T. If $n \leq 5$, then $T \in \{K_{1,2}, K_{1,3}, K_{1,4}, P_2, P_3, P_5\}$ and clearly $T \in F$. Let $n \geq 6$ and assume that for all $(\gamma_{\rm pr}, 2\beta_1)$-trees T' of order $n' < n$, it holds that $T' \in \mathcal{F}$. Let T be a $(\gamma_{\rm pr}, 2\beta_1)$-tree of order n. If T is a star, it is easily seen that T can be obtained from a finite sequence of operations $\mathcal{F}_1$. So we assume that T is not a star. Let $v_1, v_2, \ldots, v_\ell$ be a longest path in T. If $\ell = 4$, then T is a double star and $\gamma_{\rm pr}(T) \neq 2\beta_1(T)$, a contradiction. Hence, $\ell \geq 5$.

Case 1. $d(v_2) \geq 3$. Then, v_2 is adjacent to at least two leaves. Let $T' = T - v_1$. Since v_2 is a remote vertex in T', every PDS of T' contains v_2, and so $\gamma_{\rm pr}(T) \leq \gamma_{\rm pr}(T')$. Further, $\beta_1(T) \geq \beta_1(T')$. Hence, $\gamma_{\rm pr}(T) \leq \gamma_{\rm pr}(T') \leq 2\beta_1(T') \leq 2\beta_1(T) = \gamma_{\rm pr}(T)$. Consequently we must have equality throughout this inequality chain. In particular, $\gamma_{\rm pr}(T') = 2\beta_1(T')$. By the inductive hypothesis, $T \in F$. Since the vertex v_2 belongs to every $\gamma_{\rm pr}(T')$-set, the tree T can be obtained from T' by operation $\mathcal{F}_1$.

Case 2. $d(v_2) = 2$ and $d(v_3) = 2$. Let $T' = T - N[v_2]$. First we note that $\gamma_{\rm pr}(T) \leq \gamma_{\rm pr}(T') + 2$ and $\beta_1(T) \geq \beta_1(T') + 1$. Therefore, we have

$$\gamma_{\rm pr}(T) \leq \gamma_{\rm pr}(T') + 2 \leq 2\beta_1(T') + 2 \leq 2\beta_1(T).$$

Since $\gamma_{\rm pr}(T) = 2\beta_1(T)$, we must have equality throughout the above inequality chain. In particular, $\gamma_{\rm pr}(T') = 2\beta_1(T')$ and $\beta_1(T') = \beta_1(T) - 1$. We show next that the vertex v_4 is in every $\gamma_{\rm pr}(T')$-set. Suppose there exists a $\gamma_{\rm pr}(T')$-set $S' = V(M')$ such that $v_4 \notin S'$, where M' is a maximum matching of T'. Then $M' \cup \{v_1v_2, v_3v_4\}$ is a matching of T. So $\beta_1(T) = 2+\beta_1(T')$, contradicting the fact that $\beta_1(T) = \beta_1(T')+1$. Hence, the vertex v_4 is in every $\gamma_{\rm pr}(T')$-set. Therefore, the tree T can be obtained from T' by operation $\mathcal{F}_2$.

Case 3. $d(v_2) = 2$ and $d(v_3) \geq 3$. Let T' and T'' denote the components of $T - v_3v_4$ containing v_4 and v_3, respectively. Suppose v_3 is adjacent to $t \geq 1$ leaves. Then, $\beta_1(T'') \geq d(v_3) - t$ and $\gamma_{\rm pr}(T'') \leq 2(d(v_3) - t - 1)$, and so

$$\begin{array}{rcl}\gamma_{\rm pr}(T) & \leq & \gamma_{\rm pr}(T') + \gamma_{\rm pr}(T'') \\ & \leq & 2(\beta_1(T') + d(v_3) - t - 1) \\ & < & 2(\beta_1(T') + \beta_1(T'')) \\ & \leq & \beta_1(T'),\end{array}$$

which contradicts the fact that T is a $(\gamma_{\rm pr}, 2\beta_1)$-tree. Hence, v_3 is adjacent to no leaf, and so each child of v_3 is a remote vertex. If some child of v_3 has degree at least 3, then proceeding as in Case 1, $T \in \mathcal{F}$. Hence we may assume that each child of v_3 has degree 2. Let $N(v_3) - \{v_2, v_4\} = \{u_1, \ldots, u_k\}$, where $k = d(v_3) - 2 \geq 1$. For $i = 1, \ldots, k$, let $N(u_i) - \{v_3\} = \{u_i'\}$. Since $\gamma_{\rm pr}(T) \leq 2(d(v_3) - 1) + \gamma_{\rm pr}(T')$ and $\beta_1(T) \geq d(v_3) - 1 + \beta_1(T')$, we have

$$\begin{array}{rcl}\gamma_{\rm pr}(T) & \leq & 2(d(v_3) - 1) + \gamma_{\rm pr}(T') \\ & \leq & 2(d(v_3) - 1) + 2\beta_1(T') \\ & \leq & 2\beta_1(T) \\ & = & \gamma_{\rm pr}(T).\end{array}$$

Consequently, we must have equality throughout this inequality chain. In particular, $\gamma_{\rm pr}(T') = 2\beta_1(T')$ and $\beta_1(T) = \beta_1(T') + d(v_3) - 1$.

We show next that the vertex v_4 is in every $\gamma_{\rm pr}(T')$-set. Suppose there exists a $\gamma_{\rm pr}(T')$-set $S' = V(M')$ such that $v_4 \notin S'$, where M' is a maximum matching of T'. Then $M = M' \cup \{v_1v_2, v_3v_4, u_1u_1', \ldots, u_ku_k'\}$ is a matching of T, and so $\beta_1(T) \geq |M| = \beta_1(T') + d(v_3)$, contradicting the fact that $\beta_1(T) = \beta_1(T') + d(v_3) - 1$. Hence, the vertex v_4 is in every $\gamma_{\rm pr}(T')$-set. Therefore, the tree T can be obtained from T' by operation $\mathcal{F}_2$. $\Box$

As an immediate consequence of Lemmas 8 and 9, we have the main result in the section.

Theorem 10 *A tree T is a $(\gamma_{\rm pr}, 2\beta_1)$-tree if and only if $T \in \mathcal{F}$.*

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(Received 3 Sep 2002; revised 10 Feb 2004)